Comment.Math.Univ.Carolin. 39,3 (1998)609–615

On the functor of order-preserving functionals

T. Radul

Abstract. We introduce a functor of order-preserving functionals which contains some known functors as subfunctors. It is shown that this functor is weakly normal and generates a monad.

Keywords: order-preserving functional, monad

Classification: 54B30, 54C35, 18C15

0. The general theory of functors acting in the category $\mathcal{C}omp$ of compact Hausdorff spaces (compacta) and continuous mappings was founded by E.V. Shchepin [1]. He distinguished some elementary properties of such functors and defined the notion of normal functor that has become very fruitful. The class of normal functors includes many classical constructions: the hyperspace exp, the space of probability measures P, the superextension λ, the space of hyperspaces of inclusion G and many other functors ([2], [3]).

The algebraic applications of the theory of functors were discovered rather recently. They are based, mainly, on the existence of a monad structure (in the sense of S. Eilenberg and J. Moore [4]) for such functors.

For all above mentioned functors exp, P, λ and G there exist the structures of monads denoted by $\mathbb{H}$, $\mathbb{P}$, $\mathbb{L}$ and $\mathbb{G}$ respectively ([5]).

In this paper we introduce the functor of order-preserving functionals O. We show that it is a weakly normal functor generating the monad $\mathbb{O}$. Moreover, the above mentioned monads $\mathbb{H}$, $\mathbb{P}$, $\mathbb{L}$, $\mathbb{G}$ are contained as submonads in $\mathbb{O}$.

The paper is organized as follows: in Section 1 we investigate some properties of order-preserving functionals and introduce the functor O, in Section 2 we prove that O is a weakly normal functor and in Section 3 we show that the functor O generates a monad $\mathbb{O}$.

1. All spaces are assumed to be compacta, all mappings are continuous. By $w(X)$ we denote the weight of X and by $d(X)$ the density. The space of real numbers $\mathbb{R}$ is considered with the usual metric.

Let $X \in \mathcal{C}omp$. By $C(X)$ we denote the Banach space of all continuous functions $\varphi : X \to \mathbb{R}$ with the usual sup-norm: $\|\varphi\| = \sup\{|\varphi(x)| \mid x \in X\}$. For each $c \in \mathbb{R}$ we denote by c_X the constant function on $C(X)$ defined by the formula $c_X(x) = c$ for each $x \in X$. We will consider the natural partial order on $C(X)$ defined as follows: for $\varphi, \psi \in C(X)$ we have $\varphi \le \psi$ iff $\varphi(x) \le \psi(x)$ for each $x \in X$.

We are going to investigate the functionals $\nu : C(X) \to \mathbb{R}$. We do not suppose apriori that ν is linear or continuous.

A functional $\nu : C(X) \to \mathbb{R}$ is called *weakly additive* if for each $c \in \mathbb{R}$ and $\varphi \in C(X)$ we have $\nu(\varphi + c_X) = \nu(\varphi) + c$; *order-preserving* if for each $\varphi, \psi \in C(X)$ with $\varphi \leq \psi$ we have $\nu(\varphi) \leq \nu(\psi)$ ([6]).

Lemma 1. *Each order-preserving weakly additive functional is a non-expanding map.*

Proof: Let $\nu : C(X) \to \mathbb{R}$ be an order-preserving weakly-additive functional and $\varphi, \psi \in C(X)$. Let $\|\varphi - \psi\| = a \in \mathbb{R}$. Then we have $\varphi - a_X \leq \psi \leq \varphi + a_X$ and $\nu(\varphi) - a \leq \nu(\psi) \leq \nu(\varphi) + a$. Thus $|\nu(\varphi) - \nu(\psi)| \leq a$. □

Corollary 1. *Each order-preserving weakly additive functional is continuous.*

A subset $L \subset C(X)$ is called an *A-subspace* if $0_X \in L$ and for each $\varphi \in L$, $c \in \mathbb{R}$ we have $\varphi + c_X \in L$. The next lemma can be considered as an analogue of the Hahn-Banach theorem.

Lemma 2. *For each A-subspace $L \subset C(X)$ and for each order-preserving weakly additive functional $\nu : L \to \mathbb{R}$ there exists an order-preserving weakly additive functional $\nu' : C(X) \to \mathbb{R}$ such that $\nu'|L = \nu$.*

Proof: Let us consider the set of all pairs (B, μ), where $L \subset B \subset C(X)$ is an A-space and μ is an order-preserving weakly additive functional. This set can be regarded as a partially ordered set by the order $(B_1, \mu_1) \leq (B_2, \mu_2)$ iff $B_1 \subset B_2$ and μ_2 is an extension of μ_1. By Zorn Lemma there exists a maximal element (B_0, μ_0).

Suppose that $B_0 \neq C(X)$. Take any $\varphi \in C(X) \setminus B_0$. Let B^+ (B^-) be the set of all $\psi \in B_0$ with $\psi \geq \varphi$ ($\psi \leq \varphi$). Then we can choose $p \in \mathbb{R}$ with $\mu_0(B^-) \leq p \leq \mu_0(B^+)$. The set $D = B_0 \cup \{\varphi + c_X \mid c \in \mathbb{R}\}$ is an A-subset in $C(X)$. Define the functional $\mu : D \to \mathbb{R}$ as follows: $\mu|B_0 = \mu_0$ and $\mu(\varphi + c_X) = p + c$, $c \in \mathbb{R}$. It is easy to check that μ is an order-preserving weakly additive functional and we obtain the contradiction with the maximality of (B_0, μ_0). □

A functional $\nu : C(X) \to \mathbb{R}$ will be called *normed* iff $\nu(1_X) = 1$.

For a compactum X, let $O(X)$ denote the set of all order-preserving weakly additive normed functionals. It is easy to see that for each $\nu \in O(X)$ and $c \in \mathbb{R}$ we have $\nu(c_X) = c$.

We consider $O(X)$ as a subspace of the space $C_p(C(X))$ of all continuous functions on $C(X)$ equipped with the pointwise topology. The base of this topology consists of sets of the form $(\mu; \varphi_1, \dots, \varphi_n; \varepsilon) = \{\mu' \in C_p(C(X)) \mid |\mu'(\varphi_i) - \mu(\varphi_i)| < \varepsilon$ for each $i \in \{1, \dots, k\}\}$, where $\mu \in C_p(C(X))$, $\varphi_1, \dots, \varphi_k \in C(X)$, $\varepsilon > 0$.

Theorem 1. *For each compactum X, the space $O(X)$ is compact.*

Proof: Observe firstly that $O(X)$ is contained in the Tychonov product of closed intervals $P = \prod\{[-\|\varphi\|, \|\varphi\|] \mid \varphi \in C(X)\}$. Thus it is sufficient to prove that $O(X)$ is closed in P.

Consider $\mu \in P \setminus O(X)$. Then μ fails to satisfy one of the three conditions from the definition of $O(X)$.

Suppose μ is not normed. Then we have $(\mu; 1_X; \frac{|\mu(1_X)-1|}{2}) \cap O(X) = \emptyset$.

Suppose μ is not weakly additive. Then there exist $\varphi \in C(X)$ and $c \in \mathbb{R}$ such that $\mu(\varphi + c_X) \neq \mu(\varphi) + c$. Put $\delta = |\mu(\varphi + c_X) - \mu(\varphi) - c| > 0$. Then $(\mu; \varphi + c_X, \varphi, c_X, \delta/4) \cap O(X) = \emptyset$.

Finally, suppose μ is not order-preserving. Then there exist $\varphi_1, \varphi_2 \in C(X)$ such that $\varphi_1 \geq \varphi_2$ and $\mu(\varphi_1) < \mu(\varphi_2)$. Put $\varepsilon = \mu(\varphi_2) - \mu(\varphi_1)$. Then $(\mu; \varphi_1, \varphi_2; \varepsilon/2) \cap O(X) = \emptyset$. Thus $O(X)$ is a closed subset of P. $\square$

Let $X, Y \in \mathcal{C}omp$ and $f : X \to Y$ be a continuous map. Define the map $O(f) : O(X) \to O(Y)$ by the formula $(O(f)(\mu))(\varphi) = \mu(\varphi \circ f)$, where $\mu \in O(X)$ and $\varphi \in C(Y)$.

It is easy to check that $O(f)$ is well defined continuous and $O(f \circ g) = O(f) \circ O(g)$. Thus O is a covariant functor on the category $\mathcal{C}omp$.

2. In what follows we will need some notions from the general theory of functors.

Let $F : \mathcal{C}omp \to \mathcal{C}omp$ be a covariant functor. A functor F is called *monomorphic* (*epimorphic*) if it preserves monomorphisms (epimorphisms). For a monomorphic functor F and an embedding $i : A \to X$, we shall identify the space $F(A)$ and the subspace $F(i)(F(A)) \subset F(X)$.

A monomorphic functor F is said to be *preimage-preserving* if for each map $f : X \to Y$ and each closed subset $A \subset Y$ we have $(F(f))^{-1}(F(A)) = F(f^{-1}(A))$.

For a monomorphic functor F the *intersection-preserving* property is defined as follows: $F(\bigcap\{X_\alpha \mid \alpha \in \mathcal{A}\}) = \bigcap\{F(X_\alpha) \mid \alpha \in \mathcal{A}\}$ for every family $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ of closed subsets of X.

A functor F is called *continuous* if it preserves the limits of inverse systems $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, \mathcal{A}\}$ over a directed set $\mathcal{A}$.

Finally, a functor F is called *weight-preserving* if $w(X) = w(F(X))$ for every infinite $X \in \mathcal{C}omp$.

A functor F is called *normal* ([1]) if it is continuous, monomorphic, epimorphic, preserves weight, intersections, preimages, singletons and the empty space. A functor F is said to be *weakly normal* if it satisfies all the properties from the definition of a normal functor except perhaps the preimage-preserving property. Let us remark that the functors exp, P are normal and λ, G are weakly normal ([3]).

It is obvious that O preserves singletons and the empty set.

Proposition 1. *O is a monomorphic functor.*

PROOF: Let $j : X \to Y$ be an embedding. Let us show that $O(j) : O(X) \to O(Y)$ is an embedding as well. If $\mu_1, \mu_2 \in O(X)$ are two different functionals then there exists a function $\varphi \in C(X)$ with $\mu_1(\varphi) \neq \mu_2(\varphi)$. We can choose a function $\psi \in C(Y)$ such that $\psi \circ j = \varphi$. Then we have $(O(j)(\mu_i))(\psi) = \mu_i(\psi \circ j) = \mu_i(\varphi)$. Hence $O(j)(\mu_1) \neq O(j)(\mu_2)$. $\square$

Proposition 2. *The functor O is epimorphic.*

Proof: Let $f : X \to Y$ be an ephimorphism and $v \in O(Y)$. Denote by C the subset of $C(X)$ consisting of the functions $\psi \circ f$, $\psi \in C(Y)$. It is easy to see that C is an A-subset of $C(X)$. We can define a normed order-preserving weakly additive functional $\nu' : C \to R$ by the formula $\nu'(\psi \circ f) = \nu(\psi)$. By Lemma 2 we can extend ν' to a functional $\mu \in O(X)$. It is obvious that $O(f)(\mu) = \nu$. □

For each $x \in X$, let the functional $\delta_x \in O(X)$ be defined by $\delta_x(\varphi) = \varphi(x)$, $\varphi \in C(X)$. It is easy to see that the map $\delta : X \to O(X)$ defined by $\delta(x) = \delta_x$ is an embedding.

Lemma 3. *Let (X, d) be an infinite metric space and let $E_p(X)$ be the subspace of $C_p(X)$ consisting of all non-expanding maps. Then $w(E_p(X)) \leq d(E_p(X)) \times d(X)$.*

Proof: Let F be a dense set in $E_p(X)$ with $|F| \leq d(E_p(X))$ and A let be a dense set in X with $|A| \leq d(X)$. Consider the family $\mathcal{B}$ of subsets in $E_p(X)$ of the form $(\varphi; x_1, \dots, x_n; \varepsilon)$, where $\varphi \in F$, $x_i \in A$ and $\varepsilon \in \mathbb{Q}$. It is easy to see that $|\mathcal{B}| \leq d(E_p(X)) \times d(X)$. One can check that $\mathcal{B}$ is a base of the space $E_p(X)$. □

Proposition 3. *The functor O preserves weight of infinite compacta.*

Proof: Since X can be embedded by the map δ in $O(X)$, we have $w(O(X)) \geq w(X)$.

On the other hand, it follows from [7, 3.4.G] that for each subspace $Y \subset C_p(Z)$ we have $d(Y) \leq w(Z)$. It follows from [2, II.3.12] that $w(C(X)) \leq w(X)$. Using Lemmas 1 and 3 we obtain that $w(O(X) \leq w(X)$. □

Proposition 4. *O is a continuous functor.*

Proof: Let $X = \lim \mathcal{S}$, where $\mathcal{S} = \{X_\alpha, \pi_\alpha^\beta, \mathcal{A}\}$ is an inverse system and all X_α are compact. Denote by Y the limit space of the inverse system $\mathbb{O}(\mathcal{S}) = \{O(X_\alpha), O(\pi_\alpha^\beta), \mathcal{A}\}$ and by $\pi : O(X) \to Y$ the limit of the maps $O(\pi_\alpha)$, where $\pi_\alpha : X \to X_\alpha$ are limit projections of the system $\mathcal{S}$.

Let us show that π is a homeomorphism. Let $\mu_1, \mu_2 \in O(X)$ be two different functionals. There exists a function $\varphi \in C(X)$ such that $|\mu_1(\varphi) - \mu_2(\varphi)| = a > 0$. It follows from the Weierstrass-Stone theorem that the set of functions $\psi \circ \pi_\alpha$, where $\psi \in C(X_\alpha)$, $\alpha \in \mathcal{A}$ is dense in $C(X)$. Hence there exist an $\alpha \in \mathcal{A}$ and a function $\psi \in X_\alpha$ such that $|\varphi - \psi \circ \pi_\alpha| < a/3$. Since μ_i are non-expanding functionals, we have $|\mu_i(\varphi) - \mu_i(\psi \circ \pi_\alpha)| < a/3$. Then

$$\begin{aligned}
a &= |\mu_1(\varphi) - \mu_2(\varphi)| \\
&= |\mu_1(\varphi) - \mu_1(\psi \circ \pi_\alpha) + \mu_1(\psi \circ \pi_\alpha) - \mu_2(\psi \circ \pi_\alpha) + \mu_2(\psi \circ \pi_\alpha) - \mu_2(\varphi)| \\
&\leq |\mu_1(\varphi) - \mu_1(\psi \circ \pi_\alpha)| + |\mu_1(\psi \circ \pi_\alpha) - \mu_2(\psi \circ \pi_\alpha)| + |\mu_2(\psi \circ \pi_\alpha) - \mu_2(\varphi)| \\
&\leq 2a/3 + |\mu_1(\psi \circ \pi_\alpha) - \mu_2(\psi \circ \pi_\alpha)|.
\end{aligned}$$

Thus we have $(O(\pi_\alpha)(\mu_1))(\psi) \neq (O(\pi_\alpha)(\mu_2))(\psi)$ and hence $O(\pi_\alpha)(\mu_1) \neq O(\pi_\alpha)(\mu_2)$. Since π is a limit map of the maps $O(\pi_\alpha)$, we have $\pi(\mu_1) \neq \pi(\mu_2)$. We have just proved that π is an embedding. Since the functor O is epimorphic, the map π is a surjection. □

Let A be a closed subset of a compactum X. We say that $\mu \in O(X)$ is *supported on* A if $\mu \in O(A) \subset O(X)$. By $O_\omega(X)$ we denote a subset of $O(X)$ consisting of all functionals supported on finite subsets of X.

The next corollary follows from [2] and Propositions 2, 4.

Corollary 2. *$O_\omega(X)$ is a dense subset of $O(X)$.*

Lemma 4. *Let $\mu \in O(X)$ and let A be a closed subset of X. Then μ is supported on A iff for each $\varphi_1, \varphi_2 \in C(X)$ with $\varphi_1|A = \varphi_2|A$ we have $\mu(\varphi_1) = \mu(\varphi_2)$.*

Proof: Let $\mu \in O(A)$. Denote by $i : A \to X$ the identity embedding. Let $\varphi_1, \varphi_2 \in C(X)$ be functions with $\varphi_1|A = \varphi_2|A$. There exists a functional $\nu \in O(A)$ such that $O(i)(\nu) = \mu$. Then we have $\mu(\varphi_1) = \nu(\varphi_1|A) = \nu(\varphi_2|A) = \mu(\varphi_2)$.

Now let $\mu \in O(X)$ be a functional such that $\mu(\varphi_1) = \mu(\varphi_2)$ for each $\varphi_1, \varphi_2 \in C(X)$ with $\varphi_1|A = \varphi_2|A$. Then we can define a functional $\nu \in O(A)$ by $\nu(\varphi) = \mu(\varphi')$, where $\varphi \in C(A)$ and φ' is any extension of φ on X. It is easy to see that $O(i)(\nu) = \mu$. □

Proposition 5. *The functor O preserves intersections.*

Proof: Since O is a continuous functor, it is sufficient to prove the proposition for the intersection of two closed subsets A_1 and A_2 of a compactum X.

It is evident that $O(A_1 \cap A_2) \subset O(A_1) \cap O(A_2)$. Let us show the inverse inclusion. Let $\mu \in O(A_1) \cap O(A_2)$. Choose any functions $\psi_1, \psi_2 \in C(X)$ such that $\psi_1|(A_1 \cap A_2) = \psi_2|(A_1 \cap A_2)$. By Lemma 4 it is sufficient to prove that $\mu(\psi_1) = \mu(\psi_2)$. Consider a function $\varphi \in C(X)$ such that $\varphi|A_1 = \psi_1$ and $\varphi|A_2 = \psi_2$. Since $\mu \in O(A_1)$, we have $\mu(\varphi) = \mu(\psi_2)$ and, since $\mu \in O(A_2)$, $\mu(\varphi) = \mu(\psi_2)$. □

The following theorem is an immediate consequence of the results of this section.

Theorem 2. *The functor O is weakly normal.*

At the end of this section we give an example showing that the functor O does not preserve preimages, thus it is not normal.

Example. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ be finite compacta (all the points x_1, x_2, x_3, y_1, y_2 are distinct). Define the map $f : X \to Y$ as follows: $f(x_1) = y_1$ and $f(x_2) = f(x_3) = y_2$. Consider the functional $\delta_{y_2} \in O(Y)$ supported on $\{y_2\} \subset Y$. Define a functional $\mu \in O(X)$ by the formula

$$\mu(\varphi) = \max\{\min\{\varphi(x_1), \varphi(x_2)\}, \min\{\varphi(x_1), \varphi(x_3)\}, \min\{\varphi(x_2), \varphi(x_3)\}\}.$$

It is easy to check that $O(f)(\mu) = \nu$ and $\mu \notin O(\{x_2, x_3\})$. Thus O does not preserve preimages.

3. In this section we show that the functor O generates a monad on $\mathcal{C}omp$.

Let F, G be two functors in the category $\mathcal{E}$. We say that a transformation $\varphi : F \to G$ is defined if for every $X \in \mathcal{E}$ a mapping $\varphi X : FX \to GX$ is given. The transformation $\varphi = \{\varphi X\}$ is called *natural* if for every mapping $f : X \to Y$ we have $\varphi Y \circ F(f) = G(f) \circ \varphi X$.

A *monad* $\mathbb{T} = (T, \eta, \mu)$ in the category $\mathcal{E}$ consists of an endofunctor $T : \mathcal{E} \to \mathcal{E}$ and natural transformations $\eta : \mathrm{Id}_{\mathcal{E}} \to T$ (unity), $\mu : T^2 \to T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = \mathbf{1}_T$ and $\mu \circ \mu T = \mu \circ T\mu$.

A natural transformation $\psi : T \to T'$ is called a *morphism* from a monad $\mathbb{T} = (T, \eta, \mu)$ into a monad $\mathbb{T}' = (T', \eta', \mu')$ if $\psi \circ \eta = \eta'$ and $\psi \circ \mu = \mu' \circ \eta T' \circ T\psi$. If all the components of ψ are monomorphisms then the monad $\mathbb{T}$ is called a *submonad* of $\mathbb{T}'$.

Let us define the mapping $\mu X : O^2(X) \to O(X)$ by the formula $\mu X(\alpha)(g) = \alpha(\tilde{g})$, where $\alpha \in O^2(X)$, $g \in C(X, [0;1])$ and the mapping $\tilde{g} : O(X) \to [0;1]$ is given by $\tilde{g}(\mu) = \mu(g)$, $\mu \in O(X)$. It is easy to check that μX is correctly defined and continuous.

Put $\eta X = \delta$. It is easy to check that ηX and μX are the components of natural transformations $\eta : \mathrm{Id}_{\mathcal{C}omp} \to O$ and $\mu : O^2 \to O$.

Theorem 3. *The triple $\mathbb{O} = (O, \eta, \mu)$ forms a monad on the category $\mathcal{C}omp$.*

Proof: Let $\nu \in O(X)$. Consider any $\varphi \in C(X)$. Then we have $\mu X \circ \eta O(X)(\nu)(\varphi) = \eta O(X)(\nu)(\tilde{\varphi}) = \tilde{\varphi}(\nu) = \nu(\varphi)$ and $\mu X \circ O(\eta X)(\nu)(\varphi) = O(\eta X)(\nu)(\tilde{\varphi}) = \nu(\tilde{\varphi} \circ \eta X) = \nu(\varphi)$.

Now let $\mathcal{N} \in O^3(X)$ and $\varphi \in C(X)$. Then $\mu X \circ \mu O(X)(\mathcal{N})(\varphi) = \mu O(X)(\mathcal{N})(\tilde{\varphi}) = \mathcal{N}(\tilde{\tilde{\varphi}})$ and $\mu X \circ O(\mu X)(\mathcal{N})(\varphi) = O(\mu X)(\mathcal{N})(\tilde{\varphi}) = \mathcal{N}(\tilde{\varphi} \circ \mu X) = \mathcal{N}(\tilde{\tilde{\varphi}})$, where $\tilde{\tilde{\varphi}} \in C(O^2(X))$ is defined by the formula $(\tilde{\tilde{\varphi}})(\nu) = \nu(\tilde{\varphi})$, $\nu \in O^2(X)$. □

Remark. It is easy to check that the monad $\mathbb{P}$ is a submonad of $\mathbb{O}$. On the other hand, it is shown in [8] that a wide class of monads which includes monads $\mathbb{G}$, $\mathbb{H}$, $\mathbb{L}$ have a functional representation, otherwise speaking, their functional part $F(X)$ can be embedded in $\mathbb{R}^{C(X)}$. Moreover the images of $\lambda(X)$, $\exp(X)$ and $G(X)$ lie in $O(X)$. Thus the monad $\mathbb{O}$ contains $\mathbb{P}$, $\mathbb{G}$, $\mathbb{H}$, $\mathbb{L}$ as submonads.

References

[1] Shchepin E.V., *Functors and uncountable powers of compacta* (in Russian), Uspekhi Mat. Nauk **36** (1981), 3–62.

[2] Fedorchuk V.V., Filippov V.V., *General Topology. Fundamental Constructions* (in Russian), Moscow, 1988, p. 252.

[3] Fedorchuk V.V., Zarichnyi M.M., *Covariant functors in categories of topological spaces* (in Russian), Results of Science and Technics. Algebra. Topology. Geometry, vol. 28, Moscow, VINITI, pp. 47–95.

[4] Eilenberg S., Moore J., *Adjoint functors and triples*, Illinois J. Math. **9** (1965), 381–389.

[5] Radul T., Zarichnyi M.M., *Monads in the category of compacta* (in Russian), Uspekhi Mat.Nauk **50** (1995), no. 3, 83–108.

[6] Shapiro L.B., *On function extension operators and normal functors* (in Russian), Vestnik Mosk. Univer. Ser.1 (1992), no. 1, 35–42.
[7] Engelking R., *General Topology*, Warszawa, 1978.
[8] Radul T., *On functional representations of Lawson monads*, to appear.

Department of Mechanics and Mathematics, Lviv State University, Universytetska st. 1, 290602 Lviv, Ukraine

E-mail: trad@mebm.lviv.ua

(*Received* October 3, 1997, *revised* March 20, 1998)