Representation theorem for convex effect algebras

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Abstract. Effect algebras have important applications in the foundations of quantum mechanics and in fuzzy probability theory. An effect algebra that possesses a convex structure is called a convex effect algebra. Our main result shows that any convex effect algebra admits a representation as a generating initial interval of an ordered linear space. This result is analogous to a classical representation theorem for convex structures due to M.H. Stone.

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1. Introduction

An algebraic structure called an effect algebra has recently been introduced for investigations in the foundations of quantum mechanics ([3], [13], [14]). Equivalent structures called D-posets and generalized orthoalgebras have also been studied ([8], [10], [11], [15], [20], [21]). Moreover, effect algebras play a fundamental role in recent investigations of fuzzy probability theory ([1], [2], [4], [5], [18]). In the quantum mechanical framework, the elements of an effect algebra P represent quantum effects and these are important for quantum statistics and quantum measurement theory ([3], [6], [7]). One may think of a quantum effect as an elementary yes-no measurement that may be unsharp or imprecise. In the fuzzy probability setting, elements of P represent fuzzy events which are statistical events that may not be crisp or sharp. The quantum effects and fuzzy events are then used to construct general quantum measurements (or observables) and fuzzy random variables. The structure of an effect algebra is given by a partially defined binary operation $\oplus$ that is used to form a combination $a \oplus b$ of effects $a, b \in P$. The element $a \oplus b$ represents a statistical combination of a and b whose probability of occurrence equals the sum of the probabilities that a and b occur individually.

The common examples of effect algebras that are employed in practice also possess a convex structure. For example, if a is a quantum effect and $\lambda \in [0, 1]$, then λa represents the effect a attenuated by a factor of λ. A similar interpretation is given for fuzzy events. Then $\lambda a \oplus (1 - \lambda)b$ is a generalized convex combination that can be constructed in practice. Due to the operational significance of such combinations it seems desirable to investigate effect algebras that possess an additional convex structure and we call them convex effect algebras.

General convex structures have important applications to studies in color vision, decision theory, operational quantum mechanics and economics ([12], [16], [17], [25], [26]). A classical representation theorem of M.H. Stone ([16], [24]) has sometimes been useful in these studies. This theorem states that certain convex structures can be represented as convex subsets of a real linear space. In this paper, we present an analogous theorem for convex effect algebras. Although there are some similarities between our proof and that of Stone, a much more delicate argument must be used because we have to preserve the effect algebra structure as well as the convex structure. Also, since our structure is richer than a convex structure alone, we obtain a stronger theorem. In Stone's theorem, a convex structure is represented by a convex base of a positive cone K that generates an ordered linear space (V, K). Our theorem states that a convex effect algebra can be represented by an initial interval $[\theta, u]$ that generates an ordered linear space (V, K). An interval $[\theta, u]$ in (V, K) has a natural effect algebra structure and we call $[\theta, u]$ a linear effect algebra. A linear effect algebra is a special case of an interval effect algebra which has recently been investigated ([14]).

2. Definitions and basic results

An **effect algebra** is an algebraic system $(P, 0, 1, \oplus)$ where $0, 1$ are distinct elements of P and $\oplus$ is a partial binary operation on P that satisfies the following conditions.

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.

(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(E3) For every $a \in P$ there exists a unique $a' \in P$ such that $a \oplus a'$ is defined and $a \oplus a' = 1$.

(E4) If $a \oplus 1$ is defined, then $a = 0$.

We define $a \leq b$ if there exists a $c \in P$ such that $a \oplus c = b$. It can be shown that $(P, 0, 1, \leq)$ is a bounded poset and $a \oplus b$ is defined if and only if $a \leq b'$ ([11], [13]). If $a \leq b'$, we write $a \perp b$. An important property of an effect algebra is the **cancellation law** which states that $a \oplus b = a \oplus c$ implies $b = c$. Moreover, it can be shown that $a'' = a$ and that $a \leq b$ implies $b' \leq a'$ for every $a, b \in P$ ([11], [13]).

An effect algebra P is **convex** if for every $a \in P$ and $\lambda \in [0,1] \subseteq \mathbb{R}$ there exists an element $\lambda a \in P$ such that the following conditions hold.

(C1) If $\alpha, \beta \in [0,1]$ and $a \in P$, then $\alpha(\beta a) = (\alpha\beta)a$.

(C2) If $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$ and $a \in P$, then $\alpha a \perp \beta a$ and $(\alpha + \beta)a = \alpha a \oplus \beta a$.

(C3) If $a, b \in P$ with $a \perp b$ and $\lambda \in [0,1]$, then $\lambda a \perp \lambda b$ and $\lambda(a \oplus b) = \lambda a \oplus \lambda b$.

(C4) If $a \in P$, then $1a = a$.

A map $(\lambda, a) \mapsto \lambda a$ that satisfies (C1)–(C4) is an example of a bimorphism from $[0,1] \times P$ into P ([10]) and we call this map a **convex structure** on P. Notice that $0a = 0$ for every $a \in P$. Indeed, by (C2) and (C4) we have

$$0a \oplus a = (0 + 1)a = 1a = a = 0 \oplus a$$

so by the cancellation law $0a = 0$.

The effect algebras that arise in practice are usually convex. For example, let H be a complex Hilbert space and let $\mathcal{E}(H)$ be the set of operators on H that satisfy $0 \le A \le I$ where we are using the usual ordering of bounded operators. For $A, B \in \mathcal{E}(H)$, we write $A \perp B$ if $A + B \in \mathcal{E}(H)$ and in this case we define $A \oplus B = A + B$. It is clear that $(\mathcal{E}(H), 0, I, \oplus)$ is an effect algebra and we call $\mathcal{E}(H)$ a **Hilbert space effect algebra**. Hilbert space effect algebras are important in foundational studies of quantum mechanics ([6], [7], [9], [19], [22], [23]). For $\lambda \in [0,1]$ and $A \in \mathcal{E}(H)$, λA is the usual scalar multiplication for operators. This gives a convex structure on $\mathcal{E}(H)$ so $\mathcal{E}(H)$ becomes a convex effect algebra. For another example, let $(\Omega, \mathcal{A})$ be a measurable space and let $\mathcal{E}(\Omega, \mathcal{A})$ be the set of measurable functions on Ω with values in $[0,1]$. If we define $\oplus$ and scalar multiplication λf analogously as in the previous example, we see that $\mathcal{E}(\Omega, \mathcal{A})$ is a convex effect algebra. The elements of $\mathcal{E}(\Omega, \mathcal{A})$ are called **fuzzy events** and they are the basic concepts in fuzzy probability theory ([1], [2], [4], [5], [18]).

We now consider a more general type of convex effect algebra called a linear effect algebra. It is no accident that the previous two examples are linear effect algebras because we shall show that any convex effect algebra is equivalent to a linear effect algebra. A linear effect algebra is an initial interval in the positive cone of an ordered linear space. We now give the precise definitions.

Let V be a real linear space with zero θ. A subset K of V is a **positive cone** if $\mathbb{R}^+K \subseteq K$, $K + K \subseteq K$ and $K \cap (-K) = \{\theta\}$. For $x, y \in V$ we define $x \le_K y$ if $y - x \in K$. Then $\le_K$ is a partial order on V and we call (V, K) an **ordered linear space** with positive cone K. We say that K is **generating** if $V = K - K$. Let $u \in K$ with $u \ne \theta$ and form the interval

$$[\theta, u] = \{x \in K\colon\ x \le_K u\}.$$

For $x, y \in [\theta, u]$ we write $x \perp y$ if $x + y \le_K u$ and in this case we define $x \oplus y = x + y$. It is clear that $([\theta, u], \theta, u, \oplus)$ is an effect algebra with $x' = u - x$ for every $x \in [\theta, u]$. This is an example of an interval effect algebra ([14]). It is also easy to check that $[\theta, u]$ is a convex subset of K. It follows that if $\lambda \in [0,1]$ and $x \in [\theta, u]$, then

$$\lambda x = \lambda x + (1 - \lambda)\theta \in [\theta, u].$$

A straightforward verification shows that $(\lambda, x) \mapsto \lambda x$ is a convex structure on $[\theta, u]$ so that $[\theta, u]$ is a convex effect algebra which we call a **linear effect algebra**. We say that $[\theta, u]$ **generates** K if $K = \mathbb{R}^+[\theta, u]$ and we say that $[\theta, u]$ **generates** V if $[\theta, u]$ generates K and K generates V. Two ordered linear spaces (V_1, K_1) and (V_2, K_2) are **order isomorphic** if there exists a linear bijection $T\colon V_1 \to V_2$ such that $T(K_1) = K_2$.

Because of the associative law (E2), we do not have to write parentheses for orthogonal sums of three or more elements. If a is an element of an effect algebra and $a \oplus a \oplus \cdots \oplus a$ is defined (n summands), then we denote this element by na. Our first result summarizes some basic properties of a convex effect algebra.

Lemma 2.1. *Let P be a convex effect algebra.* (i) *If $a \le b$, then $\lambda a \le \lambda b$ for every $\lambda \in [0,1]$.* (ii) *If $0 \le \alpha \le \beta \le 1$, then $\alpha a \le \beta a$ for every $a \in P$.* (iii) *If $\alpha, \beta \in [0,1]$ with $\alpha + \beta \le 1$, then $\alpha a \perp \beta b$ for every $a, b \in P$.* (iv) *For $\lambda \in (0,1)$, $\lambda a = 0$ if and only if $a = 0$.* (v) *If na is defined for $n \in \mathbb{N}$ and $0 \le \lambda \le 1/n$, then $\lambda(na) = (\lambda n)a$.* (vi) *If na is defined for $n \in \mathbb{N}$ and $\lambda \in [0,1]$, then $n(\lambda a)$ is defined and $n(\lambda a) = \lambda(na)$.* (vii) *If $\lambda \in (0,1]$ and $\lambda a = \lambda b$, then $a = b$.* (viii) *If $a \ne 0$, $\alpha, \beta \in [0,1]$ and $\alpha a = \beta a$, then $\alpha = \beta$.*

Proof: (i) Since $a \oplus c = b$ for some $c \in P$, we have by (C3) that

$$\lambda a \oplus \lambda c = \lambda(a \oplus c) = \lambda b.$$

Hence, $\lambda a \le \lambda b$.

(ii) Applying (C2) gives

$$\beta a = [\alpha + (\beta - \alpha)]\, a = \alpha a \oplus (\beta - \alpha)a.$$

Hence, $\alpha a \le \beta a$.

(iii) By (C2), $\alpha 1 \perp \beta 1$. Applying (i), we have $\alpha a \le \alpha 1$ and $\beta b \le \beta 1$. We conclude that

$$\alpha a \le \alpha 1 \le (\beta 1)' \le (\beta b)'.$$

Hence, $\alpha a \perp \beta b$.

(iv) Since $\lambda \le 1$, by (ii) we have $\lambda 0 \le 10 = 0$ so $\lambda 0 = 0$. Conversely, suppose $\lambda a = 0$ and let n be the largest integer such that $n\lambda \le 1$. Then $(n+1)\lambda > 1$ so that $1 - n\lambda < \lambda$. Since

$$(n\lambda)a = (\lambda + \cdots + \lambda)a = n(\lambda a) = 0$$

and by (ii), $(1 - n\lambda)a = 0$, we have

$$a = (n\lambda)a \oplus (1 - n\lambda)a = 0.$$

(v) Since $na = a \oplus \cdots \oplus a$ (n summands) we have by (C3) that

$$\lambda(na) = \lambda(a \oplus \cdots \oplus a) = (\lambda + \cdots + \lambda)a = (n\lambda)a.$$

(vi) We proceed by induction on n. The result clearly holds for $n = 1$. Assume the result holds for $n \in \mathbb{N}$ and that $(n+1)a$ is defined. Then na is defined so $n(\lambda a)$ is defined and $n(\lambda a) = \lambda(na)$. Since $na \perp a$, we have by (C3) that $\lambda(na) \perp \lambda a$. Hence, $n(\lambda a) \perp \lambda a$ so that $(n+1)(\lambda a)$ is defined and applying (C3) gives

$$(n+1)(\lambda a) = n(\lambda a) \oplus \lambda a = \lambda(na) \oplus \lambda a = \lambda(na \oplus a) = \lambda\left((n+1)a\right).$$

The result follows by induction.

(vii) Let $m/n \in (0,1]$ be rational and suppose that $(m/n)a = (m/n)b$. Then

$$\frac{1}{n}\,a = \frac{1}{m}\left(\frac{m}{n}\right)a = \frac{1}{m}\left(\frac{m}{n}\right)b = \frac{1}{n}\,b.$$

Hence,

$$a = n\left(\frac{1}{n}\,a\right) = n\left(\frac{1}{n}\,b\right) = b.$$

Thus, the result holds if λ is rational. Suppose that $\lambda \in (0,1]$ is irrational and let $0 < r < \lambda$ be rational. Letting $\alpha = r/\lambda$ we have that $\alpha \in (0,1)$. Then if $\lambda a = \lambda b$ we conclude that

$$ra = (\alpha\lambda)a = \alpha(\lambda a) = \alpha(\lambda b) = (\alpha\lambda)b = rb.$$

Since r is rational, $a = b$.

(viii) Suppose that $\beta > \alpha$. Then

$$(\beta - \alpha)a \oplus \alpha a = \beta a = \alpha a$$

and by the cancellation law, $(\beta - \alpha)a = 0$. Applying (iv) we conclude that $a = 0$, which is a contradiction. Hence, $\beta \leq \alpha$ and by symmetry $\alpha \leq \beta$. $\square$

It follows from Lemma 2.1 (iii) that a convex effect algebra P is "convex" in the following sense. If $\lambda \in [0,1]$ and $a, b \in P$, then $\lambda a \oplus (1-\lambda)b$ is defined and hence is an element of P.

If P and Q are effect algebras, a map ϕ: $P \to Q$ is **additive** if $a \perp b$ implies that $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. An additive map ϕ that satisfies $\phi(1) = 1$ is called a **morphism**. A morphism ϕ: $P \to Q$ for which $\phi(a) \perp \phi(b)$ implies that $a \perp b$ is called a **monomorphism**. A surjective monomorphism is called an **isomorphism**. It is easy to show that if ϕ is an isomorphism, then ϕ is injective and ϕ^{-1} is an isomorphism. If P and Q are convex effect algebras, a morphism ϕ: $P \to Q$ is called an **affine morphism** if $\phi(\lambda a) = \lambda\phi(a)$ for every $\lambda \in [0,1]$, $a \in P$. It follows from Lemma 2.1 (iii) that an affine morphism preserves convex combinations in the sense that if $\lambda \in [0,1]$ and $a, b \in P$, then

$$\phi\left(\lambda a \oplus (1-\lambda)b\right) = \lambda\phi(a) \oplus (1-\lambda)\phi(b).$$

An isomorphism ϕ: $P \to Q$ that is affine is called an **affine isomorphism** and if such a ϕ exists, we say that P and Q are **affinely isomorphic**. Notice that if ϕ: $P \to Q$ is an affine isomorphism, then ϕ^{-1}: $Q \to P$ is also an affine isomorphism. Indeed, let $\lambda \in [0,1]$ and $b \in Q$. Then there exists an $a \in P$ such that $\phi(a) = b$ so that $\phi(\lambda a) = \lambda b$. Hence,

$$\phi^{-1}(\lambda b) = \lambda a = \lambda\phi^{-1}(b).$$

Lemma 2.2. *If P is a convex effect algebra, Q is an effect algebra and $\phi\colon P \to Q$ is an isomorphism, then there exists a unique convex structure on Q such that ϕ is an affine isomorphism.*

Proof: For $\lambda \in [0,1]$, $b \in Q$ define $\lambda b = \phi(\lambda a)$ where $\phi(a) = b$. Then $\phi(\lambda a) = \lambda\phi(a)$ so all we need to show is that $(\lambda, b) \mapsto \lambda b$ is a convex structure on Q. To verify (C1), let $\alpha, \beta \in [0,1]$ and $b \in Q$. Then $\beta b = \phi(\beta a)$ where $\phi(a) = b$ and $\alpha(\beta b) = \phi\left(\alpha(\beta a)\right)$. Hence,

$$\alpha(\beta b) = \phi\left(\alpha(\beta a)\right) = \phi\left((\alpha\beta)a\right) = (\alpha\beta)\phi(a) = (\alpha\beta)b.$$

To verify (C2), let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$ and let $b \in Q$. Then $\alpha b = \phi(\alpha a)$ and $\beta b = \phi(\beta a)$ where $\phi(a) = b$. Since $\alpha a \perp \beta a$, we have that $\phi(\alpha a) \perp \phi(\beta a)$. Hence, $\alpha b \perp \beta b$ and

$$\begin{aligned}\alpha b \oplus \beta b &= \phi(\alpha a) \oplus \phi(\beta a) = \phi(\alpha a \oplus \beta b) = \phi\left((\alpha+\beta)a\right)\\ &= (\alpha+\beta)\phi(a) = (\alpha+\beta)b.\end{aligned}$$

To verify (C3), let $c, d \in Q$ with $c \perp d$ and let $\lambda \in [0,1]$. We then have $\lambda c = \phi(\lambda a)$, $\lambda d = \phi(\lambda b)$ where $\phi(a) = c$ and $\phi(b) = d$. Since ϕ is an isomorphism, $a \perp b$ and hence $\lambda a \perp \lambda b$ and $\lambda(a \oplus b) = \lambda a \oplus \lambda b$. Thus, $\lambda c \perp \lambda d$ and

$$\begin{aligned}\lambda c \oplus \lambda d &= \phi(\lambda a) \oplus \phi(\lambda b) = \phi(\lambda a \oplus \lambda b) = \phi\left(\lambda(a \oplus b)\right)\\ &= \lambda\phi(a \oplus b) = \lambda\left[\phi(a) \oplus \phi(b)\right] = \lambda(c \oplus d).\end{aligned}$$

Finally, (C4) holds because for $b = \phi(a) \in Q$, we have

$$1b = \phi(1a) = \phi(a) = b.$$

To prove uniqueness, suppose that $(\lambda, b) \mapsto \lambda \cdot b$ is a convex structure on Q for which $\phi\colon P \to Q$ is an affine isomorphism. Then if $\phi(a) = b$, we have

$$\lambda \cdot b = \phi(\lambda a) = \lambda b.$$

□

3. Representation theorem

We now prove a representation theorem for convex effect algebras. This theorem is analogous to a representation theorem for convex structures due to M.H. Stone ([16], [24]). The beginning of the proof is similar to that of Stone's theorem but a more delicate argument must be used later because we have to preserve the effect algebra structure as well as the convex structure.

Theorem 3.1. *If $(P, 0, 1, \oplus)$ is a convex effect algebra, then P is affinely isomorphic to a linear effect algebra $[\theta, u]$ that generates an ordered linear space (V, K) and the effect algebra order $\leq$ on $[\theta, u]$ coincides with linear space order $\leq_K$ restricted to $[\theta, u]$. Moreover, (V, K) is unique in the sense that if P is affinely isomorphic to a linear effect algebra $[\theta_1, u_1]$ that generates (V_1, K_1), then (V_1, K_1) is order isomorphic (V, K).*

Proof: Define the set $\widehat{K} \subseteq \mathbb{R} \times P$ by

$$\widehat{K} = \{(\alpha, a)\colon \alpha \geq 1, a \in P\}.$$

For $(\alpha, a), (\beta, b) \in \widehat{K}$, define the relation $\sim$ on $\widehat{K}$ by $(\alpha, a) \sim (\beta, b)$ if $\beta^{-1}a = \alpha^{-1}b$. Clearly, $\sim$ is reflexive and symmetric. To prove transitivity, suppose that $(\alpha, a) \sim (\beta, b)$ and $(\beta, b) \sim (\gamma, c)$. Then $\beta^{-1}a = \alpha^{-1}b$ and $\gamma^{-1}b = \beta^{-1}c$. Hence,

$$\beta^{-1}(\gamma^{-1}a) = (\gamma^{-1}\alpha^{-1})b = \beta^{-1}(\alpha^{-1}c)$$

and by Lemma 2.1 (vii) we have $\gamma^{-1}a = \alpha^{-1}c$. Thus, $(\alpha, a) \sim (\gamma, c)$ so $\sim$ is an equivalence relation on $\widehat{K}$. Denote the equivalence class containing (α, a) by $[(\alpha, a)]$ and let $\widetilde{K} = \left\{[(\alpha, a)]\colon (\alpha, a) \in \widehat{K}\right\}$. For $\beta \geq 0$, define

$$\beta\,[(\alpha, a)] = \begin{cases} [(\beta\alpha, a)] & \text{if } \beta \geq 1, \\ [(\alpha, \beta a)] & \text{if } \beta \leq 1. \end{cases}$$

To show that this operation is well defined, suppose that $(\alpha_1, a_1) \sim (\alpha, a)$. If $\beta \geq 1$, then $(\beta\alpha_1, a_1) \sim (\beta\alpha, a)$ because $\alpha^{-1}a_1 = \alpha_1^{-1}a$ so that $(\beta^{-1}\alpha^{-1})a_1 = (\beta^{-1}\alpha_1^{-1})a$. If $\beta \leq 1$, then $(\alpha_1, \beta a_1) \sim (\alpha, \beta a)$ because $\alpha^{-1}(\beta a_1) = \alpha_1^{-1}(\beta a)$.

We also define an operation $+$ on $\widetilde{K}$ by

$$[(\alpha, a)] + [(\beta, b)] = \left[\left(\alpha + \beta, \frac{\alpha}{\alpha+\beta}\, a \oplus \frac{\beta}{\alpha+\beta}\, b\right)\right].$$

To show that this operation is well defined, suppose that $(\alpha_1, a_1) \sim (\alpha, a)$ and $(\beta_1, b_1) \sim (\beta, b)$. Then $\alpha^{-1}a_1 = \alpha_1^{-1}a$ and $\beta^{-1}b_1 = \beta_1^{-1}b$. Hence,

$$\begin{aligned}
\frac{1}{\alpha+\beta}&\left(\frac{\alpha_1}{\alpha_1+\beta_1}\, a_1 \oplus \frac{\beta_1}{\alpha_1+\beta_1}\, b_1\right) \\
&= \left(\frac{\alpha_1}{\alpha_1+\beta_1}\right)\left(\frac{\alpha}{\alpha+\beta}\right)\alpha^{-1}a_1 \oplus \left(\frac{\beta_1}{\alpha_1+\beta_1}\right)\left(\frac{\beta}{\alpha+\beta}\right)\beta^{-1}b_1 \\
&= \left(\frac{\alpha_1}{\alpha_1+\beta_1}\right)\left(\frac{\alpha}{\alpha+\beta}\right)\alpha_1^{-1}a \oplus \left(\frac{\beta_1}{\alpha_1+\beta_1}\right)\left(\frac{\beta}{\alpha+\beta}\right)\beta_1^{-1}b \\
&= \frac{1}{\alpha_1+\beta_1}\left(\frac{\alpha}{\alpha+\beta}\, a \oplus \frac{\beta}{\alpha+\beta}\, b\right).
\end{aligned}$$

It follows that

$$\left(\alpha_1+\beta_1, \frac{\alpha_1}{\alpha_1+\beta_1}\, a_1 \oplus \frac{\beta_1}{\alpha_1+\beta_1}\, b_1\right) \sim \left(\alpha+\beta, \frac{\alpha}{\alpha+\beta}\, a \oplus \frac{\beta}{\alpha+\beta}\, b\right).$$

We now show that $\widetilde{K}$ forms an abstract cone with a zero ([16], [24]). Notice that $(\alpha, a) \sim (1,0)$ if and only if $a=0$ and let $\widetilde{\theta} = [(1,0)]$. To show that $\widetilde{K}$ is an abstract cone with zero $\widetilde{\theta}$, we must verify that the following conditions hold for every $X, Y, Z \in \widetilde{K}$ and $\alpha, \beta \geq 0$.

(1) $X+Y=Y+X$
(2) $X+\widetilde{\theta}=X$
(3) $X+(Y+Z)=(X+Y)+Z$
(4) If $X+Y=X+Z$, then $Y=Z$
(5) $\alpha(X+Y)=\alpha X+\alpha Y$
(6) $(\alpha+\beta)X=\alpha X+\beta X$
(7) $\alpha(\beta X)=(\alpha\beta)X$
(8) $1X=X$

It is clear that (1) and (2) hold. To show that (3) holds, suppose that $X=[(\alpha,a)]$, $Y=[(\beta,b)]$ and $Z=[(\gamma,c)]$. We then have

$$\begin{aligned} X+(Y+Z) &= [(\alpha,a)] + ([\beta,b]+[\gamma,c]) \\ &= [(\alpha,a)] + \left[\left(\beta+\gamma, \frac{\beta}{\beta+\gamma}\, b \oplus \frac{\gamma}{\beta+\gamma}\, c\right)\right] \\ &= \left[\left(\alpha+\beta+\gamma, \frac{\alpha}{\alpha+\beta+\gamma}\, a \oplus \left(\frac{\beta}{\alpha+\beta+\gamma}\, b \oplus \frac{\gamma}{\alpha+\beta+\gamma}\, c\right)\right)\right] \\ &= \left[\left(\alpha+\beta+\gamma, \left(\frac{\alpha}{\alpha+\beta+\gamma}\, a \oplus \frac{\beta}{\alpha+\beta+\gamma}\, b\right) \oplus \frac{\gamma}{\alpha+\beta+\gamma}\, c\right)\right] \\ &= \left[\left(\alpha+\beta, \frac{\alpha}{\alpha+\beta}\, a \oplus \frac{\beta}{\alpha+\beta}\, b\right)\right] + [(\gamma,c)] \\ &= (X+Y)+Z. \end{aligned}$$

To show that (4) holds, if $X+Y=X+Z$, we have

$$\left[\left(\alpha+\beta, \frac{\alpha}{\alpha+\beta}\, a \oplus \frac{\beta}{\alpha+\beta}\, b\right)\right] = \left[\left(\alpha+\gamma, \frac{\alpha}{\alpha+\gamma}\, a \oplus \frac{\gamma}{\alpha+\gamma}\, c\right)\right].$$

We conclude that

$$\frac{\alpha}{(\alpha+\gamma)(\alpha+\beta)}\, a \oplus \frac{\beta}{(\alpha+\gamma)(\alpha+\beta)}\, b = \frac{\alpha}{(\alpha+\beta)(\alpha+\gamma)}\, a \oplus \frac{\gamma}{(\alpha+\beta)(\alpha+\gamma)}\, c.$$

It follows from the cancellation law that

$$\frac{\gamma^{-1}}{(\alpha+\gamma)(\alpha+\beta)}\, b = \frac{\beta^{-1}}{(\alpha+\beta)(\alpha+\gamma)}\, c.$$

By Lemma 2.1 (vii) we obtain $\gamma^{-1}b = \beta^{-1}c$ so that

$$Y = [(\beta, b)] = [(\gamma, c)] = Z.$$

It is clear that (5) holds. To verify (6), we have three cases.

Case 1. If $\beta, \gamma \geq 1$, we have

$$\begin{aligned}\beta X + \gamma X &= \beta\,[(\alpha, a)] + \gamma\,[(\alpha, a)] = [(\beta\alpha, a)] + [(\gamma\alpha, a)] \\ &= \left[\left(\beta\alpha + \gamma\alpha, \frac{\beta}{\beta+\gamma}\, a \oplus \frac{\gamma}{\beta+\gamma}\, a\right)\right] = [(\beta\alpha + \gamma\alpha, a)] \\ &= (\beta+\gamma)\,[(\alpha, a)] = (\beta+\gamma)X.\end{aligned}$$

Case 2. If $\beta \geq 1$, $\gamma \leq 1$, we have

$$\begin{aligned}\beta X + \gamma X &= [(\beta\alpha, a)] + [(\alpha, \gamma a)] = \left[\left(\beta\alpha + \alpha, \frac{\beta}{\beta+1}\, a \oplus \frac{1}{\beta+1}\, \gamma a\right)\right] \\ &= \left[\left(\beta\alpha + \alpha, \frac{\beta+\gamma}{\beta+1}\, a\right)\right] = [(\beta\alpha + \gamma\alpha, a)] \\ &= (\beta+\gamma)\,[(\alpha, a)] = (\beta+\gamma)X.\end{aligned}$$

Case 3. If $\beta, \gamma \leq 1$ and $\beta + \gamma \leq 1$, we have

$$\begin{aligned}\beta X + \gamma X &= [(\alpha, \beta a)] + [(\alpha, \gamma a)] = \left[\left(2\alpha, \frac{1}{2}\beta a \oplus \frac{1}{2}\gamma a\right)\right] \\ &= \left[\left(2\alpha, \frac{1}{2}(\beta+\gamma)a\right)\right] = [(\alpha, (\beta+\gamma)a)] \\ &= (\beta+\gamma)\,[(\alpha, a)] = (\beta+\gamma)X.\end{aligned}$$

If $\beta, \gamma \leq 1$ and $\beta + \gamma \geq 1$, we have

$$\begin{aligned}\beta X + \gamma X &= \left[\left(2\alpha, \frac{1}{2}(\beta+\gamma)a\right)\right] = [((\beta+\gamma)\alpha, a)] \\ &= (\beta+\gamma)\,[(\alpha, a)] = (\beta+\gamma)X.\end{aligned}$$

Finally, it is clear that (7) and (8) hold.

We next show that $\widetilde{K}$ can be extended to a real linear space. Let

$$V_0 = \left\{(X, Y)\colon\ X, Y \in \widetilde{K}\right\}.$$

Define the relation $\approx$ on V_0 by $(X_1, Y_1) \approx (X, Y)$ if $X_1 + Y = X + Y_1$. It is clear that $\approx$ is reflexive and symmetric. To prove transitivity, suppose that $(X_1, Y_1) \approx$

(X,Y) and $(X,Y) \approx (X_2,Y_2)$. Then $X_1+Y = X+Y_1$ and $X+Y_2 = X_2+Y$. Hence,

$$X_1+Y_2+Y = X+Y_1+Y_2 = X_2+Y_1+Y$$

and it follows from (4) that $X_1+Y_2 = X_2+Y_1$. Thus, $(X_1,Y_1) \approx (X_2,Y_2)$ and $\approx$ is an equivalence relation on V_0. Denote the equivalence class containing (X,Y) by $[(X,Y)]$ and let

$$V = \{[(X,Y)] \colon (X,Y) \in V_0\}\,.$$

If $(X,Y) \approx (\widetilde{\theta},\widetilde{\theta})$, then

$$X = X+\theta = Y+\theta = Y$$

so that

$$\left[(\widetilde{\theta},\widetilde{\theta})\right] = \left\{(X,X)\colon\ X \in \widetilde{K}\right\}.$$

Define addition on V by

$$[(X,Y)] + [(X_1,Y_1)] = [(X+X_1, Y+Y_1)]\,.$$

To show that $+$ is well defined, suppose that $(X_2,Y_2) \approx (X,Y)$ and $(X_3,Y_3) \approx (X_1,Y_1)$. Then $X_2+Y = X+Y_2$ and $X_3+Y_1 = X_1+Y_3$. Hence,

$$X_2+X_3+Y+Y_1 = X+X_1+Y_2+Y_3$$

so that

$$(X_2+X_3, Y_2+Y_3) \approx (X+X_1, Y+Y_1).$$

It is now easy to verify that $(V,+)$ is an abelian group with zero $\theta = \left[(\widetilde{\theta},\widetilde{\theta})\right]$. Define a scalar multiplication by real numbers as follows. If $\lambda \geq 0$, then $\lambda\,[(X,Y)] = [(\lambda X, \lambda Y)]$ and if $\lambda < 0$, then $\lambda\,[(X,Y)] = [((-\lambda)Y, (-\lambda)X)]$. It is straightforward to show that this operation is well defined and using Properties (1)–(8) that V is a real linear space.

We now define $K \subseteq V$ by

$$K = \left\{\left[(X,\widetilde{\theta})\right] \colon\ X \in \widetilde{K}\right\}.$$

To show that K is a positive cone in V it is clear that $\mathbb{R}^+K \subseteq K$ and $K+K \subseteq K$. To show that $K \cap (-K) = \{0\}$, suppose $[(X,Y)] \in K \cap (-K)$. Then $[(X,Y)] \in K$ and

$$[(Y,X)] = -\,[(X,Y)] \in K.$$

Hence, there exist $Z, Z_1 \in K$ such that $[(X,Y)] = \left[(Z,\widetilde{\theta})\right]$ and $[(Y,X)] = \left[(Z_1,\widetilde{\theta})\right]$. It follows that $X = Z+Y$ and $Y = Z_1+X$. Then $X = Z+Z_1+X$ and applying (4) gives $Z + Z_1 = \widetilde{\theta}$. If $Z = [(\alpha, a)]$ and $Z_1 = [(\alpha_1, a_1)]$ we conclude that

$$\left(\alpha + \alpha_1, \frac{\alpha}{\alpha + \alpha_1}\, a \oplus \frac{\alpha_1}{\alpha + \alpha_1}\, a_1\right) \sim (1, 0).$$

Hence,

$$\frac{\alpha}{\alpha + \alpha_1}\, a \oplus \frac{\alpha_1}{\alpha + \alpha_1}\, a_1 = 0.$$

It follows from Lemma 2.1 (iv) that $a = a_1 = 0$. Thus, $Z = \widetilde{\theta}$ so $[(X, Y)] = \theta$. We conclude that (V, K) is an ordered linear space. Since any $[(X, Y)] \in V$ has the form

$$[(X, Y)] = \left[(X, \widetilde{\theta})\right] + \left[(\widetilde{\theta}, Y)\right] = \left[(X, \widetilde{\theta})\right] - \left[Y, \widetilde{\theta})\right],$$

it follows that $V = K - K$ so K generates V.

Define $u \in K$ by $u = \left[\left([(1,1)], \widetilde{\theta}\right)\right]$ and form the interval $[\theta, u] \subseteq K$. We first show that $u \neq \theta$. If $u = \theta$, then $\left([(1,1)], \widetilde{\theta}\right) \approx (\widetilde{\theta}, \widetilde{\theta})$ so that $[(1,1)] = \widetilde{\theta} = [(1,0)]$. Hence, $(1,1) \sim (1,0)$ and $1 = 0$ which is a contradiction. Thus, $[\theta, u]$ is a linear effect algebra under the induced partial operation $\oplus$. We next show that $\mathbb{R}^+[\theta, u] = K$ so that $[\theta, u]$ generates (V, K). It is clear that $\mathbb{R}^+[\theta, u] \subseteq K$. For the opposite inclusion, suppose that $\left[(X, \widetilde{\theta})\right] \in K$ where $X = [(\alpha, a)]$. Then

$$\alpha^{-1}\left[(X, \widetilde{\theta})\right] = \left[(\alpha^{-1}X, \widetilde{\theta})\right] = \left[\left([\alpha, \alpha^{-1}a], \widetilde{\theta}\right)\right] = \left[\left([(1,a)], \widetilde{\theta}\right)\right].$$

Hence,

$$\begin{aligned}
\alpha^{-1}\left[(X, \widetilde{\theta})\right] + \left[\left([(1,a')], \widetilde{\theta}\right)\right] &= \left[\left([(1,a)], \widetilde{\theta}\right)\right] + \left[\left([(1,a')], \widetilde{\theta}\right)\right] \\
&= \left[\left([(1,a)] + [(1,a')], \widetilde{\theta}\right)\right] \\
&= \left[\left(\left[\left(2, \frac{1}{2}\, a \oplus \frac{1}{2}\, a'\right)\right], \widetilde{\theta}\right)\right] \\
&= \left[\left(\left[2, \frac{1}{2}\, 1\right], \widetilde{\theta}\right)\right] = \left[\left([(1,1)], \widetilde{\theta}\right)\right] = u.
\end{aligned}$$

It follows that $\alpha^{-1}\left[(X, \widetilde{\theta})\right] \leq_K u$ so that $\alpha^{-1}\left[(X, \widetilde{\theta})\right] \in [\theta, u]$.

To show that P is affinely isomorphic to $[\theta, u]$ we define ϕ: $P \to [\theta, u]$ by $\phi(a) = \left[\left([(1,a)], \widetilde{\theta}\right)\right]$. It follows from the last computation in the previous paragraph that $\phi(a)$ is indeed in $[\theta, u]$. We now show that ϕ is an affine isomorphism. If $a \perp b$, then

$$\begin{aligned}
\phi(a) + \phi(b) &= \left[\left([(1,a)], \widetilde{\theta}\right)\right] + \left[\left([(1,b)], \widetilde{\theta}\right)\right] \\
&= \left[\left(\left[\left(2, \frac{1}{2}\, a \oplus \frac{1}{2}\, b\right)\right], \widetilde{\theta}\right)\right] = \left[\left([(1, a \oplus b)], \widetilde{\theta}\right)\right] \\
&= \phi(a \oplus b).
\end{aligned}$$

Hence, ϕ is additive. It is clear that $\phi(1) = u$ so ϕ is a morphism. If $\lambda \in [0,1]$ and $a \in P$, then

$$\phi(\lambda a) = \left[\left([(1,\lambda a)]\,,\widetilde{\theta}\right)\right] = \left[\left(\lambda\,[(1,a)]\,,\widetilde{\theta}\right)\right] = \lambda\left[\left([(1,a)]\,,\widetilde{\theta}\right)\right] = \lambda\phi(a)$$

so ϕ is an affine morphism. Suppose that $\phi(a) + \phi(b) = \phi(c)$. We then have

$$\left[\left([(1,a)] + [(1,b)]\,,\widetilde{\theta}\right)\right] = \left[\left([(1,c)]\,,\widetilde{\theta}\right)\right].$$

It follows that

$$\left[\left(2, \frac{1}{2}\,a \oplus \frac{1}{2}\,b\right)\right] = [(1,c)]$$

and $\frac{1}{2}\,a \oplus \frac{1}{2}\,b = \frac{1}{2}\,c$. Hence, $\frac{1}{2}\,a \oplus \frac{1}{2}\,b \perp \frac{1}{2}\,a \oplus \frac{1}{2}\,b$ and we conclude that $a \perp b$. Therefore, if we can prove that ϕ is surjective, then it follows that ϕ is a monomorphism and hence ϕ is an affine isomorphism.

We now show that if $\phi(a) \leq_K \phi(b)$, then there exists a $c \in P$ such that $\phi(a) + \phi(c) = \phi(b)$. Since $\phi(a) \leq_K \phi(b)$, there exists an $x \in K$ such that $\phi(b) - \phi(a) = x$. Suppose that $x = \left[\left([(\alpha,c)]\,,\widetilde{\theta}\right)\right]$. If $\alpha = 1$, then $x = \phi(c)$ and we are finished so suppose that $\alpha > 1$. Let $n \in \mathbb{N}$ with $\alpha \leq n$ and let $d = (\alpha/n)c$. Then $n \geq 2$, $d \in P$ and

$$\phi(a) + n\phi(d) = \phi(a) + n\phi\left(\frac{\alpha}{n}\,c\right) = \phi(a) + \alpha\phi(c) = \phi(a) + x = \phi(b).$$

Now $\frac{1}{n}\,a \perp d$ because

$$\phi\left(\frac{1}{n}\,a\right) + \phi(d) = \frac{1}{n}\,\phi(a) + \phi(d) = \frac{1}{n}\,\phi(b) = \phi\left(\frac{1}{n}\,b\right).$$

Moreover, $\left(\frac{1}{n}\,a \oplus d\right) \perp \left(\frac{1}{n}\,a \oplus d\right)$ because

$$\phi\left(\frac{1}{n}\,a \oplus d\right) + \phi\left(\frac{1}{n}\,a \oplus d\right) = 2\phi\left(\frac{1}{n}\,b\right) = \phi\left(\frac{1}{n}\,b\right) + \phi\left(\frac{1}{n}\,b\right) = \phi\left(\frac{2}{n}\,b\right).$$

It follows from associativity that $2d = d \oplus d$, $\frac{1}{n}\,a \oplus 2d$ and $\frac{2}{n}\,a \oplus 2d$ are defined in P. If $n \geq 3$, then

$$\left(\frac{2}{n}\,a \oplus 2d\right) \perp \left(\frac{1}{n}\,a \oplus d\right)$$

because

$$\phi\left(\frac{2}{n}\,a \oplus 2d\right) + \phi\left(\frac{1}{n}\,a \oplus d\right) = 3\phi\left(\frac{1}{n}\,b\right) = \phi\left(\frac{3}{n}\,b\right).$$

As before, $3d$, $\frac{2}{n}\,a \oplus 3d$ and $\frac{3}{n}\,a \oplus 3d$ are defined in P. Continuing this process, we conclude that nd is defined in P. Hence,

$$\phi(a) + \phi(nd) = \phi(a) + n\phi(d) = \phi(b).$$

For arbitrary $x, y \in [\theta, u]$, it is clear that $x \leq y$ implies $x \leq_K y$. It follows that if ϕ is surjective, then the order $\leq$ and $\leq_K$ coincide on $[\theta, u]$.

Finally, we show that ϕ is surjective. Let $x \in [\theta, u]$ where $x = \left[\left([(\alpha, a)], \widetilde{\theta}\right)\right]$. If $\alpha = 1$, then $x = \phi(a)$ and we are finished so suppose that $\alpha > 1$. Let $n \in \mathbb{N}$ with $\alpha \leq n$ and let $b = \frac{\alpha}{n}\,a \in P$. Then

$$n\phi(b) = n\phi\left(\frac{\alpha}{n}\,a\right) = \alpha\phi(a) = x \leq_K u.$$

We now show by induction on n that if $n\phi(b) \leq_K u$, then nb is defined in P. The result clearly holds for $n = 1$. Suppose the result holds for n and assume that $(n+1)\phi(b) \leq_K u$. Then $n\phi(b) \leq (n+1)\phi(b) \leq_K u$ so by the induction hypothesis, nb is defined in P. Since ϕ is a morphism, we have

$$\phi(nb) = n\phi(b) \leq_K u - \phi(b) = \phi(b').$$

It follows that there exists a $c \in P$ such that

$$\phi(nb) + \phi(c) = \phi(b') = u - \phi(b).$$

Hence,

$$\phi(nb) + \phi(b) = u - \phi(c) = \phi(c').$$

We conclude from our previous work that $nb \perp b$ so $nb \oplus b$ is defined. Hence, $(n+1)b = nb \oplus b$ is defined which completes the induction proof. Thus, $nb \in P$ and

$$x = n\phi(b) = \phi(nb).$$

Therefore, ϕ is surjective.

To prove uniqueness, suppose $\phi_1\colon\ P \to [\theta_1, u_1]$ is an affine isomorphism, where $[\theta_1, u_1]$ is a linear effect algebra that generates (V_1, K_1). It is easy to check that $\psi = \phi_1 \circ \phi^{-1}$ is an affine bijection from $[\theta, u]$ onto $[\theta_1, u_1]$. Define $T\colon\ V \to V_1$ as follows. If $x \in V$, then x has the form $x = \alpha y - \beta z$ where $\alpha, \beta \geq 0$ and $y, z \in [\theta, u]$. We define $T(x)$ by

$$T(x) = \alpha\psi(y) - \beta\psi(z).$$

To show that T is well defined, suppose that

$$\alpha y - \beta z = \alpha_1 y_1 - \beta_1 z_1$$

where $\alpha_1, \beta_1 \geq 0$ and $y_1, z_1 \in [\theta, u]$. Then

$$\alpha y + \beta_1 z_1 = \alpha_1 y_1 + \beta z.$$

Letting $\gamma = \alpha + \beta_1 + \alpha_1 + \beta$, we have

$$\frac{\alpha+\beta_1}{\gamma}\left(\frac{\alpha}{\alpha+\beta_1}\,y + \frac{\beta_1}{\alpha+\beta_1}\,z_1\right) = \frac{\alpha_1+\beta}{\gamma}\left(\frac{\alpha_1}{\alpha_1+\beta}\,y_1 + \frac{\beta}{\alpha_1+\beta}\,z\right).$$

Hence,

$$\frac{\alpha+\beta_1}{\gamma}\left(\frac{\alpha}{\alpha+\beta_1}\,\psi(y) + \frac{\beta_1}{\alpha+\beta_1}\,\psi(z_1)\right) = \frac{\alpha_1+\beta}{\gamma}\left(\frac{\alpha_1}{\alpha_1+\beta}\,\psi(y_1) + \frac{\beta}{\alpha_1+\beta}\,\psi(z)\right).$$

It follows that

$$\alpha\psi(y) - \beta\psi(z) = \alpha_1\psi(y_1) - \beta_1\psi(z_1).$$

It is straightforward to check that T is an order isomorphism. $\square$

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