DiPerna–Majda measures and uniform integrability

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Abstract. The purpose of this note is to discuss the relationship among Rosenthal's modulus of uniform integrability, Young measures and DiPerna-Majda measures. In particular, we give an explicit characterization of this modulus and state a criterion of the uniform integrability in terms of these measures. Further, we show applications to Fatou's lemma.

Keywords: bounded sequences, DiPerna-Majda measures, Fatou's lemma, relative sequential weak compactness, uniform integrability, Young measures

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1. Introduction

We will consider a problem of uniform integrability of bounded sequences in $L^1(\Omega;\mathbb{R}^m)$ where $\Omega\subset\mathbb{R}^n$ is a bounded domain. Let us recall that a bounded sequence $\{u_k\}_{k\in\mathbb{N}}\subset L^1(\Omega;\mathbb{R}^m)$ is said to be uniformly integrable if

$$\forall\,\varepsilon>0\ \ \exists\,K>0\ :\ \ \sup_{k\in\mathbb{N}}\int_{\{x\in\Omega;\ |u_k(x)|>K\}}|u_k(x)|\,\mathrm{d}x\le\varepsilon. \tag{1}$$

The uniform integrability is equivalent to the relative (sequential) weak L^1-compactness of the sequence in question via the Dunford-Pettis compactness criterion; cf. e.g. [13, Section IV.8] or [30]. We refer e.g. to [9], [11], [13], [22] for other criteria ensuring the relative weak compactness. Briefly, any bounded and uniformly integrable sequence $\{u_k\}_{k\in\mathbb{N}}$ in $L^1(\Omega;\mathbb{R}^m)$ contains a subsequence converging weakly in $L^1(\Omega;\mathbb{R}^m)$. The opposite implication is also valid: weakly converging sequences in $L^1(\Omega;\mathbb{R}^m)$ are uniformly integrable. This additional requirement, namely the uniform integrability, on bounded sequences in $L^1(\Omega;\mathbb{R}^m)$ to be relatively weakly compact reflects the non-reflexiveness of $L^1(\Omega;\mathbb{R}^m)$.

Saadoune and Valadier [25] introduced the so-called Rosenthal modulus of uniform integrability η; cf. also [7], [19]. Taking a bounded sequence $\{u_k\}_{k\in\mathbb{N}}$ in $L^1(\Omega;\mathbb{R}^m)$ then ("meas" stands for the Lebesgue measure on $\mathbb{R}^n$)

$$\eta(\{u_k\}_{k\in\mathbb{N}})=\lim_{\varepsilon\to 0_+}\left[\sup_{k\in\mathbb{N}}\left\{\int_A|u_k(x)|\,\mathrm{d}x;\ \mathrm{meas}\,(A)\le\varepsilon,\ A\subset\Omega\right\}\right].$$

It is proved in [25] (see also [27]) that $\eta(\{u_k\}_{k\in\mathbb{N}})$ can be equivalently expressed as

$$\eta(\{u_k\}_{k\in\mathbb{N}}) = \lim_{K\to\infty}\left[\sup_{k\in\mathbb{N}}\int_{\{x\in\Omega;\ |u_k(x)|>K\}}|u_k(x)|\,\mathrm{d}x\right]. \tag{2}$$

We remark that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is uniformly integrable if and only if $\eta(\{u_k\}_{k\in\mathbb{N}})=0$.

To understand better the meaning of Rosenthal's modulus let us suppose that $\{u_k\}_{k\in\mathbb{N}}$ in $L^1(\Omega;\mathbb{R}^m)$ is *not* uniformly integrable. This means that

$$\exists\,\varepsilon>0\ \ \forall\,K>0\ :\ \sup_{k\in\mathbb{N}}\int_{\{x\in\Omega;\ |u_k(x)|>K\}}|u_k(x)|\,\mathrm{d}x>\varepsilon. \tag{3}$$

Then $\eta(\{u_k\}_{k\in\mathbb{N}})$ is the supremum of all of these ε's. This means that for Rosenthal's modulus instead of ε the sharp inequality (3) changes to $\geq$. In fact, this is the most convenient definition for our purposes and we are going to use it. Let us just mention that we will sometimes speak about the uniform integrability without saying explicitly that we mean that one in $L^1(\Omega;\mathbb{R}^m)$.

In what follows $L^p(\Omega;\mathbb{R}^m)$, $1\leq p\leq+\infty$ is the usual Lebesgue space of measurable functions $\Omega\to\mathbb{R}^m$ that are integrable with their p-th power (for $1\leq p<+\infty$) or essentially bounded on Ω (if $p=+\infty$). If $m=1$, we write only $L^p(\Omega)$ instead of $L^p(\Omega;\mathbb{R}^1)$. For more information we refer e.g. to [13].

The aim of the note is twofold. Firstly, to show how modern mathematical apparatus of Young measures and their generalizations fits in the classical topic as the uniform integrability, secondly, to provide better understanding of these generalizations. The plan of this paper is as follows. First, we briefly introduce Young measures (see [4], [20], [26], [31], [32]) and their generalization called DiPerna-Majda measures; cf. [12], [21]. Afterwards, we study the relation between Rosenthal's modulus and measures of DiPerna and Majda. In particular, we give an explicit characterization of this modulus and show its intimate relationship to the support of these measures. We also touch properties of DiPerna-Majda measures which were analyzed in detail in [21]. This enables us to find a new characterization of uniformly integrable sequences. Further, we apply our results to the Fatou lemma getting thus simple and straightforward proofs of interesting inequalities involving Young measures.

Young measures. The Young measures [31] represent a modern mathematical tool to hold certain "limit" information about oscillations in nonlinear problems arising in optimal control theory, variational calculus, partial differential equations, game theory, etc.; more details about Young measures can be found, e.g., in [4], [6], [10], [15], [20], [21], [23], [24], [28], [29], [30]. The Young measures on a domain $\Omega\subset\mathbb{R}^n$ are weakly measurable mappings $x\mapsto\nu_x:\Omega\to rca\,(\mathbb{R}^m)$ with values in probability measures; "*rca*" denotes the set of regular countably additive set functions on the Borel σ-algebra on $\mathbb{R}^m$ (cf. [13]) with a bounded total variation and the adjective "weakly measurable" means that, for any $v \in C_0(\mathbb{R}^m)$, the mapping $\Omega \to \mathbb{R} : x \mapsto \langle \nu_x, v\rangle = \int_{\mathbb{R}^m} v(\lambda)\nu_x(\mathrm{d}\lambda)$ is measurable in the usual sense. Let us remind that, by the Riesz theorem, $rca(\mathbb{R}^m)$, normed by the total variation, is a Banach space which is isometrically isomorphic with $C_0(\mathbb{R}^m)^*$, where $C_0(\mathbb{R}^m)$ stands for the space of all continuous functions $\mathbb{R}^m \to \mathbb{R}$ vanishing at infinity. Let us denote the set of all Young measures by $\mathcal{Y}(\Omega;\mathbb{R}^m)$. It is known that $\mathcal{Y}(\Omega;\mathbb{R}^m)$ is a convex subset of $L^\infty_{\mathrm{w}}(\Omega; rca(\mathbb{R}^m)) \cong L^1(\Omega; C_0(\mathbb{R}^m))^*$, where the subscript "w" indicates the property "weakly measurable". A classical result [10], [28], [31] is that, for every sequence $\{u_k\}_{k\in\mathbb{N}}$ bounded in $L^\infty(\Omega;\mathbb{R}^m)$, there exists its subsequence (denoted by the same indices for notational simplicity) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^m)$ such that

$$\forall\, v \in C_0(\mathbb{R}^m) \ : \quad \lim_{k\to\infty} v\circ u_k = v_\nu \qquad \text{weakly* in } \ L^\infty(\Omega), \tag{4}$$

where $[v\circ u_k](x) = v(u_k(x))$ and

$$v_\nu(x) = \int_{\mathbb{R}^m} v(\lambda)\nu_x(\mathrm{d}\lambda). \tag{5}$$

Let us denote by $\mathcal{Y}^\infty(\Omega;\mathbb{R}^m)$ the set of all Young measures which are created by this way, i.e. by taking all bounded sequences in $L^\infty(\Omega;\mathbb{R}^m)$. Note that (4) actually holds for any $v : \mathbb{R}^m \to \mathbb{R}$ continuous.

A generalization of this result was formulated by Maria Schonbek [26] (cf. also [4], for $p = 1$ especially [20] and [24] where further generalization in this direction has been performed) for the case $1 \le p < +\infty$: for every sequence $\{u_k\}_{k\in\mathbb{N}}$ bounded in $L^p(\Omega;\mathbb{R}^m)$ there exists its subsequence (denoted by the same indices) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^m)$ such that

$$\forall\, v \in C_p(\mathbb{R}^m) : \quad \lim_{k\to\infty} v\circ u_k = v_\nu \qquad \text{weakly in } \ L^1(\Omega), \tag{6}$$

where

$$C_p(\mathbb{R}^m) = \{v \in C(\mathbb{R}^m); v(\lambda) = o(|\lambda|^p) \ \text{ for } \ |\lambda| \to \infty\}.$$

We denote by $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$ the set of all Young measures which are created by this way, i.e. by taking all bounded sequences in $L^p(\Omega;\mathbb{R}^m)$. The reader can find in [20] that $\mathcal{Y}^p(\Omega;\mathbb{R}^m) = \{\nu \in \mathcal{Y}(\Omega;\mathbb{R}^m);\ \int_\Omega\int_{\mathbb{R}^m} |\lambda|^p\nu_x(\mathrm{d}\lambda)\,\mathrm{d}x < +\infty\}$. We call a sequence $\{u_k\}_{k\in\mathbb{N}}$ satisfying (6) a generating sequence of $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$.

DiPerna-Majda measures. Sometimes nonlinear problems may exhibit, beside the rapid-oscillation phenomena, also concentration effects which were previously neglected because the L^p-Young measures admits only test functions with the growth strictly lower than p. DiPerna and Majda ([12]) developed a tool to handle both oscillation and concentration effects simultaneously. Let us take a complete (i.e. containing constants and separating points from closed subsets) separable (i.e. containing a countable subset which is dense with respect to the supremum norm) ring $\mathcal{R}$ of continuous bounded functions $\mathbb{R}^m \to \mathbb{R}$. It is known (cf. [14, § 3.12.21]) that there is a one-to-one correspondence $\mathcal{R} \mapsto \beta_{\mathcal{R}}\mathbb{R}^m$ between such rings and metrizable compactifications of $\mathbb{R}^m$; by a compactification we mean here a compact set, denoted by $\beta_{\mathcal{R}}\mathbb{R}^m$ into which $\mathbb{R}^m$ is embedded homeomorphically and densely. We will not distinguish between $\mathbb{R}^m$ and its image in $\beta_{\mathcal{R}}\mathbb{R}^m$. If we take, for example, the smallest complete ring containing only continuous functions possessing limits for $|\lambda| \to \infty$, then the corresponding compactification is the one point (Alexandroff) compactification, i.e., $\beta_{\mathcal{R}}\mathbb{R}^m = \mathbb{R}^m \cup \{\infty\}$. See [12], [16], [24] for other examples.

DiPerna and Majda showed that, having a bounded sequence $\{u_k\}_{k\in\mathbb{N}}$ in $L^p(\Omega;\mathbb{R}^m)$ with $1 \le p < +\infty$ and Ω an open domain in $\mathbb{R}^n$, there exists its subsequence (denoted by the same indices), a positive Radon measure $\sigma \in rca(\bar{\Omega})$, and a Young measure $\hat{\nu} \in \mathcal{Y}(\bar{\Omega},\sigma;\beta_{\mathcal{R}}\mathbb{R}^m)$ (i.e. we consider here the closure $\bar{\Omega}$ of Ω endowed with the Radon measure σ instead of the Lebesgue measure as previously) such that

$$\forall\, g \in C(\bar{\Omega})\ \forall\, v_0 \in \mathcal{R}:\ \lim_{k\to\infty}\int_\Omega g(x)v(u_k(x))\,\mathrm{d}x = \int_{\bar{\Omega}}\int_{\beta_{\mathcal{R}}\mathbb{R}^m} g(x)v_0(\lambda)\hat{\nu}_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x), \tag{7}$$

where $v(\lambda) = v_0(\lambda)(1+|\lambda|^p)$. In particular, putting $v_0 = 1 \in \mathcal{R}$ we can see that

$$\lim_{k\to\infty}(1+|u_k|^p) \;=\; \sigma \quad \text{weakly* in} \;\; rca(\bar{\Omega}).$$

Let us again denote by $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ the set of all pairs $(\sigma,\hat{\nu}) \in rca(\bar{\Omega}) \times \mathcal{Y}(\bar{\Omega},\sigma;\beta_{\mathcal{R}}\mathbb{R}^m)$ created by this way, i.e. $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ contains just such $(\sigma,\hat{\nu})$ for which there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ such that (7) holds; note that, taking $v_0 = 1$, we can see that such a sequence must be inevitably bounded in $L^p(\Omega;\mathbb{R}^m)$. Also here the sequence appearing in (7) is called the generating sequence of $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$. We refer to [21] for properties and the full explicit description of $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$.

The next paragraph shows how DiPerna-Majda measures are related to non-uniformly integrable sequences.

Remarks on concentrations. Let $p = 1$, $n = m = 1$, $\Omega = (0,1)$ and $\beta_{\mathcal{R}}\mathbb{R} = \mathbb{R}\cup\{\infty\}$ for a moment. There are two basic types of non-uniformly integrable bounded sequences in $L^1(\Omega)$ generating a DiPerna-Majda measure.

Example 1.

$$u_k(x) = \begin{cases} k & \text{if } \ x \in \left(\frac{1}{2}-\frac{1}{k}\,,\,\frac{1}{2}+\frac{1}{k}\right) \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.

$$v_k(x) = \begin{cases} k & \text{if } x \in \left(\frac{l}{k} - \frac{1}{2k^2}, \frac{l}{k} + \frac{1}{2k^2}\right) \cap (0,1),\ l \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Similar examples to these have been already used in [24]; cf. also [5]. It can be shown (see [24]) that $\{u_k\}_{k\in\mathbb{N}}$ generates a DiPerna-Majda measure $(\sigma^1, \hat{\nu}^1) \in \mathcal{D}M^1_{\mathcal{R}}(\Omega;\mathbb{R})$ such that $\sigma^1(\mathrm{d}x) = \mathrm{d}x + 2\delta_{0.5}$ and $\hat{\nu}^1_x = \delta_0$, $x \neq 0.5$ and $\hat{\nu}^1_{1/2} = \delta_\infty$.

The sequence $\{v_k\}_{k\in\mathbb{N}}$ generates $(\sigma^2, \hat{\nu}^2) \in \mathcal{D}M^1_{\mathcal{R}}(\Omega;\mathbb{R})$ where $\sigma^2(\mathrm{d}x) = 2\mathrm{d}x$ and $\hat{\nu}^2_x = 0.5\delta_0 + 0.5\delta_\infty$ for $x \in (0,1)$; cf. [24].

The sequence $\{u_k\}$ concentrates around the point $x = 0.5$ meanwhile $\{v_k\}$ exhibits a "continuous" concentration smeared out uniformly throughout the whole Ω. In [21] it is shown that these two different situations are reflected in the properties of σ^1 and σ^2. The measure σ^1 is not absolutely continuous with respect to the Lebesgue measure but σ^2 is.

Taking now $(\sigma, \hat{\nu}) \in \mathcal{D}M^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ (for simplicity $\beta_{\mathcal{R}}\mathbb{R}^m = \mathbb{R}^m \cup \{\infty\}$) generated by some $\{w_k\}_{k\in\mathbb{N}}$ and having $E \subset \bar{\Omega}$, σ-measurable, with the characteristic function χ_E then it follows from [21, Theorem 2] that the following three basic situations can take place:

(i) $\{\chi_E w_k\}_{k\in\mathbb{N}}$ is uniformly integrable if and only if $\hat{\nu}_x(\infty) = 0$ for σ-almost all $x \in E$,

(ii) $\{\chi_E w_k\}_{k\in\mathbb{N}}$ exhibits a "point" concentration at $x \in E$ if and only if $\hat{\nu}_x(\infty) = 1$ and $\sigma(\{x\}) > 0$, (see Example 1),

(iii) $\{\chi_E w_k\}_{k\in\mathbb{N}}$ shows a "continuous" concentration on E if and only if $0 < \hat{\nu}_x(\infty) < 1$ for σ-almost all $\sigma^2(\mathrm{d}x) = 2\mathrm{d}x$ and $\hat{\nu}^2_x = 0.5\delta_0 + 0.5\delta_\infty$ for $x \in (0,1)$; cf. [24].

If (i), or (iii) is valid then the restriction of σ on E is absolutely continuous with respect to the Lebesgue measure.

Recently Roubíček (see [20], [24]) proved the following result.

Proposition 1. *Let $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ be a generating sequence of $(\sigma, \hat{\nu}) \in \mathcal{D}M^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$. Then $\{|u_k|^p\}_{k\in\mathbb{N}}$ is uniformly integrable if and only if*

$$\int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^m \setminus \mathbb{R}^m} \hat{\nu}_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x) = 0. \tag{8}$$

In particular, taking $p = 1$ we have that $\{u_k\}_{k\in\mathbb{N}}$ is uniformly integrable if and only if (8) is valid.

2. DiPerna-Majda measures and the Rosenthal modulus

Let us start with the following lemma.

Lemma 1. *Let $\{u_k\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ be bounded. Then $\eta(\{|u_k|\}_{k\in\mathbb{N}}) = \eta(\{1+|u_k|\}_{k\in\mathbb{N}})$.*

Proof: This is easy. □

Now we show that if $(\sigma,\hat\nu)\in\mathcal{D}\mathcal{M}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ then $\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m}\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x)$ does not depend on the particular compactification of $\mathbb{R}^m$, i.e., on $\beta_{\mathcal{R}}\mathbb{R}^m$. In other words, we show that it is only related to the generating sequence.

Proposition 2. *Let $\mathcal{R}$ and $\mathcal{R}'$ be two separable complete and closed rings of continuous bounded functions $\mathbb{R}^m\to\mathbb{R}$ and let $\{u_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^m)$, $1\le p<+\infty$ generate $(\sigma,\hat\nu)\in\mathcal{D}\mathcal{M}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and also $(\sigma',\hat\nu')\in\mathcal{D}\mathcal{M}^p_{\mathcal{R}'}(\Omega;\mathbb{R}^m)$. Then*

$$\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m}\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x)=\int_{\bar\Omega}\int_{\beta_{\mathcal{R}'}\mathbb{R}^m\setminus\mathbb{R}^m}\hat\nu'_x(\mathrm{d}\lambda)\sigma'(\mathrm{d}x).$$

Proof: Let us take $g=1$ and $v_0=1$ in (7). Then we have

$$\lim_{k\to\infty}\int_\Omega(1+|u_k(x)|^p)\,\mathrm{d}x = \int_{\bar\Omega}\int_{\mathbb{R}^m}\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x)+\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m}\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x). \tag{9}$$

On the other hand, we can write Lebesgue's decomposition of $\sigma(\mathrm{d}x)=d_\sigma(x)\mathrm{d}x+\sigma^s(\mathrm{d}x)$, where $d_\sigma\in L^1(\Omega)$ is the density of the absolutely continuous part of σ with respect to the Lebesgue measure and σ^s is the singular part of σ. It follows from [21, Theorem 2] that $\sigma^s(\{x\in\bar\Omega;\ \int_{\mathbb{R}^m}\hat\nu_x(\mathrm{d}\lambda)>0\})=0$. Therefore, we have

$$\int_{\bar\Omega}\int_{\mathbb{R}^m}\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x)=\int_{\bar\Omega}\int_{\mathbb{R}^m}\hat\nu_x(\mathrm{d}\lambda)d_\sigma(x)\mathrm{d}x=\int_{\Omega}\int_{\mathbb{R}^m}\hat\nu_x(\mathrm{d}\lambda)d_\sigma(x)\mathrm{d}x.$$

The second equality is due to the fact that we assume the Lebesgue measure of $\partial\Omega=\bar\Omega\setminus\Omega$ being zero.

Finally, it follows from [20, Formulae (13-15)] and [21, Theorem 1] that $\{u_k\}_{k\in\mathbb{N}}$ generates a Young measure $\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^m)$ given for almost all $x\in\Omega$ by

$$\nu_x(\mathrm{d}\lambda)=d_\sigma(x)\frac{\hat\nu_x(\mathrm{d}\lambda)}{1+|\lambda|^p}\,.$$

Eventually, we can write (9) as

$$\lim_{k\to\infty}\int_\Omega(1+|u_k(x)|^p)\,\mathrm{d}x = \int_{\Omega}\int_{\mathbb{R}^m}(1+|\lambda|^p)\nu_x(\mathrm{d}\lambda)\,\mathrm{d}x+\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m}\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x).$$

The same procedure can be done also for $(\sigma', \hat{\nu}')$ and we obtain

$$\lim_{k\to\infty} \int_\Omega (1 + |u_k(x)|^p)\,\mathrm{d}x = \\ = \int_\Omega \int_{\mathbb{R}^m} (1 + |\lambda|^p)\nu_x(\mathrm{d}\lambda)\,\mathrm{d}x + \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}'}\mathbb{R}^m\setminus\mathbb{R}^m} \hat{\nu}'_x(\mathrm{d}\lambda)\sigma'(\mathrm{d}x).$$

The subtraction of the last equality from the last but one gives the assertion of the proposition. □

We will need the following auxiliary and easy lemma.

Lemma 2. *Let $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ be bounded and generate $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$. Then also any subsequence of $\{u_k\}_{k\in\mathbb{N}}$ generates the same $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$.*

Proof: The proof is quite the same as Step 5 in the proof of [24, Proposition 3.2.9].

□

Remark 1. (i) Proposition 2 and Lemma 2 lead us to the following conclusion. If $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ is bounded, generates a Young measure $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ and, moreover, there exists $\lim_{k\to\infty}\int_\Omega |u_k(x)|^p\,\mathrm{d}x$, then

$$\lim_{k\to\infty} \int_\Omega |u_k(x)|^p\,\mathrm{d}x = \\ = \int_\Omega \int_{\mathbb{R}^m} |\lambda|^p \nu_x(\mathrm{d}\lambda)\,\mathrm{d}x + \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m} \hat{\nu}_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x) = \sigma(\bar{\Omega}) - \mathrm{meas}\,(\Omega),$$

where $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ is an arbitrary DiPerna-Majda measure generated by some subsequence of $\{u_k\}_{k\in\mathbb{N}}$. Indeed, for such subsequence, say $\{u_{k_l}\}_{l\in\mathbb{N}}$, the last equality obviously holds and also $\lim_{k\to\infty}\int_\Omega |u_k(x)|^p\,\mathrm{d}x = \lim_{l\to\infty}\int_\Omega |u_{k_l}(x)|^p\,\mathrm{d}x$.
(ii) Following the same proof as that of the previous lemma we can show that once $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$, $1 \le p < +\infty$ is bounded and generates a Young measure $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ than this sequence generates also $\tilde{\nu} \in \mathcal{Y}^1(\Omega;\mathbb{R}^m)$ and $\nu = \tilde{\nu}$.

The next proposition characterizes η.

Proposition 3. *Let $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$, $1 \le p < +\infty$ generate $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$. Then*

$$\eta(\{|u_k|^p\}_{k\in\mathbb{N}}) = \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m} \hat{\nu}_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x).$$

Proof: The main idea is basically taken from Roubíček's proof of Proposition 1. Utilizing Lemma 1 it is sufficient to look for Rosenthal's modulus of $\{1+|u_k|^p\}_{k\in\mathbb{N}}$.

If $\{|u_k|^p\}_{k\in\mathbb{N}}$ is uniformly integrable the assertion follows from Proposition 1. Let us suppose that $\{|u_k|^p\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ is *not* uniformly integrable. First, let us abbreviate

$$T = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m} \hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x).$$

Further, we define for any $\varrho \geq 0$ the function $v_0^\varrho : \mathbb{R}^m \to \mathbb{R}$

$$v_0^\varrho(\lambda) = \begin{cases} 0 & \text{if } \ |\lambda| \leq \varrho \\ |\lambda| - \varrho & \text{if } \ \varrho \leq |\lambda| \leq \varrho + 1 \\ 1 & \text{if } \ |\lambda| \geq \varrho + 1. \end{cases}$$

Note that always $v_0^\varrho \in \mathcal{R}$. We can estimate for any $\varrho > 0$

$$\begin{aligned} T = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m} \hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x) &\leq \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m} v_0^\varrho(\lambda)\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x) \\ &= \lim_{k\to\infty}\int_\Omega v_0^\varrho(u_k(x))(1+|u_k(x)|^p)\,\mathrm{d}x \\ &\leq \sup_{k\in\mathbb{N}}\int_{\{x\in\Omega;\ |u_k(x)|\geq\varrho\}} (1+|u_k(x)|^p)\,\mathrm{d}x. \end{aligned}$$

This gives us that $\eta(\{u_k\}_{k\in\mathbb{N}}) \geq T$.

To finish the proof we have to show that T is the supremum of all of $\tilde T$'s satisfying

$$\forall\, K > 0 \ : \ \sup_{k\in\mathbb{N}}\int_{\{x\in\Omega;\ |u_k(x)|\geq K\}} (1+|u_k(x)|^p)\,\mathrm{d}x > \tilde T. \tag{10}$$

This will be done if for any $\delta > 0$ we find $K(\delta) > 0$ that $\sup_{k\in\mathbb{N}}\int_{\{x\in\Omega;\ |u_k(x)|\geq K(\delta)\}}(1+|u_k(x)|^p)\,\mathrm{d}x < T+\delta$.

Let us define $B_\varrho = \{\lambda\in\mathbb{R}^m;\ |\lambda|\leq\varrho\}$. We have from the Lebesgue dominated convergence theorem

$$\lim_{\varrho\to\infty}\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus B_\varrho} \hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x) = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m} \hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x) = T.$$

For any $\delta > 0$ we can find $\varrho > 0$ sufficiently large that

$$\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus B_\varrho} \hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x) < T + \frac{\delta}{4}\,.$$

On the other hand, there is $k_\varrho > 0$ that for any $k > k_\varrho$

$$\left|\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m} v_0^\varrho(\lambda)\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x) \ - \ \int_\Omega v_0^\varrho(u_k(x))(1+|u_k(x)|^p)\,\mathrm{d}x\right| < \frac{\delta}{4}\,.$$

As $v_0^\varrho = 0$ on B_ϱ we have also

$$\int_{\bar\Omega}\int_{\beta_{\mathcal R}\mathbb{R}^m} v_0^\varrho(\lambda)\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x) = \int_{\bar\Omega}\int_{\beta_{\mathcal R}\mathbb{R}^m\setminus B_\varrho} v_0^\varrho(\lambda)\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x).$$

Altogether we obtain that $\int_\Omega v_0^\varrho(u_k(x))(1+|u_k(x)|^p)\,\mathrm{d}x < T+\delta/2$ for any $k > k_\varrho$. Thus, for any $k > k_\varrho$ also

$$\int_{\{x\in\Omega;\ |u_k(x)|\geq\varrho+1\}}(1+|u_k(x)|^p)\,\mathrm{d}x \ \leq\ \int_\Omega v_0^\varrho(u_k(x))(1+|u_k(x)|^p)\,\mathrm{d}x < T+\frac{\delta}{2}$$

and finally,

$$\sup_{k>k_\varrho}\int_{\{x\in\Omega;\ |u_k(x)|\geq\varrho+1\}}(1+|u_k(x)|^p)\,\mathrm{d}x < T+\frac{3\delta}{4}\,.$$

We end up the proof recalling that the finite set $\{1+|u_k|, k = 1,\dots,k_\varrho\}$ is obviously uniformly integrable, hence, we can take $\tilde\varrho > 0$ such that $\sup_{k\in\{1,\dots,k_\varrho\}}\int_{\{x\in\Omega;\ |u_k(x)|\geq\tilde\varrho\}}(1+|u_k(x)|^p)\,\mathrm{d}x \leq \delta/4$. Eventually, we get

$$\begin{aligned}
&\sup_{k\in\mathbb{N}}\int_{\{x\in\Omega;\ |u_k(x)|\geq\max(\varrho+1,\tilde\varrho)\}}(1+|u_k(x)|^p)\,\mathrm{d}x\\
&\leq \sup_{k\in\{1,\dots,k_\varrho\}}\int_{\{x\in\Omega;\ |u_k(x)|\geq\max(\varrho+1,\tilde\varrho)\}}(1+|u_k(x)|^p)\,\mathrm{d}x\\
&+\sup_{k>k_\varrho}\int_{\{x\in\Omega;\ |u_k(x)|\geq\max(\varrho+1,\tilde\varrho)\}}(1+|u_k(x)|^p)\,\mathrm{d}x\\
&< T+\frac{\delta}{4}+\frac{3\delta}{4} = T+\delta.
\end{aligned}$$

As $\delta > 0$ has been arbitrary we see that $T = \sup\{\tilde T;\ \tilde T$ satisfies (10)$\}$ and thus $T = \eta(\{|u_k|^p\}_{k\in\mathbb{N}})$. The proposition is proved. ☐

Remark 2. If $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ generates except a Young measure $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ also some DiPerna-Majda measure, we can write

$$\lim_{k\to\infty}\int_\Omega |u_k(x)|^p\,\mathrm{d}x = \int_\Omega\int_{\mathbb{R}^m}|\lambda|^p\nu_x(\mathrm{d}\lambda)\,\mathrm{d}x + \eta(\{|u_k|^p\}_{k\in\mathbb{N}}).$$

Now we give a criterion of the uniform integrability.

Proposition 4. *Let $\mathcal R$ be a separable complete and closed ring of continuous functions $\mathbb{R}^m \to \mathbb{R}$. Let $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$, $1 \leq p < +\infty$ be bounded. Then*

$$S := \sup_{(\sigma,\hat\nu)\in\mathcal U}\int_{\bar\Omega}\int_{\beta_{\mathcal R}\mathbb{R}^m\setminus\mathbb{R}^m}\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x) = \eta(\{|u_k|^p\}_{k\in\mathbb{N}}), \tag{11}$$

where $\mathcal{U}$ is a set of all $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ that are generated by some subsequence of $\{u_k\}_{k\in\mathbb{N}}$.

Proof: Indeed, we have $\eta(\{|u_k|^p\}_{k\in\mathbb{N}})\ge S$. If $\eta(\{|u_k|^p\}_{k\in\mathbb{N}})>S$, then we would be able to extract a subsequence from $\{u_k\}_{k\in\mathbb{N}}$ which generates $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and for which $\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m}\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x)>S$ contrary to the definition of S.

□

Corollary 1. *Under the assumptions of the above proposition the sequence $\{|u_k|^p\}_{k\in\mathbb{N}}$ is uniformly integrable if and only if $S=0$.*

Proof: It follows from the previous proposition or, alternatively, from Proposition 1.

□

In particular, if $\beta_{\mathcal{R}}\mathbb{R}^m$ is just the one point Alexandroff compactification of $\mathbb{R}^m$, i.e., if $\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m=\{\infty\}$, then (11) reduces to

$$\forall\ (\sigma,\hat\nu)\in\mathcal{U}\quad \hat\nu_x(\infty)=0\quad\text{for }\sigma\text{-almost all }\ x\in\bar\Omega.$$

3. Applications to Fatou's lemma

Proposition 5 (see [19], [25]). *Let $\{u_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^m)$, $1\le p<+\infty$ be bounded and such that there exists $\lim_{k\to\infty}\int_\Omega|u_k(x)|^p\,\mathrm{d}x$. Then*

$$\int_\Omega\liminf_{k\to\infty}|u_k(x)|^p\,\mathrm{d}x\le\lim_{k\to\infty}\int_\Omega|u_k(x)|^p\,\mathrm{d}x-\eta(\{|u_k|^p\}_{k\in\mathbb{N}}).\tag{12}$$

Taking our characterization of the Rosenthal modulus into the play we can come up with the following assertions.

Corollary 2. *Under the assumptions of Proposition 5, (12) is equivalent to*

$$\int_\Omega\liminf_{k\to\infty}|u_k(x)|^p\,\mathrm{d}x\le\lim_{k\to\infty}\int_\Omega|u_k(x)|^p\,\mathrm{d}x-\sup_{(\sigma,\hat\nu)\in\mathcal{U}}\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m}\hat\nu_x(\mathrm{d}\lambda)\sigma(\mathrm{d}x),$$

where $\mathcal{U}\subset\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ contains all of DiPerna-Majda measures generated by some subsequence of $\{u_k\}_{k\in\mathbb{N}}$.

Proof: It follows immediately from Propositions 4 and 5. □

Proposition 6. *Let $\{u_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^m)$, $1\le p<+\infty$ be bounded and generate a Young measure $\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^m)$. Then*

$$\int_\Omega\liminf_{k\to\infty}|u_k(x)|^p\,\mathrm{d}x\le\int_\Omega\int_{\mathbb{R}^m}|\lambda|^p\nu_x(\mathrm{d}\lambda)\,\mathrm{d}x\le\liminf_{k\to\infty}\int_\Omega|u_k(x)|^p\,\mathrm{d}x.$$

PROOF: The second inequality is standard and can be found e.g. in [20], [29]. Let us prove the first one. Let $\{u_{k_l}\}_{l\in\mathbb{N}}$ be such subsequence of $\{u_k\}_{k\in\mathbb{N}}$ that generates except the Young measure ν also some DiPerna-Majda measure. This means due to Remark 2 that

$$\lim_{l\to\infty}\int_\Omega |u_{k_l}(x)|^p\,\mathrm{d}x = \int_\Omega\int_{\mathbb{R}^m}|\lambda|^p\nu_x(\mathrm{d}\lambda)\,\mathrm{d}x + \eta(\{|u_{k_l}|^p\}_{l\in\mathbb{N}}),$$

where $\eta(\{|u_{k_l}|^p\}_{l\in\mathbb{N}})$ is given by Proposition 3. Applying now Proposition 5 to this subsequence we have

$$\int_\Omega \liminf_{l\to\infty}|u_{k_l}(x)|^p\,\mathrm{d}x \le \int_\Omega\int_{\mathbb{R}^m}|\lambda|^p\nu_x(\mathrm{d}\lambda)\,\mathrm{d}x$$

and because $\int_\Omega \liminf_{k\to\infty}|u_k(x)|^p\,\mathrm{d}x \le \int_\Omega \liminf_{l\to\infty}|u_{k_l}(x)|^p\,\mathrm{d}x$ we obtain the assertion. □

Corollary 3. *Let $\{u_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^m)$, $1\le p<+\infty$ be bounded. Then*

$$\int_\Omega \liminf_{k\to\infty}|u_k(x)|^p\,\mathrm{d}x \le \inf_{\nu\in\mathcal{U}}\int_\Omega\int_{\mathbb{R}^m}|\lambda|^p\nu_x(\mathrm{d}\lambda)\,\mathrm{d}x \le \liminf_{k\to\infty}\int_\Omega |u_k(x)|^p\,\mathrm{d}x,$$

where $\mathcal{U}\subset\mathcal{Y}^p(\Omega;\mathbb{R}^m)$ contains all of Young measures generated by some subsequence of $\{u_k\}_{k\in\mathbb{N}}$.

PROOF: Let $\{v_k\}_{k\in\mathbb{N}}$ be a subsequence of $\{u_k\}_{k\in\mathbb{N}}$ that generates $\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^m)$. According to the Proposition 6

$$\int_\Omega \liminf_{k\to\infty}|v_k(x)|^p\,\mathrm{d}x \le \int_\Omega\int_{\mathbb{R}^m}|\lambda|^p\nu_x(\mathrm{d}\lambda)\,\mathrm{d}x,$$

which gives

$$\int_\Omega \liminf_{k\to\infty}|u_k(x)|^p\,\mathrm{d}x \le \int_\Omega\int_{\mathbb{R}^m}|\lambda|^p\nu_x(\mathrm{d}\lambda)\,\mathrm{d}x.$$

The first inequality follows straightforwardly.

Now let $\{w_k\}_{k\in\mathbb{N}}$ be such subsequence of $\{u_k\}_{k\in\mathbb{N}}$ that $\lim_{k\to\infty}\int_\Omega |w_k(x)|^p\,\mathrm{d}x = \liminf_{k\to\infty}\int_\Omega |u_k(x)|^p\,\mathrm{d}x$ and that it generates $\tilde\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^m)$. Then due to the previous proposition

$$\int_\Omega\int_{\mathbb{R}^m}|\lambda|^p\tilde\nu_x(\mathrm{d}\lambda)\,\mathrm{d}x \le \liminf_{k\to\infty}\int_\Omega |u_k(x)|^p\,\mathrm{d}x$$

from which we have the second inequality. □

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