Existence of nonzero nonnegative solutions of semilinear equations at resonance

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Abstract. The existence of nonzero nonnegative solutions are established for semilinear equations at resonance with the zero solution and possessing at most linear growth. Applications are given to nonlinear boundary value problems of ordinary differential equations.

Keywords: semilinear equations at resonance, boundary value problems

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Introduction

The existence of solutions in convex sets for abstract semilinear equations at resonance are recently studied by Gaines and Santanilla [1], Nieto [2], Przeradzki [3] and Santanilla [4]. Like in these papers, we consider the operator equation

$$Lu = N(u), \quad u \in C,$$

where C is a cone, $L\colon \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero, $N\colon X \to Y$ is continuous, nonlinear satisfying a compact property with respect to L, and X, Y are Banach spaces. We note that C is a cone provided that C is a nonempty closed convex subset of X such that $\alpha C \subset C\,\forall\,\alpha \geq 0$. Hence $0 \in C$. We suppose that $C \neq X$. Results of [1]–[4] imply the existence of a solution for the above equation in C.

In this paper, we assume $N(0) = 0$. Consequently, $Lu = N(u)$ has always a trivial solution $u = 0$ in C. We derive results giving another (nonzero) solutions of $Lu = N(u)$ belonging to a cone shell of C (see a set Ω below). We use the alternative method like in [4] together with the retraction method. To illustrate our theory, we show nonzero nonnegative solutions for boundary value problems of higher order ordinary differential equations motivated by [1]–[4].

Notation and main results

In this paper, L is a Fredholm operator of index zero. It is well known that there are projections $P\colon X \to X$ and $Q\colon Y \to Y$ such that $\operatorname{im} P = \ker L$ and $\ker Q = \operatorname{im} L$. We denote by $K_p\colon \operatorname{im} L \to \operatorname{dom} L \cap \ker P$ the partial inverse of L. We assume that $0 < \|K_p(I - Q)\| < \infty$ and $K_p(I - Q)$ maps bounded sets into relative compact ones.

Furthermore, following [4], we suppose the existence of a continuous bilinear form $\langle\cdot,\cdot\rangle$ on $Y \times X$ such that

$$z \in \operatorname{im} L \quad \text{if and only if} \quad \langle z, u_0\rangle = 0$$

for any $u_0 \in \ker L$. Let $\{v_1, v_2, \cdots, v_n\}$ be a basis of $\ker L$. We define the mapping $J\colon \operatorname{im} Q \to \ker L$ as follows

$$z \to \sum_{i=1}^{n} \langle z, v_i\rangle v_i.$$

It is clear that J is an isomorphism satisfying $\langle J^{-1}u_0, u_0\rangle > 0$ if $u_0 \neq 0$.

The next result is an extension of [4, Theorem 1] for showing the existence of a nonzero solution of $Lu = N(u)$ in C when $N(0) = 0$.

Theorem 1. *Suppose that the following conditions are satisfied.*

(i) *There are constants $c_1 > 0$ and $0 \le c_2 < \|K_p(I-Q)\|^{-1}$ such that*

$$|N(u)| \le c_1 + c_2|u|, \qquad \forall\, u \in C.$$

(ii) *There is $R > 0$ such that*

$$\langle QN(u_0 + u_1), u_0\rangle \le 0$$

for all $u = u_0 + u_1 \in C$, where $u_0 \in \ker L$, $|u_0| = R$, $u_1 \in \ker P$, and

$$|u_1| \le \rho = (c_1 + c_2 R)/\big(\|K_p(I-Q)\|^{-1} - c_2\big).$$

(iii) *There is $0 < r < R$ such that $|u| \le r$ implies $|u_0| \le R$ and $|u_1| \le \rho$, and*

$$u \neq \lambda\big(P + JQN + K_p(I-Q)N\big)(u)$$

for all $u \in C$, $|u| = r$ and $\lambda > 1$.

(iv) $\big(P + JQN + K_p(I-Q)N\big)(\Omega) \subset C \setminus \{0\}$, *where*

$$\Omega = \Big\{u = u_0 + u_1 \in C : |u| \ge r,\ |u_0| \le R,\ |u_1| \le \rho\Big\}.$$

Then $Lu = N(u)$ has a solution $u \in \Omega$.

Proof: We take the retraction $\sigma\colon C \setminus \{0\} \to \Omega$ given by

$$\sigma(u) = \begin{cases} u & \text{for} \quad u \in \Omega \\ \frac{r}{|u|}u & \text{for} \quad 0 < |u| \le r \\ \frac{R}{|u_0|}u & \text{for} \quad |u_0| \ge R,\ |u_1| \le \rho \\ \min\Big\{\frac{R}{|u_0|}, \frac{\rho}{|u_1|}\Big\}u & \text{for} \quad |u_0| \ge R,\ |u_1| \ge \rho \\ \frac{\rho}{|u_1|}u & \text{for} \quad |u_0| \le R,\ |u_1| \ge \rho. \end{cases}$$

According to (iv), we can consider the mapping $M_1 = \sigma M_2 \colon \Omega \to \Omega$, where

$$M_2 = P + JQN + K_p(I - Q)N.$$

It is well known that M_1 has a fixed point $u \in \Omega$. We show that u is also a solution of $Lu = N(u)$. By decomposing

$$M_2(u) = M_2(u)_0 + M_2(u)_1, \quad M_2(u)_0 \in \ker L, \quad M_2(u)_1 \in \ker P,$$

we have the following possibilities:

1. If $M_2(u) \in \Omega$ then $u = \sigma(M_2(u)) = M_2(u)$ and so $Lu = N(u)$.

2. If $0 < |M_2(u)| < r$ then $u = \frac{r}{|M_2(u)|} M_2(u)$ and hence $u = \lambda M_2(u)$ for $\lambda = \frac{r}{|M_2(u)|} > 1$ and $|u| = r$. Contradiction to (iii).

3. If $|M_2(u)_0| \geq R$ and $|M_2(u)_1| \leq \rho$ then $\frac{R}{|M_2(u)_0|} M_2(u) = u$. If $\lambda = \frac{R}{|M_2(u)_0|}=1$ then we have the case **1**. Hence for $\lambda = \frac{R}{|M_2(u)_0|} < 1$, we have

$$u_0 = \lambda\big(Pu + JQN(u)\big), \quad |u_1| \leq \rho, \quad |u_0| = R.$$

This implies

$$0 < (1 - \lambda)\langle J^{-1}u_0, u_0\rangle = \lambda\langle QN(u), u_0\rangle \leq 0,$$

a contradiction to (ii).

4. If $|M_2(u)_0| \geq R$ and $|M_2(u)_1| \geq \rho$ then

$$\lambda M_2(u) = u, \quad \lambda = \min\left\{\frac{R}{|M_2(u)_0|}, \frac{\rho}{|M_2(u)_1|}\right\}, \quad u \in \Omega.$$

Hence either $|u_0| = R$, $|u_1| \leq \rho$ or $|u_0| \leq R$, $|u_1| = \rho$. When $\lambda = 1$ then we have the case **1**. For $|u_0| = R$, $|u_1| \leq \rho$, $\lambda < 1$ we have the case **3**. If $|u_0| \leq R$, $|u_1| = \rho$, $\lambda < 1$ then

$$\begin{aligned}
\rho = |u_1| &= \lambda|K_p(I - Q)N(u)| \\
&\leq \lambda\|K_p(I - Q)\|(c_1 + c_2|u|) \\
&\leq \lambda\|K_p(I - Q)\|(c_1 + c_2R + c_2\rho) \\
&< c_2\|K_p(I - Q)\|\rho + (c_1 + c_2R)\|K_p(I - Q)\| = \rho,
\end{aligned}$$

a contradiction.

5. If $|M_2(u)_0| \leq R$ and $|M_2(u)_1| \geq \rho$ then $u = \lambda M_2(u)$ with $\lambda = \frac{\rho}{|M_2(u)_1|} \leq 1$. When $\lambda = 1$ then we have the case **1**. If $\lambda < 1$ then $|u_0| \leq R$, $|u_1| = \rho$ and like in the end of the case **4**, we arrive at a contradiction.

Summarizing we see that only the case **1** is valid and the proof is finished. $\square$

Proposition 2. (a) *Assume that there is $r > 0$ such that*

$$\langle QN(u_0 + u_1), u_0\rangle \geq 0$$

for all $u = u_0 + u_1 \in C$, $u_0 \in \ker L$, $u_1 \in \ker P$ and $|u| = r$. Then

$$u \neq \lambda\big(P + JQN + K_p(I - Q)N\big)(u)$$

for all $u \in C$, $|u| = r$ and $\lambda > 1$ provided that it holds

$$u \in C \cap \ker P \quad \Longrightarrow \quad u = 0.$$

(b) *Assume that $|N(u)| \leq c_1 \,\forall\, u \in C$ and for any $K_1 > 0$ there is $K_2 > 0$ such that*

$$\langle QN(u_0 + u_1), u_0\rangle \leq 0$$

for all $u = u_0 + u_1 \in C$, $u_0 \in \ker L$, $u_1 \in \ker P$ and $|u_0| = K_2$, $|u_1| \leq K_1$. Then (i) *and* (ii) *of Theorem* 1 *hold.*

Proof: To prove (a), we assume

$$u = \lambda\big(P + JQN + K_p(I - Q)N\big)(u)$$

for some $u \in C$, $|u| = r$ and some $\lambda > 1$. Then

$$\begin{aligned} &u_0 = \lambda u_0 + JQN(u) \\ &0 \leq \langle QN(u_0 + u_1), u_0\rangle = (1 - \lambda)\langle J^{-1}u_0, u_0\rangle \leq 0. \end{aligned}$$

Hence $\langle J^{-1}u_0, u_0\rangle = 0$ and so $u_0 = 0$. By using $u \in C \cap \ker P$, we obtain $u = 0$ which contradicts to $|u| = r > 0$. The assertion (b) is clear. We note that (b) is a certain Landesman-Lazer type condition (see [3]). □

Corollary 3. *Suppose that $\ker L = \{0\}$ and N maps bounded sets of C into bounded ones of Y. If there are $r_{1,2} > 0$ and $R > r_1$ such that*

$$\begin{aligned} &Lu = \lambda(N(u) + \epsilon u),\ \lambda < 1,\ 0 < \epsilon \leq r_2 \quad \Longrightarrow \quad |u| \neq R \\ &Lu = \lambda(N(u) + \epsilon u),\ \lambda > 1,\ 0 < \epsilon \leq r_2 \quad \Longrightarrow \quad |u| \neq r_1 \\ &L^{-1}(N(u) + \epsilon u) \in C \setminus \{0\} \quad \forall\, u \in C \setminus \{0\}, \quad \forall\, 0 < \epsilon \leq r_2, \end{aligned}$$

then $Lu = N(u)$ has a nonzero solution $u \in C$ satisfying $r_1 \leq |u| \leq R$.

Proof: Let us fix $r_2 \geq \epsilon > 0$. The proof of Theorem 1 is applicable when M_2 is replaced by $u \to L^{-1}(N(u) + \epsilon u)$, Ω is replaced by the set

$$\Gamma = \big\{u \in C : r_1 \leq |u| \leq R\big\},$$

and the retraction σ is replaced by the retraction $\tau \colon C \setminus \{0\} \to \Gamma$ given as follows

$$\tau(u) = \begin{cases} u & \text{for} \quad u \in \Gamma \\ \frac{u}{|u|}r & \text{for} \quad 0 < |u| \leq r \\ \frac{u}{|u|}R & \text{for} \quad |u| \geq R. \end{cases}$$

So we have $u_\epsilon \in C$, $r_1 \leq |u_\epsilon| \leq R$ for $\epsilon > 0$ sufficiently small such that $Lu_\epsilon = N(u_\epsilon) + \epsilon u_\epsilon$. By passing to the limit $\epsilon \to 0_+$ and using the compactness of L^{-1}, we arrive at the desired solution of $Lu = N(u)$. □

Examples

We consider the following boundary value problem

$$\mathcal{L}u = f(x,u), \tag{1}$$

where $f\colon [0,1]\times[0,\infty)\to\mathbb{R}$ is continuous and $\mathcal{L}$ represents a linear boundary value problem for an ordinary differential equation on $[0,1]$ with continuous coefficients. We assume that there is $\omega \in C([0,1],[0,\infty))$ such that ω is nonzero almost everywhere on $[0,1]$ and

$$(A1)\qquad \begin{aligned} &\ker\mathcal{L} = \mathbb{R}\omega, \quad \int_0^1 \omega^2(s)\,ds = 1,\\ &h\in\operatorname{im}\mathcal{L} \iff \int_0^1 h(s)\omega(s)\,ds = 0. \end{aligned}$$

Moreover, we suppose that there is $H\in C([0,1]\times[0,1],\mathbb{R})$ such that

$$\mathcal{L}u = h \in \operatorname{im}\mathcal{L}, \quad \int_0^1 u(s)\omega(s)\,ds = 0 \iff u(x) = \int_0^1 H(x,t)h(t)\,dt.$$

We put

$$\begin{aligned} X &= Y = C([0,1],\mathbb{R}),\\ Lu &= \mathcal{L}u, \quad N(u) = f(\cdot,u(\cdot)),\\ \langle u,z\rangle &= \beta\int_0^1 u(x)z(x)\,dx, \quad \beta>0 \ \text{ is a constant},\\ C &= \big\{u\in X:\ u(\cdot)\ge 0\big\}. \end{aligned}$$

Since $(A1)$ holds, we take

$$(Qu)(x) = (Pu)(x) = \omega(x)\int_0^1 u(s)\omega(s)\,ds, \quad J=\beta I.$$

Furthermore, we have

$$(K_p(I-Q)z)(x) = \int_0^1 G(x,s)z(s)\,ds,$$

where

$$G(x,s)=H(x,s)-\omega(s)\int_0^1 H(x,t)\omega(t)\,dt.$$

So we arrive at

$$\begin{aligned}
&Pu(x)+JQN(u)(x)+K_p(I-Q)N(u)(x)=\\
&\omega(x)\int_0^1 u(s)\omega(s)\,ds+\beta\omega(x)\int_0^1 f(s,u(s))\omega(s)\,ds\\
&+\int_0^1 G(x,s)f(s,u(s))\,ds=\omega(x)\int_0^1 u(s)\omega(s)\,ds\\
&+\int_0^1 \big(\beta\omega(x)\omega(s)+G(x,s)\big)f(s,u(s))\,ds.
\end{aligned}$$

By assuming

$$(A2)\qquad \begin{gathered}K_1\omega(x)\omega(s)\geq -G(x,s)\geq K_2\omega(x)\omega(s)\\ \forall\,(x,s)\in[0,1]\times[0,1]\end{gathered}$$

for constants $K_1>0>K_2$, the condition (iv) holds for $\beta=K_1$ provided that

$$(A3)\qquad f(s,0)\geq 0,\quad f(s,u)>\frac{-u}{K_1-K_2}\quad \forall\,(s,u)\in[0,1]\times(0,\infty).$$

We also suppose that

$$(A4)\qquad \sup_{[0,1]\times[0,\infty)}|f(\cdot,\cdot)|<\infty.$$

Then (i) holds with $c_2=0$.

Since

$$\langle QN(u_0+u_1),u_0\rangle=\beta\int_0^1 f(s,u(s))\omega(s)\,ds\int_0^1 u(s)\omega(s)\,ds,$$

by assuming

$$(A5)\qquad f(s,u)\geq 0\quad \forall\,s\in[0,1],\ \forall\,0\leq u\leq r$$

for a constant $r>0$, the condition (a) of Proposition 2 is valid.

Finally, by supposing the Landesman-Lazer type condition

$$(A6) \qquad \limsup_{u\to\infty} \int_0^1 f(s,u)\omega(s)\,ds < 0$$

uniformly with respect to $s \in [0,1]$, it is well known ([3]) that the condition (b) of Proposition 2 holds as well.

Summarizing, Theorem 1 and Proposition 2 give the following result:

(1) *has a nonzero nonnegative solution provided that* $(A1)$–$(A6)$ *are satisfied.*

But it is worth to point out that if $(A1)$, $(A2)$, $(A4)$–$(A6)$ hold then by taking — if it is necessary — another sufficiently large constants K_1, $-K_2$, the conditions $(A1)$–$(A6)$ are satisfied. Consequently, we obtain the following theorem.

Theorem 4. (1) *has a nonzero nonnegative solution provided that* $(A1)$, $(A2)$, $(A4)$–$(A6)$ *are satisfied.*

When f has a linear growth in u, we have like in [4] the following result.

Theorem 5. (1) *has a nonzero nonnegative solution provided that* $(A1)$, $(A2)$, $(A3)$, $(A5)$ *hold and moreover, there are constants* $a > 0 > b$ *such that*

$$(A7) \qquad f(s,u) \le bu + a \quad \forall\,(s,u) \in [0,1]\times[0,\infty),$$

and as well as, it holds

$$(A8) \qquad K_1 - K_2 > \max\{K_1, -K_2\}\,\max\omega(\cdot)\int_0^1 \omega(s)\,ds.$$

Proof: $(A2)$ implies

$$\|K_p(I-Q)\| \le \max\{K_1,-K_2\}\,\max\omega(\cdot)\int_0^1 \omega(s)\,ds.$$

$(A3)$ and $(A7)$ give $-\frac{1}{K_1-K_2} \le b < 0$. Consequently by $(A8)$, (i) holds with $c_2 = \frac{1}{K_1-K_2}$.

We verify (ii) by putting $R = -\frac{a}{b}\int_0^1 \omega(s)\,ds$, since for

$$u(x) = u_0(x) + u_1(x), \qquad \int_0^1 u_1(s)\omega(s)\,ds = 0, \quad u_0(x) = R\omega(x),$$

it holds

$$\langle QN(u_0+u_1),u_0\rangle = \beta R\int_0^1 f(s,u(s))\omega(s)\,ds$$
$$\le \beta R\int_0^1 \big(a+bu(s)\big)\omega(s)\,ds = 0.$$

Consequently, (ii) holds and Theorem 1 is applicable. $\square$

By rewriting (1) in the form

$$-\mathcal{L}u = -f(x,u), \tag{2}$$

and applying Theorems 4, 5 to (2), the conditions $(A3)$ and $(A5)$–$(A7)$ are replaced by

$(A3')$ $\quad f(s,0)\le 0,\quad f(s,u) < \dfrac{u}{K_1-K_2}\quad \forall\,(s,u)\in[0,1]\times(0,\infty),$

$(A5')$ $\quad f(s,u)\le 0\quad \forall\, s\in[0,1],\ \forall\, 0\le u\le r,$

$(A6')$ $\quad \liminf\limits_{u\to\infty}\displaystyle\int_0^1 f(s,u)\omega(s)\,ds > 0$ uniformly with respect to $s\in[0,1]$,

$(A7')$ $\quad$ there are constants $b>0>a$ such that $f(s,u)\ge bu+a\quad \forall\,(s,u)\in[0,1]\times[0,\infty)$,

respectively.

Theorem 6. (1) *has a nonzero nonnegative solution provided that either* $(A1)$, $(A2)$, $(A4)$, $(A5')$, $(A6')$ *or* $(A1)$, $(A2)$, $(A3')$, $(A5')$, $(A7')$, $(A8)$ *hold.*

We note that clearly $(A2)$ is satisfied for some constants $K_1>0>K_2$ when $\omega(\cdot)>0$. So in this case we can construct f satisfying the rest conditions of either Theorem 4, 5 or 6.

Now we consider

$$\begin{aligned} &u'' = f(x,u),\\ &u(0)-u(1) = u'(0)-u'(1) = 0, \end{aligned} \tag{3}$$

where $f\colon[0,1]\times[0,\infty)\to\mathbb{R}$ is continuous.

Theorem 7. *Assume that there are constants $R > r > 0$ such that either*

$$(A9) \qquad f(\cdot, r) \le 0 \ \ \textit{and} \ \ (A4), (A6') \ \ \textit{hold},$$

or

$$(A10) \qquad f(\cdot, r) \le 0 \ \ \textit{and} \ \ f(\cdot, R) \ge 0 \ \ \textit{hold}.$$

Then (3) *has a solution satisfying either $u(\cdot) \ge r$ in the case* $(A9)$*, or $R \ge u(\cdot) \ge r$ in the case* $(A10)$*.*

Proof: In the case $(A9)$, we can modify f to $\tilde{f}$ such that $\tilde{f}$ coincides with f on the set $\{(x,u) : x \in [0,1],\, r \le u\}$, $\tilde{f}(x,u) < 0$ for $u < r$ and $\lim_{u \to -\infty} \tilde{f}(x,u) = -1$ uniformly for $x \in [0,1]$.

In the case $(A10)$, we can modify f to $\tilde{f}$ such that $\tilde{f}$ coincides with f on the set $\{(x,u) : x \in [0,1],\, r \le u \le R\}$, $\tilde{f}(x,u) < 0$ for $u < r$, $\tilde{f}(x,u) > 0$ for $u > R$ and $\lim_{u \to \pm\infty} \tilde{f}(x,u) = \pm 1$ uniformly for $x \in [0,1]$.

Then [3, Theorem 4] and [4, Theorems 1 and 5] give in both cases a solution of the problem

$$\begin{aligned} &u'' = \tilde{f}(x,u), \\ &u(0) - u(1) = u'(0) - u'(1) = 0. \end{aligned}$$

If $u(x_0) < r$ for some x_0, then $\min u = u(z_0) < r$ and

$$0 \le u''(z_0) = \tilde{f}(z_0, u(z_0)) < 0.$$

This contradiction implies $u(\cdot) \ge r$ and consequently, u is the desired solution in the case $(A9)$.

If $u(x_0) > R$ for some x_0 in the case $(A10)$, then $\max u = u(z_0) > R$ and

$$0 \ge u''(z_0) = \tilde{f}(z_0, u(z_0)) > 0.$$

This contradiction implies $u(\cdot) \le R$ and consequently, u is the desired solution also in the case $(A10)$. □

Finally, we consider

$$(4) \qquad \begin{aligned} &u'' + f(x,u) = 0, \\ &u(0) = u(\pi) = 0, \end{aligned}$$

where $f\colon [0,1] \times [0,\infty) \to \mathbb{R}$ is continuous and satisfying $f(\cdot, 0) = 0$.

Theorem 8. *Suppose that there are constants* $c_1 > 0$, $0 \le c_2 < 1$ *and* $r > 0$ *such that*

$$0 \le f(x,u) \le c_1 + c_2 u \quad \forall\,(x,u) \in [0,\pi] \times [0,\infty)$$
$$f(x,u) \ge u \quad \forall\, x \in [0,\pi], \quad \forall\, u \in [0,r].$$

Then (4) *has a nonzero nonnegative solution.*

Proof: We apply Corollary 3. Clearly $\ker L = \{0\}$ and the inverse of $Lu = -u''$ has the form

$$(L^{-1}z)(x) = \int_0^\pi G(x,s)z(s)\,ds,$$

where

$$G(x,s) = \begin{cases} (\pi - s)x/\pi & 0 \le x \le s \le \pi \\ (\pi - x)s/\pi & 0 \le s \le z \le \pi. \end{cases}$$

To show the first assumption of Corollary 3, we consider

$$\begin{aligned} &u'' + \lambda(f(s,u) + \epsilon u) = 0, \quad \lambda < 1, \quad \frac{1-c_2}{2} > \epsilon > 0 \\ &u(0) = u(\pi) = 0, \quad u(\cdot) \ge 0, \end{aligned} \tag{5}$$

with $|u| = R$. Let $|\cdot|_{L_2}$ denote the norm of $L_2(0,1)$. The proof of [4, Theorem 7] gives a constant $\overline{K}_1 > 0$ such that any solution of (5) satisfies $|u|_{L_2} \le \overline{K}_1$. Then we have

$$\begin{aligned} |u(x)| &\le \int_0^\pi G(x,s)\big(f(s,u(s)) + \epsilon u(s)\big)\,ds \le \pi \int_0^\pi \big(c_1 + (c_2+1)u(s)\big)\,ds \\ &\le \pi^2 c_1 + 2\pi \int_0^\pi u(s)\,ds \le \pi^2 c_1 + 2\pi\sqrt{\pi}|u|_{L_2} \le \pi^2 c_1 + 2\pi\sqrt{\pi}\,\overline{K}_1 = \overline{K}_2. \end{aligned}$$

So $|u|$ is bounded by the constant $\overline{K}_2$ uniformly for any solution of (5). By taking $R > \max\{\overline{K}_2, r\}$, the first assumption of Corollary 3 holds.

To show the second assumption of Corollary 3, we consider

$$\begin{aligned} &u'' + \lambda(f(s,u) + \epsilon u) = 0, \quad \lambda > 1, \quad \frac{1-c_2}{2} > \epsilon > 0 \\ &u(0) = u(\pi) = 0, \quad u(\cdot) \ge 0, \end{aligned}$$

with $|u| = r$. Then we have

$$-\int_0^\pi u(s)\sin s\,ds = \int_0^\pi u''(s)\sin s\,ds = -\int_0^\pi \lambda(f(s,u(s)) + \epsilon u(s))\sin s\,ds.$$

Hence

$$
\begin{aligned}
0 &= \int_0^\pi \big(-u(s) + \lambda(f(s,u(s)) + \epsilon u(s))\big) \sin s \, ds \\
&\geq \int_0^\pi \big(-u(s) + u(s) + \epsilon u(s)\big) \sin s \, ds = \epsilon \int_0^\pi u(s) \sin s \, ds.
\end{aligned}
$$

This contradiction implies the validity of the second assumption. The third one is clearly satisfied. □

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