Metric–fine uniform frames

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Abstract. A locallic version of Hager's metric-fine spaces is presented. A general definition of $\mathcal{A}$-fineness is given and various special cases are considered, notably $\mathcal{A}$ = all metric frames, $\mathcal{A}$ = complete metric frames. Their interactions with each other, quotients, separability, completion and other topological properties are discussed.

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We present a locallic version of metric-fine spaces which were introduced by Hager in "Some nearly fine uniform spaces". These spaces and related ideas were studied in detail by a group of topologists in the seventies in seminars, "Seminar Uniform Spaces", under the direction of Zdeněk Frolik in Prague. Specifically, the work done by Frolik, Hager and Rice is relative to this paper. A general definition of $\mathcal{A}$-fineness will be given and $\mathcal{A}$-fine frames are shown to be ubiquitous. Special cases will be considered, notably $\mathcal{A}$ = all metric frames, and $\mathcal{A}$ = all complete metric frames. Their behaviour and interactions with each other, quotients, separability, completeness and other topological properties will be discussed. In particular, Lindelöf and pseudocompact frames will be characterised in terms of this $\mathcal{M}$-fine property. In this setting, the interactions and some of the proofs are more elegant and perspicuous than their spatial counterparts.

Preliminaries on uniform frames

A *frame* is a bounded lattice L with top e and bottom 0, which is complete and satisfies $x \wedge \bigvee S = \bigvee\{ x \wedge t \mid t \in S \}$ for $x \in L$ and any $S \subseteq L$. A *frame map* is a function which preserves e, 0, $\wedge$ and $\bigvee$. The resulting category will be denoted **Frm**. A standard reference for frames is Johnstone [17]. A *cover* of a frame L is a subset $A \subseteq L$ with $\bigvee A = e$. Let $Cov(L)$ denote all the covers of L. L is said to be *compact* (respectively, *Lindelöf*) if each cover has a finite (respectively, countable) subcover. For $A, B \subseteq L$, A is called a *refinement* of B, written $A \leq B$, if for each $a \in A$ there exists $b \in B$ with $a \leq b$. For any $x \in L$

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the *star* of x relative to a cover A is the element $Ax = \bigvee\{a \in A \mid a \wedge x \neq 0\}$. If $\mathcal{A}$ is a subset of $Cov(L)$ then $x \lhd_{\mathcal{A}} y$ in L if there exists $A \in \mathcal{A}$ with $Ax \leq y$. Let $L_{\mathcal{A}} = \{x \in L \mid x = \bigvee\{y \mid y \lhd_{\mathcal{A}} x\}\}$. A *star refinement* of a cover B of L is a cover A of L such that the cover $\{Ax \mid x \in A\}$ refines B. We write this as $A \leq^* B$. A cover A is said to be *normal* whenever there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of covers such that $A = A_0$ and $A_{n+1} \leq^* A_n$ for all $n \in \mathbb{N}$. Further, L is called *fully normal* if every cover is normal.

A *preuniformity* on L is a filter of covers μ (relative to $\leq$) such that each $A \in \mu$ is star refined by some $B \in \mu$, and a *uniformity* is a preuniformity which satisfies the compatibility condition that $L = L_\mu$, that is, each $a \in L$ is the join of all $x \in L$ such that $Ax \leq a$ for some $A \in \mu$. If the latter condition holds, x is said to be *uniformly below* a, written $x \lhd a$. A *uniform frame* is a frame L together with a specified uniformity μ and is denoted (L, μ). A *uniform map* is a frame map which preserves uniform covers. The resulting category will be denoted by **UniFrm**. Every uniform frame is completely regular, and every completely regular frame admits a uniformity ([23]). In fact, every completely regular frame L admits a finest uniformity which consists of all the normal covers. This uniformity will be called the *fine uniformity* of L and will be denoted by α_L. A uniform frame is *subfine* if it is a quotient of a fine uniform frame, and *cozfine* if it has a base consisting of countable covers of uniform cozero elements ([36]). See below for details on uniform cozero elements. A uniform map $h : (L, \mu) \longrightarrow (M, \upsilon)$ is a *surjection* or (*uniform*) *quotient* if it is onto, and υ is generated by the image covers $h[A]$, $A \in \mu$. We say a uniform frame is *complete* if every dense surjection is an isomorphism. It has been shown that every uniform frame has a unique (up to isomorphism) completion which will be denoted by $\gamma_L : C(L, \mu) \twoheadrightarrow (L, \mu)$. See [6] or [15]. We will need the following proposition (for a proof see [34]) which follows from the result of Banaschewski and Pultr [6] that the completion functor C takes dense surjections to isomorphisms:

Proposition. *In* **UniFrm**, *if a reflection preserves quotients and completeness, and the underlying frame, then it commutes with the completion functor.*

A uniform frame (L, μ) is *separable*, or enumerable, if μ has a basis of countable covers. For any uniform frame (L, μ), let $e\mu$ be the uniformity generated by all countable uniform covers, then the separable uniform frames **SepUniFrm** may be shown to be a coreflective subcategory of **UniFrm** with coreflection functor e, and coreflection map the identity. For any uniform frame (L, μ) the "uniform cozero part", written $Coz_u L$, consists of those elements $a \in L$ such that $a = h((0, 1])$ for some uniform $h : \mathcal{O}[0, 1] \longrightarrow (L, \mu)$. (Note that since $\mathcal{O}[0, 1]$ is compact it admits a unique uniformity.) The elements of $Coz_u L$ are called *uniform cozero elements*, or often simply cozero elements, if the context is clear. If the uniformity is the fine one, then these are precisely the cozero elements with respect to all frame maps. That is, $Coz L = Coz_u(L, \alpha_L)$. The category of uniform σ-frames, denoted **UniσFrm**, can also be defined ([32]), and then Coz_u may be regarded as a functor from **UniFrm** to **UniσFrm**, by taking the uniformity generated by countable uniform covers consisting of cozero elements. It is interesting to note that $Coz_u(L, \mu)$ generates the separable coreflection of (L, μ), in the sense that each element of L is a join of elements from $Coz_u L$, and each uniform cover in $e\mu$ is refined by a uniform cover of $Coz_u(L, \mu)$ ([32]). A uniform map is *coz codense* if the top element is the only cozero element mapped to the top. The subspace topology $\mathcal{O}A$ on a subset A of a Tychonoff space X is a coz codense quotient of $\mathcal{O}X$ if and only if A is G_δ-dense in X ([34]).

Some basics on metric frames

A uniform frame is said to be *metric* if the uniformity has a countable base, or equivalently, it admits a metric diameter ([24]). The metric frames can be thought of as "generating" the uniform frames in the sense that each uniform frame is a quotient of a coproduct of metric frames. Using the result that for metric frames, Lindelöf and second countable (that is, the frame has a countable base) are equivalent, it can be seen that separable metric frames are Lindelöf. Moreover, the fine uniformity on a separable metric frame is separable. Also, a metric subframe of a separable uniform frame is separable. We also have that every element of a metric frame is a uniformly cozero element, that is, $Coz_u(M, \rho) = M$ for a metric frame (M, ρ). Using properties of metric frames it is possible to get a frame version of Gleason's theorem for metric spaces ([16]):

Proposition. *Every uniform map from a metric frame to a quotient of an arbitrary coproduct of metric frames factors through a countable subcoproduct.*

The detailed proofs of these results can be found in the Doctoral Thesis of the author ([34]).

$\mathcal{A}$-fine uniform frames

Let $\mathcal{A}$ be any class of uniform frames, then a given uniform frame (L, μ) is *$\mathcal{A}$-fine* if whenever $f : (M, \upsilon) \longrightarrow (L, \mu)$ is uniform with $(M, \upsilon) \in \mathcal{A}$, $f : (M, \alpha_M) \longrightarrow (L, \mu)$ is also uniform. Recall that α_M denotes the fine uniformity on M. Clearly the fine uniformity on a uniformizable frame is $\mathcal{A}$-fine for any class $\mathcal{A}$, so each uniformizable frame admits $\mathcal{A}$-fine uniformities. Moreover, if $\mathcal{A}$ is the class of all uniform frames, then "$\mathcal{A}$-fineness" is simply "fineness". To consider some special cases of "$\mathcal{A}$-fineness" we need to recall the definition of a projective frame (in the categorical sense): a uniform frame (L, μ) is projective if whenever (N, δ) is a quotient of any uniform frame (M, υ), every uniform map $(L, \mu) \longrightarrow (N, \delta)$ can be extended to a uniform map $(L, \mu) \longrightarrow (M, \upsilon)$. Let $\mathcal{P}$ denote the collection of all projective frames, then the following proposition shows that the subfine uniform frames are precisely the $\mathcal{P}$-fine uniform frames.

Proposition. *A uniform frame (L, μ) is subfine iff it is $\mathcal{P}$-fine.*

PROOF: Consider any uniform map $f : (P, \upsilon) \longrightarrow (L, \mu)$ with (P, υ) projective. Suppose $h : (K, \alpha_K) \longrightarrow (L, \mu)$ makes (L, μ) subfine. Since (P, υ) is projective, there exists a uniform map $\bar{f}$ such that the diagram commutes:

$$\begin{tikzcd} (P,\upsilon) \arrow[r, "f"] \arrow[dr, dashed, "\bar{f}"'] & (L,\mu) \\ & (K,\alpha_K) \arrow[u, "h"'] \end{tikzcd}$$

But α_K is the fine uniformity so $\bar{f} : (P, \alpha_P) \longrightarrow (K, \alpha_K)$ is also uniform. Since $f = h \circ \bar{f}$, $f : (P, \alpha_P) \longrightarrow (L, \mu)$ is uniform. The converse follows directly from the construction of the subfine reflection ([34]): (L, μ) is a quotient of a projective frame, say (P, υ) with quotient map h. Since (L, μ) is $\mathcal{P}$-fine, this map is still uniform for the fine uniformity on P, that is, $h : (P, \alpha_P) \longrightarrow (L, \mu)$ is uniform, and hence (L, μ) is subfine. □

Now let $\mathcal{L}$ denote the collection of all Lindelöf uniform frames, then we show that the cozfine uniform frames are precisely the $\mathcal{L}$-fine uniform frames.

Proposition. *(L, μ) is $\mathcal{L}$-fine iff (L, μ) is cozfine.*

Proof: Suppose that (L, μ) is cozfine, and consider any uniform map $f : (K, \upsilon) \longrightarrow (L, \mu)$ with K Lindelöf. Since K is Lindelöf, the fine uniformity α_K has a basis of countable cozero covers and thus $f : (K, \alpha_K) \longrightarrow (L, \mu)$ is still uniform. Conversely, suppose (L, μ) is $\mathcal{L}$-fine, and consider the following uniform map:

$$\mathcal{H}Coz_u(L, \mu) \xrightarrow{\varepsilon_L} (L, \mu).$$

Now $\mathcal{H}Coz_u(L, \mu)$ is Lindelöf, so the map is still uniform when the fine uniformity is on $\mathcal{H}Coz_u(L, \mu)$:

$$(\mathcal{H}Coz_u L, \alpha_{\mathcal{H}Coz_u L}) \xrightarrow{\varepsilon_L} (L, \mu).$$

Take any countable cozero cover (a_n) of L, then $(\downarrow a_n)$ is a countable cozero cover of $\mathcal{H}Coz_u L$, and hence in $\alpha_{\mathcal{H}Coz_u L}$. Thus $\varepsilon_L(\downarrow a_n) \in \mu$. Since $\varepsilon_L(\downarrow a_n) = (a_n)$, this shows that (L, μ) is cozfine. □

The $\mathcal{A}$-fine reflection.

Since all three examples of $\mathcal{A}$-fine subcategories mentioned above form reflective subcategories in **UniFrm**, it is not surprising that a general construction should exist. Let (L, μ) be any uniform frame and define $\tilde{\mu}$ as follows: $\tilde{\mu}$ is generated by images of all normal covers of uniform frames $(M, \delta) \in \mathcal{A}$ under uniform maps $(M, \delta) \longrightarrow (L, \mu)$. Let $\hat{\mu} = \tilde{\mu} \vee \mu$, then $\hat{\mu}$ is a compatible uniformity so $(L, \hat{\mu})$ is a uniform frame. Note that $\mu \leq \hat{\mu} \leq \alpha_L$ and that $\mu \leq \upsilon$ implies $\hat{\mu} \leq \hat{\upsilon}$. Moreover, if μ is $\mathcal{A}$-fine, then $\mu = \hat{\mu}$, and conversely. Let $a\mu$ be the uniformity constructed by transfinite iteration of this hat operation. This process must terminate since $\hat{\mu} \leq \alpha_L$ for any compatible μ. Moreover, it can be shown that $a\mu = \bigcap\{\upsilon \mid \mu \leq \upsilon = \hat{\upsilon}\}$: by construction $\mu \leq a\mu = \widehat{a\mu}$, so one inclusion holds. For the converse, take any υ with $\mu \leq \upsilon = \hat{\upsilon}$. Then $\hat{\mu} \leq \upsilon = \hat{\upsilon}$, and repeating this process transfinitely gives $a\mu \leq \upsilon = \hat{\upsilon}$. Thus $a\mu \subseteq \bigcap\{\upsilon \mid \mu \leq \upsilon = \hat{\upsilon}\}$. We also show that uniform maps preserve the hat operation:

Lemma. *If $f:(L,\mu)\longrightarrow(K,\upsilon)$ is uniform then so is $f:(L,\hat\mu)\longrightarrow(K,\hat\upsilon)$.*

Proof: Suppose $f:(L,\mu)\longrightarrow(K,\upsilon)$ is uniform, and take any $A\in\hat\mu$ with A coming from $\tilde\mu$, say $A=g[B]$ with $g:(M,\delta)\longrightarrow(L,\mu)$ uniform, B a normal cover of M and $(M,\delta)\in\mathcal{A}$. Consider the following diagram:

$$\begin{tikzcd}
(M,\delta) \arrow[r, "g"] & (L,\mu) \arrow[r, "id_L"] \arrow[d, "f"'] & (L,\hat\mu) \arrow[d, dashed] \\
 & (K,\upsilon) \arrow[r, "id_K"] & (K,\hat\upsilon)
\end{tikzcd}$$

Obviously $f\circ g[B]\in\hat\upsilon$, and thus so is $f[A]$. $\square$

If we consider the uniformity $a\mu$ constructed above on any given uniform frame (L,μ), then $(L,\mu)\longrightarrow(L,a\mu)$ is obviously uniform. Moreover, the latter frame is $\mathcal{A}$-fine: take any uniform map $f:(M,\delta)\longrightarrow(L,a\mu)$ with $(M,\delta)\in\mathcal{A}$ and consider the following diagram:

$$\begin{tikzcd}
(M,\delta) \arrow[r, "f"] \arrow[d] & (L,a\mu) \arrow[r] & (L,\upsilon)=(L,\hat\upsilon) \\
(M,\alpha_M) \arrow[ur, dashed, "f"] \arrow[urr, bend right, "f"'] & &
\end{tikzcd}$$

f is uniform for each uniformity $\upsilon=\hat\upsilon$ since (L,υ) is $\mathcal{A}$-fine, and thus it is uniform for the intersection of all these uniformities. Now take any uniform map $f:(L,\mu)\longrightarrow(K,\upsilon)$ with (K,υ) $\mathcal{A}$-fine, then $f:(L,\hat\mu)\longrightarrow(K,\upsilon)$ remains uniform since $\upsilon=\hat\upsilon$. And thus by the definition of $a\mu$, $f:(L,a\mu)\longrightarrow(K,\upsilon)$ is uniform. Thus we have shown:

Proposition. *$\mathcal{A}$-fine uniform frames form a reflective subcategory of* **UniFrm** *with reflection functor a.*

Metric-fine uniform frames

Let $\mathcal{M}$ be the subcategory of **UniFrm** consisting of all metric frames, that is, uniform frames with countable bases, and consider the frames which are *Metric-fine*, or *$\mathcal{M}$-fine* (for the equivalent notion in spaces see Hager [11]). Using the general result from above, we have that the $\mathcal{M}$-fine uniform frames form a reflective subcategory of **UniFrm**. We will denote the reflection functor by m. Various properties of metric frames can be used to simplify the construction of the $\mathcal{A}$-fine reflection in the special case where $\mathcal{A}=\mathcal{M}$. For example, $\mu\subseteq\tilde\mu$, so $\tilde\mu=\hat\mu$ in this case. The next few results will make this more precise.

Lemma. *The following are equivalent for a uniform frame (L,μ):*

(1) *(L,μ) is $\mathcal{M}$-fine;*
(2) *whenever $f:(M,\rho)\longrightarrow(L,\mu)$ is uniform for metric (M,ρ), $f[A]\in\mu$ for each cover A of M;*
(3) *whenever (M,ρ) is metric and a uniform subframe of (L,μ), then (M,α_M) is also a uniform subframe.*

Proof: Since metric frames are paracompact, the fine uniformity on a metric frame consists of all covers, and thus (1) implies (2). The converse is obviously true. (1) implies (3) is trivial. Suppose (3) holds for a uniform frame (L,μ). Let $f:(M,\rho)\longrightarrow(L,\mu)$ be a uniform map with (M,ρ) metric. Now consider the factorisation

$$(M,\rho)\xrightarrow{f}(f[M],\ f[\rho])\xhookrightarrow{i}(L,\mu).$$

Clearly both the maps are uniform, and $(f[M],f[\rho])$ is metric. By hypothesis, every cover of $f[M]$ is in μ. Take any cover A of M, then $f[A]$ is a cover of $f[M]$ and thus $f[A]\in\mu$. Thus (3) implies (1). ☐

The next proposition shows that the transfinite iteration which is used in the general case is not required here, in fact, $(L,\tilde{\mu})$ is already $\mathcal{M}$-fine:

Proposition. *If, for any uniform frame (L,μ), $\tilde{\mu}$ is generated by uniform images of all covers of metric frames (M,ρ), then $(L,\tilde{\mu})$ is an $\mathcal{M}$-fine uniform frame. Thus $m\mu=\tilde{\mu}$.*

Proof: It should be noted that basic elements of $\tilde{\mu}$ are finite meets of uniform images of covers of metric frames. Let $f:(M,\rho)\longrightarrow(L,\tilde{\mu})$ be a uniform map with (M,ρ) metric. We must show that $f:(M,\alpha_M)\longrightarrow(L,\tilde{\mu})$ is uniform. Since (M,ρ) is metric, it has a countable basis, say $A_1,A_2\dots$. For each n, $f[A_n]\in\tilde{\mu}$, and so by the definition of $\tilde{\mu}$, there exists a finite set $F(n)$ and maps $g_s:(M_s,\rho_s)\longrightarrow(L,\mu)$ with (M_s,ρ_s) metric for each $s\in F(n)$ such that $\bigwedge_{s\in F(n)}g_s[C_s]\le f[A_n]$ with C_s covers of M_s. Let $(N,\delta)=\bigoplus_{s\in F(n),n\in\mathbb{N}}(M_s,\rho_s)$, a countable coproduct of metric frames, hence metric. Let $(N,\delta)\xrightarrow{g}(L,\mu)$ be the induced map. By general properties of the coproduct, for each n there exists a cover C_n of N with $g[C_n]\le f[A_n]$. Now consider the factorisation:

$$\begin{tikzcd}
(N,\delta) \arrow[rr, "g"] \arrow[dr, two heads, "\bar{g}"'] & & (L,\mu) \\
& (Z,\upsilon) \arrow[ur, hook, "e"'] &
\end{tikzcd}$$

Then (Z,υ) is also metric, and hence, by the definition of $\tilde{\mu}$, $e:(Z,\alpha_Z)\hookrightarrow(L,\tilde{\mu})$ is uniform. By an argument similar to that used in the proof of the locallic version

Gleason's theorem ([34]), since each $f(A_n)$ is refined by a cover in α_Z, $f[M] \subseteq Z$. Thus as a uniform map f factors through (Z, α_Z):

$$\begin{tikzcd} (Z,\alpha_Z) \arrow[r, hook] & (L,\tilde{\mu}) \\ & (M,\rho) \arrow[u, "f"'] \arrow[ul, dashed, "\bar{f}"] \end{tikzcd}$$

But then $\bar{f} : (M, \alpha_M) \longrightarrow (Z, \alpha_Z)$ is uniform and thus so is $f : (M, \alpha_M) \longrightarrow (L, \tilde{\mu})$. □

The next proposition shows that the one above extends the corresponding result for spaces obtained by Rice [28]. However, Rice proved this using the existence of the $\mathcal{M}$-fine coreflection and his proof depended on some technical results and the notion of "metrically determined categories" (which has been explored in the frame setting ([34]) but will not be discussed in this paper). These complications are avoided through the proof given above.

Proposition. *For any uniform space μX, μX is $\mathcal{M}$-fine iff $\mathcal{O}\mu X$ is $\mathcal{M}$-fine. That is, $\mathcal{M}$-fineness is a conservative notion.*

Proof: Suppose μX is an $\mathcal{M}$-fine uniform space. Consider any uniform frame map $(M, \rho) \xrightarrow{h} \mathcal{O}\mu X$ with (M, ρ) metric. Then, since $\Sigma(M, \rho)$ is a metric space, the $\mathcal{M}$-fineness of μX gives

$$\begin{tikzcd} \Sigma(M,\rho) & \Sigma\mathcal{O}\mu X \arrow[l, "\Sigma h"'] & \mu X \arrow[l, "O_X"'] \arrow[dll] \\ (\Sigma M, \alpha_{\Sigma M}) \arrow[u, "id"] & & \end{tikzcd}$$

And hence $(M, \alpha_M) \longrightarrow \mathcal{O}(\Sigma M, \alpha_{\Sigma M}) \longrightarrow \mathcal{O}\mu X$ is uniform.
Conversely, suppose that $\mathcal{O}\mu X$ is an $\mathcal{M}$-fine uniform frame, and take any uniformly continuous map $\rho Y \xleftarrow{f} \mu X$ with ρY a metric space. Then $\mathcal{O}\rho Y \xrightarrow{\mathcal{O}f} \mathcal{O}\mu X$ is uniform, and thus so is $(\mathcal{O}Y, \alpha_{\mathcal{O}Y}) \xrightarrow{\mathcal{O}f} \mathcal{O}\mu X$, and hence $(\Sigma\mathcal{O}Y, \alpha) \xleftarrow{\Sigma\mathcal{O}f} \Sigma\mathcal{O}\mu X \xleftarrow{O_X} \mu X$ is uniformly continuous. The result follows since metric spaces are sober, that is $\Sigma\mathcal{O}\rho Y \cong \rho Y$ and hence $(\Sigma\mathcal{O}Y, \alpha) \cong \alpha Y$. □

$\mathcal{M}$-fineness and separability.

As mentioned in the preliminaries of this paper, the separable uniform frames form a coreflective subcategory of **UniFrm**. We now explore how the property of separability interacts with that of $\mathcal{M}$-fineness. Since metric subframes of separable frames are separable, and hence Lindelöf, if $(M, \rho) \hookrightarrow (L, e\mu)$ is uniform with (L, μ) $\mathcal{M}$-fine and (M, υ) metric, then (M, α_M) is separable, so $(M, \alpha_M) \hookrightarrow (L, \mu)$

is uniform and factors through the separable coreflection. That is, $(M, \alpha_M) \hookrightarrow (L, e\mu)$ is uniform. Thus we have that $\mathcal{M}$-fineness is preserved by the separable coreflection. Moreover, since $\mu \subseteq m\mu$, if μ is separable then $\mu \subseteq em\mu$. By the argument above, $em\mu$ is $\mathcal{M}$-fine, so $em\mu = m\mu$. This shows that separability, in turn, preserves the $\mathcal{M}$-fine reflection. One consequence of the above is that the separable coreflection of the fine uniformity, that is the Shirota uniformity, is $\mathcal{M}$-fine: by the above $me\alpha$ is separable, so $me\alpha \subseteq e\alpha$. Hence the two are equal. Moreover, using some basic diagram chasing, we can show the following:

Proposition. *The separable coreflection commutes with the $\mathcal{M}$-fine reflection. That is, $(L, em\mu) = (L, me\mu)$.*

This well-behaved interaction makes it possible to restrict to the subcategory of separable uniform frames to get the following result:

Proposition. *The separable $\mathcal{M}$-fine frames form a reflective subcategory of the separable uniform frames.*

Now consider the case where $\mathcal{A}$ is the subcategory $\mathcal{SM}$ of separable metric fames, with reflection given by m_s. Clearly every $\mathcal{M}$-fine frame is $\mathcal{SM}$-fine, but the converse need not true. However when restricting to separable frames it is:

Proposition. *For separable (L, μ), if (L, μ) is $\mathcal{SM}$-fine then it is $\mathcal{M}$-fine. Hence $m_s\mu = m\mu$ for separable (L, μ).*

Proof: Suppose (L, μ) is separable and $\mathcal{SM}$-fine. Take any metric subframe (M, ρ) of (L, μ), then (M, ρ) is separable, and so by hypothesis $(M, \alpha_M) \hookrightarrow (L, \mu)$ is uniform. $\square$

Since every separable metric frame is Lindelöf, and the connection between cozfine and Lindelöf frames has been established, it is clear that there should be a connection between $\mathcal{M}$-fine and cozfine uniform frames. This is indeed the case. We recall that the cozfine uniform frames form a reflective subcategory, with the reflection map given by $(L, \mu) \longrightarrow (L, \alpha_\mu)$, where α_μ is the uniformity generated by all countable covers consisting of uniform cozero elements [36]. The following theorem is analogous to the spatial result obtained by Hager [11]:

Theorem. *For (L, μ) separable, (L, μ) is $\mathcal{M}$-fine iff (L, μ) is cozfine. In fact $m\mu = \alpha_\mu$ for separable (L, μ).*

Proof: Suppose (L, μ) is $\mathcal{M}$-fine. Let (a_n) be a countable cozero cover of (L, μ), with $a_n = h_n(\mathbb{R} - 0)$ where $h_n : \mathcal{O}\mathbb{R} \longrightarrow (L, \mu)$ is uniform, taking the standard uniformity on $\mathcal{O}\mathbb{R}$. This gives rise to a uniform map

$$\bigoplus_n \mathcal{O}\mathbb{R} \xrightarrow{h} (L, \mu)$$

where $\bigoplus_n \mathcal{O}\mathbb{R}$ is the coproduct. Let ε_n be the coproduct maps. Since the countable coproduct of metric frames is metric ([6]) and (L, μ) is $\mathcal{M}$-fine,

$$(\bigoplus_n \mathcal{O}\mathbb{R}, \alpha_{\bigoplus_n \mathcal{O}\mathbb{R}}) \xrightarrow{h} (L, \mu)$$

is also uniform. Now $(\varepsilon_n(\mathbb{R}-0))$ covers $\bigoplus_n \mathcal{O}\mathbb{R}$, and hence is in the fine uniformity, since the fine uniformity consists of all covers. Thus $h[(\varepsilon_n(\mathbb{R}-0))] \in \mu$. But $h[(\varepsilon_n(\mathbb{R}-0))] = (h_n(\mathbb{R}-0)) = (a_n)$, so $(a_n) \in \mu$. Hence $\alpha_\mu = \mu$.
For the converse, suppose (L,μ) is cozfine. Let (M,ρ) be any metric subframe of (L,μ) with $f:(M,\rho) \hookrightarrow (L,\mu)$ uniform. Then (M,ρ) is separable and so is α_M. Take any basic cover A of the fine uniformity on M, that is, a countable cover of cozero elements. Then $f[A]$ is a countable cozero cover of L, but (L,μ) is cozfine, so $f[A] \in \mu$, hence $f:(M,\alpha_M) \longrightarrow (L,\mu)$ is uniform, and so (L,μ) is $\mathcal{M}$-fine. □

The above characterisation of the separable $\mathcal{M}$-fine frames can be used to further explore the interaction between them and the $\mathcal{SM}$-fine frames:

Proposition. *For any uniform frame (L,μ), $me\mu = em_s\mu$. That is, the $\mathcal{M}$-fine reflection of the separable coreflection is the separable coreflection of the $\mathcal{SM}$-fine reflection.*

Proof: Note that since $e\mu$ is separable, $me\mu = \alpha_{e\mu}$. It is clear that $(L, me\mu) \xrightarrow{id_L} (L, m_s\mu)$ is uniform. And thus, so is $(L, me\mu) \xrightarrow{id_L} (L, em_s\mu)$. Now take any countable cozero cover $A \in m_s\mu$. Then if $A \in \mu$, it is obviously in α_μ. Otherwise $A = f[B]$ for some uniform cover B of a separable metric frame. So there exists a countable uniform cozero cover $C \leq B$, and thus $f[C] \in \alpha_{e\mu}$, and therefore so is A. Hence $\alpha_{e\mu} = em_s\mu$. □

This is the frame analogue of the result that Frolik proves for spaces in [10].

$\mathcal{M}$-fineness and completion.
It can be shown that the cozfine reflection commutes with the completion functor iff the completion map is coz codense [36], and thus the same is true for the $\mathcal{M}$-fine reflection when restricted to the separable uniform frames. The solution is not known for the general case in the frame setting. However it may be shown that completeness is preserved by the $\mathcal{M}$-fine reflection and conversely, that $\mathcal{M}$-fineness is preserved by the completion functor:

Proposition. *If (L,μ) is complete then so is $(L, m\mu)$. Moreover, if (L,μ) is $\mathcal{M}$-fine then so is $C(L,\mu)$.*

Proof: Since $m\mu$ is finer than μ, if (L,μ) is complete, then so is $(L, m\mu)$.
Suppose (L,μ) is $\mathcal{M}$-fine, and take $f:(M,\rho) \longrightarrow C(L,\mu)$ uniform with (M,ρ) metric. Consider

$$\begin{tikzcd} (M,\rho) \arrow[r, "f"] \arrow[dr, "\gamma_L \circ f"'] & C(L,\mu) \arrow[d, "\gamma_L"] \\ & (L,\mu) \end{tikzcd}$$

Since (L,μ) is $\mathcal{M}$-fine, $(M,\alpha_M) \xrightarrow{\gamma_L \circ f} (L,\mu)$ is uniform. But M is metric hence paracompact, and thus (M, α_M) is complete [7], so $f : (M, \alpha_m) \longrightarrow C(L, \mu)$ is uniform. Thus $C(L, \mu)$ is $\mathcal{M}$-fine. $\square$

(Complete metric)-fine uniform frames

We now consider the case where $\mathcal{A}$ is the subcategory $\mathcal{CM}$ of complete metric frames. Clearly every $\mathcal{M}$-fine frame is $\mathcal{CM}$-fine. The general result for reflectivity can be applied to show that the $\mathcal{CM}$-fine uniform frames form a reflective subcategory of **UniFrm**. The reflection will be denoted by m_c. In the setting of spaces the analogous subcategory is called $\mathcal{M}_1$-fine ([11]) or sub-$\mathcal{M}$-fine ([28]). This latter terminology is justified by the last theorem of this section which shows that these frames are precisely the quotients of $\mathcal{M}$-fine uniform frames. Using the results that complete metric frames are spatial, and hence (M, ρ) complete metric implies $\Sigma(M, \rho)$ complete metric, and conversely if ρY is a complete metric space then $\mathcal{O}\rho Y$ is a complete metric frame (see [6]), it can be proved that the notion of $\mathcal{CM}$-fine is also conservative. To prove the following we need a series of results about $\mathcal{CM}$-fine frames.

Theorem. *The $\mathcal{CM}$-fine reflection commutes with completion.*

This will follow from the proposition mentioned in the preliminaries if the $\mathcal{CM}$-fine reflection can be shown to preserve quotients and completeness. It is clear that all the $\mathcal{A}$-fine reflections preserve the underlying frame. To that end, we first consider the behaviour relative to quotients; unlike the $\mathcal{M}$-fine frames which do not preserve quotients ([34]), these frames are well behaved:

Proposition. *A quotient of a $\mathcal{CM}$-fine frame is $\mathcal{CM}$-fine.*

Proof: Suppose (L, μ) is $\mathcal{CM}$-fine and consider any quotient $h : (L, \mu) \twoheadrightarrow (N, \delta)$. Let $f : (M, \rho) \longrightarrow (N, \delta)$ be a uniform map with (M, ρ) complete metric. Then (M, ρ) is the closed quotient of a projective metric frame ([34]), say (P, υ):

$$\begin{tikzcd} (L,\mu) \arrow[r, two heads, "h"] & (N,\delta) \\ (P,\upsilon) \arrow[u, dashed, "\widehat{fg}"] \arrow[r, two heads, "g", "closed"'] & (M,\rho) \arrow[u, "f"'] \end{tikzcd}$$

Now $\widehat{fg}$ exists since (P, υ) is projective. Moreover (P, υ) is complete and since (L, μ) is $\mathcal{CM}$-fine, $\widehat{fg}$ remains uniform when P has the fine uniformity. But g closed implies that $g : (P, \alpha_P) \longrightarrow (M, \alpha_M)$ is a uniform quotient, and hence $f : (M, \alpha_M) \longrightarrow (N, \delta)$ is also uniform. $\square$

Proposition. *The $\mathcal{CM}$-fine reflection preserves quotients.*

Proof: Consider any quotient $h : (L, \mu) \twoheadrightarrow (N, \delta)$. It suffices to check the result only for those covers which come from a complete metric frame. Take any such cover A of N with $f[B] \leq A$ for uniform $f : (M,\rho) \longrightarrow (N,\delta)$ and B a cover of (M,ρ), a complete metric frame. Consider the following diagram:

$$
\begin{tikzcd}
 & (P,\rho) \arrow[rr, two heads, "g", "closed"'] \arrow[dl] \arrow[ddr, dashed, "\widehat{fg}"] & & (M,\rho) \arrow[dl] \arrow[ddr, "f"] & \\
(P,\alpha_P) \arrow[rr, two heads, "g"] \arrow[ddr, dashed, "\widehat{fg}"'] & & (M,\alpha_M) & & \\
 & & (L,\mu) \arrow[rr, two heads, "h"] \arrow[dl] & & (N,\delta) \arrow[dl] \\
 & (L,m_c\mu) \arrow[rr, "h"] & & (N,m_c\delta) &
\end{tikzcd}
$$

Since g is a closed quotient it remains a uniform quotient relative to the fine uniformities. Hence there exists a cover C of P with $g[C] \leq B$. Now $\widehat{fg}$ exists since P is projective, but projective frames are complete, and hence $\widehat{fg} : (P,\alpha_P) \longrightarrow (L,m_c\mu)$ is uniform. Thus $\widehat{fg}[C] \in m_c\mu$. The result follows since $h \circ \widehat{fg}[C] = f \circ g[C] \leq f[B] \leq A$. $\square$

To complete the proof of the theorem of this section, it remains to show that completeness is preserved by this reflection. In fact, more is true:

Proposition. *If (L,μ) is $\mathcal{CM}$-fine then so is $C(L,\mu)$. Furthermore, if (L,μ) is complete then so is $(L,m_c\mu)$.*

Proof: Suppose (L,μ) is $\mathcal{CM}$-fine and consider any uniform map $f : (M,\rho) \longrightarrow C((L,\mu))$ with (M,ρ) complete metric. Then the composite:

$$
(M,\rho) \xrightarrow{\ f\ } C((L,\mu)) \xrightarrow{\ \gamma_L\ } (L,\mu)
$$

is uniform, and thus so is $(M,\alpha_M) \overset{\gamma_L \circ f}{\longrightarrow} (L,\mu)$. Since (M,ρ) is complete, so is (M,α_M), and hence this map factors through the completion of (L,μ). That is, $(M,\alpha_M) \overset{f}{\longrightarrow} C((L,\mu))$ is uniform.
Conversely, suppose (L,μ) is complete and consider the following diagram:

$$
\begin{tikzcd}
C((L,\mu)) \arrow[r, two heads, "\simeq"] \arrow[d] & (L,\mu) \arrow[d] \\
C(L,m_c\mu) \arrow[r, two heads] & (L,m_c\mu)
\end{tikzcd}
$$

Since $C(L,m_c\mu)$ is $\mathcal{CM}$-fine it follows that $C(L,m_c\mu) \cong (L,m_c\mu)$. $\square$

Subfine and $\mathcal{CM}$-fine frames.

We saw earlier that the subfine frames could be described in terms of $\mathcal{A}$-fine frames; namely they are precisely the $\mathcal{P}$-fine frames. Thus it is not surprising that the subfine frames form a reflective subcategory. This reflection, however, was defined independently from the notion of $\mathcal{A}$-fineness [34]. It is also shown that for complete metric frames, the subfine reflection is the fine uniformity, and so we can prove the following:

Proposition. *If (L,μ) is subfine then it is $\mathcal{CM}$-fine.*

The converse of this is not true in general; however, when restricting to separable uniform frames, these two subcategories are equivalent:

Proposition. *For a separable uniform frame (L,μ), (L,μ) is $\mathcal{CM}$-fine iff it is subfine.*

Proof: We need only prove the one direction: suppose (L,μ) is separable and $\mathcal{CM}$-fine. The separable uniform frames are generated by the separable metric frames, that is, **SepUniFrm** = Proj [**eMFrm**], so by taking the completion of each metric frame, μ can be thought of as "generated" by a family $\mathcal{F}$ of uniform maps $f : (M,\rho)_f \longrightarrow (L,\mu)$ with (M,ρ) a complete separable metric frame. Let $(K,\kappa) = \bigoplus_{f\in\mathcal{F}}(M,\rho)_f$ and then the induced map $h : (K,\kappa) \longrightarrow (L,\mu)$ is a uniform quotient. It will be shown that $h : (K,\alpha_K) \longrightarrow (L,\mu)$ is uniform, and hence (L,μ) is subfine. It suffices to show that $h\circ g$ is uniform whenever $g : (M,\rho) \longrightarrow (K,\alpha_K)$ is uniform for (M,ρ) complete metric. So take any uniform $g : (M,\rho) \longrightarrow (K,\alpha_K)$ with (M,ρ) complete metric. By the frame version of Gleason's Theorem, g factors through a countable subcoproduct:

$$\begin{tikzcd}
(M,\rho) \arrow[rr, "g"] \arrow[dr, "\bar{g}"'] & & (K,\alpha_K) \arrow[r, "h"] & (L,\mu) \\
& \bigoplus_n (M,\rho)_{f_n} \arrow[ur, "e"'] & &
\end{tikzcd}$$

Thus $h\circ e : \bigoplus_n(M,\rho)_{f_n} \longrightarrow (L,\mu)$ is uniform. Since $\bigoplus_n(M,\rho)_{f_n}$ is complete metric, and (L,μ) is $\mathcal{CM}$-fine, it follows that $h\circ e : \alpha(\bigoplus_n(M,\rho)_{f_n}) \longrightarrow (L,\mu)$ is uniform. Clearly $\bar{g} : (M,\rho) \longrightarrow \alpha(\bigoplus_n(M,\rho)_{f_n})$ is uniform, and thus so is $h\circ e\circ\bar{g}$. Hence $h\circ g$ is uniform. $\square$

$\mathcal{CM}$-fine versus $\mathcal{M}$-fine.

As mentioned, $\mathcal{M}$-fine frames are obviously $\mathcal{CM}$-fine, and $m_c\mu \subseteq m\mu$ for any uniformity μ. But a natural question is whether anything more can be said about their relationship. We first prove a technical lemma which gives two cases when these reflections coincide:

Lemma. *If (L,μ) is either a coproduct of complete metric frames or a projective uniform frame then $m_c\mu = m\mu$.*

Proof: Suppose $(L,\mu) = \bigoplus_{i\in I}(M_i,\rho_i)$ with the (M_i,ρ_i) complete metric frames. It suffices to show that $\tilde{\mu} \subseteq m_c(\mu)$. That is, if any $f:(M,\rho)\longrightarrow(L,\mu)$ is uniform with (M,ρ) metric, then so is $f:(M,\alpha_M)\longrightarrow(L,m_c\mu)$. Take any such map f, then applying the locallic Gleason's theorem gives the following factorisation:

$$\begin{tikzcd}
(M,\rho) \arrow[r, "f"] \arrow[dr, "g"'] & (L,\mu) \arrow[r, equal] & \bigoplus_{i\in I}(M_i,\rho_i) \\
& \bigoplus_n(M_n,\rho_n) \arrow[u, hook, "e"'] &
\end{tikzcd}$$

where $\bigoplus_n(M_n,\rho_n)$ is a countable subcoproduct, and thus is a complete metric frame. Therefore $\alpha(\bigoplus_n(M_n,\rho_n)) = m_c(\bigoplus_n(M_n,\rho_n))$. It follows that

$$(M,\alpha_M) \xrightarrow{g} \alpha(\textstyle\bigoplus_n(M_n,\rho_n)) = m_c(\textstyle\bigoplus_n(M_n,\rho_n)) \xrightarrow{e} (L,m_c\mu)$$

is uniform. That is, $f:(M,\alpha_M)\longrightarrow(L,m_c\mu)$ is uniform.

Now suppose (L,μ) is a projective uniform frame, then consider (L,μ) as a quotient of a coproduct of complete metric frames, say $q:(Z,\upsilon)\twoheadrightarrow(L,\mu)$. By projectivity there is a uniform map $h:(L,\mu)\longrightarrow(Z,\upsilon)$ such that $h\circ e = id_L$, and hence

$$(L,m\mu) \xrightarrow{h} (Z,m\upsilon) = (Z,m_c\upsilon) \xrightarrow{e} (L,m_c\mu)$$

is uniform. So $m\mu \subseteq m_c\mu$. □

Take any $\mathcal{CM}$-fine frame (L,μ) and consider it as a quotient of a coproduct of complete metric frames (Z,υ). Since the $\mathcal{CM}$-fine reflection preserves quotients, and $\mu = m_c\mu$, (L,μ) is a quotient of $(Z,m_c\upsilon)$. By the above result $m_c\upsilon = m\upsilon$ so (L,μ) is a quotient of an $\mathcal{M}$-fine frame. Using a similar argument to the one used in the proof that the $\mathcal{CM}$-fine reflection preserves quotients, it can be shown that any quotient of a $\mathcal{M}$-fine frame is $\mathcal{CM}$-fine.

Thus we have the following:

Theorem. *$\mathcal{CM}$-fine frames are precisely the quotients of $\mathcal{M}$-fine frames.*

Interactions with topological properties

The Lindelöf property.

We recall that a frame is Lindelöf if every cover has a countable subcover. For regular frames, it has been shown that Lindelöf is equivalent to being complete in the separable fine uniformity. This is the frame version of one of Shirota's theorems, and shows that in the frame setting, Lindelöf frames are the appropriate analogue to realcompact spaces ([35]). We first note that for any regular Lindelöf frame L, given any compatible uniformity μ on L, we have that $Coz_u(L,\mu) = CozL$. Thus the $\mathcal{M}$-fine reflection of a Lindelöf uniform frame is always complete. This follows from the lemma below, and Shirota's theorem:

Lemma. *If L is Lindelöf then $m\mu = e\alpha$ for any compatible μ.*

PROOF: L Lindelöf implies that (L,μ) is separable for any compatible uniformity μ, that is $\mu = e\mu$. Therefore, $m\mu = me\mu = \alpha_{e\mu} = \alpha_\mu$. Since α_μ has a basis of all the countable covers of $Coz_u(L,\mu)$, and by the above lemma, $Coz_u(L,\mu) = CozL$, it follows that $\alpha_\mu = e\alpha$. □

In fact, the Lindelöf frames can be characterised in terms of completeness and $\mathcal{M}$-fineness, an extension of the Shirota theorem for frames:

Proposition. *The following are equivalent for any completely regular frame L:*

(i) *L is complete in each of its separable $\mathcal{M}$-fine uniformities;*
(ii) *L is Lindelöf.*

PROOF: Suppose L is Lindelöf. Then $m\mu = e\alpha$ for each compatible uniformity μ, and thus $\mu = m\mu = e\alpha$ for each separable $\mathcal{M}$-fine μ, and so (L,μ) is complete. Conversely, since $e\alpha$ is separable $\mathcal{M}$-fine, $(L, e\alpha)$ is complete, and by Shirota's theorem, L is Lindelöf. □

Pseudocompactness.

A frame L is said to be *pseudocompact* if each frame map from $\mathcal{O}\mathbb{R}$ to L is bounded. In the case of L being completely regular, this is equivalent to $CozL$ being compact ([4]). Banaschewski and Pultr [7] show that a frame is pseudocompact iff every normal cover has a finite normal refinement, but their proof of this result actually shows the following:

Lemma. *For completely regular L, L is pseudocompact iff $\alpha_L = p\alpha_L$ iff all compatible uniformities are precompact.*

The final proposition lists characterisations of pseudocompactness. Each condition depends on one or more of the properties of compactness, $\mathcal{M}$-fineness, precompactness and coz-codensity. We recall that $\beta L \longrightarrow L$ denotes the Stone-Čech compactification (compact coreflection) of a completely regular frame ([5]), and $\beta_\sigma L \longrightarrow L$ denotes the Stone-Čech compactification (compact coreflection) of a regular σ-frame ([32]):

Proposition. *For a completely regular frame L, the following are equivalent:*

(1) *each of L's precompact uniformities is $\mathcal{M}$-fine (equivalently, cozfine);*
(2) *$(L, p\alpha)$ is $\mathcal{M}$-fine (equivalently, cozfine);*
(3) *$\beta L \twoheadrightarrow L$ is coz codense;*
(4) *every compactification of L is coz codense;*

(5) *if* $M \twoheadrightarrow L$ *is dense and* υ *is a compatible uniformity on* M*, then* (M, υ) *is* $\mathcal{M}$*-fine;*
(6) L *is pseudocompact.*

PROOF: (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are trivial.
(2) $\Rightarrow$ (6): Take any countable cozero cover A of L. Then $A \in p\alpha$ since $(L, p\alpha)$ is cozfine. But $p\alpha$ is precompact so there exists a finite uniform cover $B \leq A$. And this identifies a finite subcover of A.
(6) $\Rightarrow$ (2): Suppose L is pseudocompact, then $\alpha_L = p\alpha_L$. The result follows since (L, α_L) is always $\mathcal{M}$-fine.
(3) $\Rightarrow$ (2): If $\beta L \twoheadrightarrow L$ is coz codense then so is $C(L, p\alpha_L) \twoheadrightarrow (L, p\alpha_L)$. Apply the cozfine reflection: since $\beta L = C(L, p\alpha_L)$ is compact it has a unique uniformity, and so is cozfine. Thus $C(L, p\alpha_L) \twoheadrightarrow (L, \alpha_{p\alpha_L})$ is also a quotient, and hence $\alpha_{p\alpha_L} = p\alpha_L$, and so $(L, p\alpha_L)$ is $\mathcal{M}$-fine.
(3) $\Rightarrow$ (4): This follows immediately from the fact that every compactification of L can be thought of as a subframe of βL.
(5) $\Rightarrow$ (1): This follows directly by taking $M = L$.
(6) $\Rightarrow$ (3): Suppose L is pseudocompact, that is $CozL$ is compact. Now consider $\beta L \twoheadrightarrow L$. Applying Coz gives: $Coz\beta L \twoheadrightarrow CozL$. But $Coz\beta L = \beta_\sigma CozL$, and since $CozL$ is compact, $CozL = \beta_\sigma CozL$. Thus the compactification is coz codense.
(4) $\Rightarrow$ (1): Let (L, μ) be a precompact uniform frame. Then, by assumption, any compactification $\rho_L : \mathcal{S}(L, \mu) \twoheadrightarrow (L, \mu)$ is coz codense, that is $Coz\mathcal{S}(L, \mu) \cong Coz(L, \mu)$. Now take any countable cover A of $Coz(L, \mu)$, then it is isomorphic to a cover A' of $Coz\mathcal{S}(L, \mu)$. Since $\mathcal{S}(L, \mu)$ is fine, hence $\mathcal{M}$-fine, A' is uniform, and thus so is $\rho_L(A')$. But $\rho_L(A') = A$, so $A \in \mu$ which shows that (L, μ) is cozfine.
(6) $\Rightarrow$ (5): Suppose L is pseudocompact. That is, all uniformities are precompact. Let $h : M \twoheadrightarrow L$ be dense, then M is pseudocompact, so $\beta(M, \upsilon) = C(M, \upsilon) \twoheadrightarrow (M, \upsilon) \twoheadrightarrow (L, h[\upsilon])$ is dense, and hence by (4), it is coz codense. Thus $C(M, \upsilon) \twoheadrightarrow (M, \upsilon)$ is coz codense, and since (4) $\Rightarrow$ (1), (M, υ) is $\mathcal{M}$-fine. $\square$

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