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A retractible non-locally connected dendroid

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Abstract. A retractible non-locally connected dendroid is constructed.

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A *continuum* is a compact connected metric space. A continuum X is *retractible* if for every subcontinuum A of X there exists a retraction $r : X \to A$. Retractible continua were introduced by J.J. Charatonik in [1], where he posed the following problem:

Problem. Give a structural (internal) characterization of retractible continua.

In the same paper, this problem is partially solved by showing that a locally connected continuum is retractible if and only if it is hereditarily locally connected.

A different approach for attacking this problem is to add requirements to the retractions. A continuum X is said to be *d-retractible* (*resp., sd-retractible, m-retractible, c-retractible, o-retractible*), provided that for each subcontinuum A of X, there exists a deformation (resp. strong deformation, monotone, confluent, open) retraction from X onto A. In [2], G.R. Gordh and L. Lum proved that a continuum is *m-retractible* if and only if it is a dendrite. Recently, the author has shown the following results:

Theorem ([3]). *If X is a continuum, then the following assertions are equivalent:*

(a) *X is a dendrite,*
(b) *X is d-retractible and,*
(c) *X is sd-retractible.*

Theorem ([4]). *If X is a pathwise connected c-retractible continuum, then X is hereditarily locally connected.*

Theorem ([4]). *If X is a pathwise connected continuum, then the following assertions are equivalent:*

(a) *X is o-retractible and,*
(b) *X is homeomorphic to an interval or to a simple closed curve.*

In [1] and [5], J.J. Charatonik and L. Lum, respectively, asked the following question:

Question. Does there exist an arcwise connected retractible continuum which is not locally connected?

In [5, p. 337], L. Lum mentioned that A. Lelek had a candidate for answering this question in the positive.

In this paper, we answer the question in the positive by constructing a non-locally connected retractible dendroid.

In a recent private communication with J.J. Charatonik, A. Lelek told him that his example had a similar construction as the example presented here and that his example was never written for publication.

Preliminary constructions

Given two points p and q in the Euclidean plane R^2, denote by $\langle p,q\rangle$ the segment joining them, if $p \neq q$ and $\langle p,q\rangle = \{p\}$, if $p = q$. For a point $p = (x,y) \in R^2$, define $p' = (-x,y)$. Given a subset B in R^2, define $B' = \{p' \in R^2 : p \in B\}$. The origin in R^2 is denoted by Θ. Let $\pi_1 : R^2 \to R$ be the projection on the first coordinate. We will define, inductively, a sequence $A_0, \ldots, A_n, \ldots$ of subsets of R^2 such that, for each integer $n > 0$, A_n is a polygon joining Θ to a point $a_n = (u_n, v_n)$. Let $a_0 = \Theta$.

Let $A_0 = \{\Theta\}$, $A_1 = \big\langle \Theta, (1, -\frac{1}{4})\big\rangle$. Suppose that $A_0, \ldots, A_n$ have been defined and $n \geq 1$. Define $b_n = a_n + a'_{n-1}$ and $A_{n+1} = A_n \cup (a_n + A'_{n-1}) \cup (b_n + A_n)$. See Figure 1.

It is easy to prove the following:

Assertion 1. *For each $n > 0$, A_n is a polygon, $v_n < 0$, $A_n \subset (0,n] \times [v_n, 0] \cup \{\Theta\}$, $u_n = n$, $a_{n+1} = 2a_n + a'_{n-1}$ and $A_n = a_n - A_n$.*

Given points p and q in A_n, $\langle\langle p,q\rangle\rangle$ will denote the subarc in A_n joining p and q, if $p \neq q$ and $\langle\langle p,q\rangle\rangle = \{p\}$, if $p = q$.

Assertion 2. *For each $n > 0$, there exists a homeomorphism $\alpha_n : A_n \to A_{n+1}$ such that $\alpha_n(\Theta) = \Theta$, $\alpha_n(a_n) = a_{n+1}$ and, for each $p \in A_n$, $|\pi_1(p) - \pi_1(\alpha_n(p))| \leq 2$.*

Proof: We proceed by induction. Define $\alpha_1 : A_1 \to A_2$ by $\alpha_1(p) = 2p$. Define $\alpha_2 : A_2 \to A_3$ by sending homeomorphically the segment $A_1 \subset A_2$ onto the arc $A_2 \cup (a_2 + A'_1)$ in such a way that $\alpha_2(\Theta) = \Theta$ and then sending linearly the segment $a_1 + A_1$ onto $b_2 + A_2$. Suppose that $\alpha_1, \ldots, \alpha_n$, have been constructed, define $\alpha_{n+1} : A_{n+1} \to A_{n+2}$ by:

$$\alpha_{n+1}(p) = \begin{cases} \alpha_n(p) & \text{if } p \in A_n, \\ b_{n+1} - (\alpha_{n-1}(-p' + b'_n))' & \text{if } p \in a_n + A'_{n-1} \text{ and,} \\ b_{n+1} + \alpha_n(p - b_n) & \text{if } p \in b_n + A_n. \end{cases}$$

It is easy to verify that α_{n+1} has the required properties. □

The key for proving the retractibility of the continuum presented in this paper is the following:

Assertion 3. *For each $n > 0$ and for each $q \in A_n$, there exists a map $\sigma : A_n \to A_n$ (which depends on q) such that:*

(a) $\sigma|\langle\langle\Theta, q\rangle\rangle = Id_{\langle\langle\Theta,q\rangle\rangle}$ *(the identity map on $\langle\langle\Theta, q\rangle\rangle$);*

(b) $|\pi_1(p) - \pi_1(\sigma(p))| \leq 2$ *for every $p \in A_n$;*

(c) *if $q \in A_{n-1}$, then $\sigma^{-1}(\Theta) \cap \langle\langle q, a_n\rangle\rangle \neq \emptyset$: and if r is the first point in $\sigma^{-1}(\Theta) \cap \langle\langle q, a_n\rangle\rangle$ (in the natural ordering of $\langle\langle q, a_n\rangle\rangle$ from q to a_n), then $\sigma\langle\langle q, r\rangle\rangle \subset \langle\langle\Theta, q\rangle\rangle$ and $\sigma(a_n) = a_n$;*

(d) *if $q \in A_n - A_{n-1}$, then $\sigma(A_n) \subset \langle\langle\Theta, q\rangle\rangle$.*

In order to prove Assertion 3, we will use the following assertion which is easy to prove.

Assertion 4. *If C and D are two arcs and $\alpha, \beta : C \to D$ are maps such that α is onto, then there exists $p \in C$ such that $\alpha(p) = \beta(p)$.*

Proof of Assertion 3: We apply induction. It is easy to prove the assertion for $n = 1$, $n = 2$ and $n = 3$. Now, suppose that, for every $i = 1, \ldots, n$ and for every $q \in A_i$, it is possible to construct σ and take $n \geq 3$. Take a point $q \in A_{n+1} = A_n \cup (a_n + A'_{n-1}) \cup (b_n + A_n)$. For defining σ we consider seven cases, in each one of which it is easy to check that the map defined has the required properties. A geometric representation of Cases 2–6 is given in Figures 2 and 3.

Case 1. $q \in A_{n-1} \subset A_n$. Apply the induction hypothesis to q and obtain the corresponding map $\sigma_0 : A_n \to A_n$. Define $\sigma : A_{n+1} \to A_{n+1}$ by:

$$\sigma(p) = \begin{cases} \sigma_0(p) & \text{if } p \in A_n \text{ and,} \\ p & \text{if } p \in (a_n + A'_{n-1}) \cup (b_n + A_n). \end{cases}$$

Case 2. $q \in A_n - A_{n-1}$. The induction hypothesis implies the existence of $\sigma_0 : A_n \to A_n$. Define α, $\beta : a_n + A'_{n-1} \to A_n$ by $\alpha(p) = a_n - \alpha_{n-1}((p - a_n)')$ and $\beta(p) = \sigma_0(-(p-a_n)' + a_n)$. Applying Assertion 4, there exists $q_0 \in a_n + A'_{n-1}$ such that $\alpha(q_0) = \beta(q_0)$.

Define $\sigma : A_{n+1} \to A_{n+1}$ by:

$$\sigma(p) = \begin{cases} \sigma_0(p) & \text{if } p \in A_n, \\ \beta(p) & \text{if } \langle\langle a_n, q_0\rangle\rangle, \\ \alpha(p) & \text{if } \langle\langle q_0, b_n\rangle\rangle \text{ and,} \\ a_{n+1} - \alpha_n(a_{n+1} - p) & \text{if } p \in b_n + A_n. \end{cases}$$

Case 3. $q \in a_n + A'_{n-2} \subset a_n + A'_{n-1}$ and $q \neq a_n$. Define $q_1 = (q - a_n)' \in A_{n-2} \subset A_{n-1}$. Apply the induction hypothesis to q_1 to obtain a map $\sigma_0 : A_{n-1} \to A_{n-1}$. Let r be the first point in $\langle\langle q_1, a_{n-2}\rangle\rangle$, in the ordering from q_1 to a_{n-2}, such that $\sigma_0(r) = \Theta$.

Define $\sigma : A_{n+1} \to A_{n+1}$ by:

$$\sigma(p) = \begin{cases} p & \text{if } p \in A_n, \\ a_n + (\sigma_0((p-a_n)'))' & \text{if } p \in \langle\langle a_n, a_n + r' \rangle\rangle, \\ a_n - \sigma_0((p-a_n)') & \text{if } p \in \langle\langle a_n + r', b_n \rangle\rangle \text{ and,} \\ b_{n-1} + \alpha_{n-1}^{-1}(p - b_n) & \text{if } p \in b_n + A_n. \end{cases}$$

Case 4. $q \in a_n + A'_{n-1} - (a_n + A'_{n-2})$. Define $q_1 = (q - a_n)' \in A_{n-1} - A_{n-2}$. Apply the induction hypothesis to q_1 to obtain a map $\sigma_0 : A_{n-1} \to A_{n-1}$.

Define $\sigma : A_{n+1} \to A_{n+1}$ by:

$$\sigma(p) = \begin{cases} p & \text{if } p \in A_n, \\ a_n + (\sigma_0((p-a_n)'))' & \text{if } p \in a_n + A'_{n-1}, \\ a_n - (\sigma_0(-(p-(b_n + a_{n-1}))))' & \text{if } p \in b_n + A_{n-1}, \\ a_n - \alpha_{n-2}((p-(b_n + a_{n-1}))') & \text{if } p \in b_n + a_{n-1} + A'_{n-2} \text{ and,} \\ b_{n-1} + p - (b_n + b_{n-1}) & \text{if } p \in b_n + b_{n-1} + A_{n-1}. \end{cases}$$

Case 5. $q \in b_n + A_{n-2} \subset b_n + A_n$. Define $q_1 = q - a_n - a'_{n-1} \in A_{n-2} \subset A_{n-1}$. Apply the induction hypothesis to q_1 to obtain a map $\sigma_0 : A_{n-1} \to A_{n-1}$. Let r be the first point in $\langle\langle q_1, a_{n-1} \rangle\rangle$, in the ordering from q_1 to a_{n-1}, such that $\sigma_0(r) = \Theta$.

Define $\sigma : A_{n+1} \to A_{n+1}$ by:

$$\sigma(p) = \begin{cases} p & \text{if } p \in A_n \cup (a_n + A'_{n-1}), \\ b_n + \sigma_0(p - b_n) & \text{if } p \in \langle\langle b_n, b_n + r \rangle\rangle, \\ b_n - (\sigma_0(p-b_n))' & \text{if } p \in \langle\langle b_n + r, b_n + a_{n-1} \rangle\rangle, \\ a_n - \alpha_{n-2}((p-(b_n + a_{n-1}))') & \text{if } p \in b_n + a_{n-1} + A'_{n-2} \text{ and,} \\ b_{n-1} + p - (b_n + b_{n-1}) & \text{if } p \in b_n + b_{n-1} + A_{n-1}. \end{cases}$$

Case 6. $q \in (b_n + A_{n-1}) - (b_n + A_{n-2})$. Define $q_1 = q - b_n \in A_{n-1} - A_n$. Apply the induction hypothesis to q_1 to obtain a map $\sigma_0 : A_{n-1} \to A_{n-1}$. Define $\alpha, \beta : b_n + a_{n-1} + A'_{n-2} \to b_n + A_{n-1}$ by: $\alpha(p) = b_n + a_{n-1} - \alpha_{n-2}((p - b_n - a_{n-1})')$ and $\beta(p) = b_n + \sigma_0(a_{n-1} - (p - b_n - a_{n-1})')$. From Assertion 4, there exists $p_0 \in b_n + a_{n-1} + A'_{n-2}$ such that $\alpha(p_0) = \beta(p_0)$.

Define $\sigma : A_{n+1} \to A_{n+1}$ by:

$$\sigma(p) = \begin{cases} p & \text{if } p \in A_n \cup (a_n + A'_{n-1}), \\ b_n + \sigma_0(p - b_n) & \text{if } p \in b_n + A_{n-1}, \\ \beta(p) & \text{if } p \in \langle\langle b_n + a_{n-1}, p_0 \rangle\rangle, \\ \alpha(p) & \text{if } p \in \langle\langle p_0, b_n + b_{n-1} \rangle\rangle \text{ and,} \\ b_n - (p - b_n - b_{n-1})' & \text{if } p \in b_n + b_{n-1} + A_{n-1}. \end{cases}$$

Case 7. $q \in (b_n + A_n) - (b_n + A_{n-1})$. Define $q_1 = q - b_n \in A_n - A_{n-1}$. Apply the induction hypothesis to q_1 to obtain a map $\sigma_0 : A_n \to A_n$.

Define $\sigma : A_{n+1} \to A_{n+1}$ by:

$$\sigma(p) = \begin{cases} p & \text{if } p \in A_n \cup (a_n + A'_{n-1}), \text{ and} \\ b_n + \sigma_0(p - b_n) & \text{if } p \in b_n + A_n. \end{cases}$$

This completes the construction of σ and the proof of Assertion 3. □

For each $n > 1$, define $C_n = (0, \frac{1}{2^{n-2}}) + \{(\frac{x}{n}, \frac{-y}{2^n v_n}) \in R^2 : (x, y) \in A_n\}$ and $B_n = C_n \cup \left\langle (0, \frac{1}{2^{n-2}}), \Theta \right\rangle \cup \langle \Theta, (1, 0) \rangle$. Given points p and q in B_n, $\langle\langle\langle p, q \rangle\rangle\rangle$ will denote the subarc in B_n joining p and q if $p \neq q$ and $\langle\langle\langle p, q \rangle\rangle\rangle = \{p\}$ if $p = q$.

Assertion 5. *For each $n \geq 2$ and $q \in C_n$ there exists a retraction $\phi : B_n \to \langle\langle\langle q, (1, 0) \rangle\rangle\rangle$ (which depends on q) such that $|\pi_1(p) - \pi_1(\phi(p))| \leq \frac{3}{n}$ for all $p \in B_n$.*

Proof: Let $\lambda : A_n \to C_n$ be the homeomorphism defined by $\lambda(x, y) = (0, \frac{1}{2^{n-2}}) + (\frac{x}{n}, \frac{-y}{2^n v_n})$. Let $q_0 = \lambda^{-1}(q) \in A_n$. Let $\sigma : A_n \to A_n$ be as in Assertion 3 applied to q_0. Define $\sigma_1 : C_n \to C_n$ by $\sigma_1 = \lambda \circ \sigma \circ \lambda^{-1}$. We consider two cases. In both cases, it is easy to check that ϕ has the mentioned properties.

Case 1. $q_0 \in A_{n-1}$.

Let $r \in \langle\langle q_0, a_n \rangle\rangle$ be the first point, in the ordering from q_0 to a_n, such that $\sigma(r) = \Theta$. Choose a homeomorphism $\delta : [0, \frac{1}{n}] \to \left\langle\left\langle\left\langle (0, \frac{1}{2^{n-2}}), (\frac{1}{n}, 0) \right\rangle\right\rangle\right\rangle$ such that $\delta(0) = (0, \frac{1}{2^{n-2}})$ and $\delta(\frac{1}{n}) = (\frac{1}{n}, 0)$.

Define $\phi : B_n \to \langle\langle\langle q, (1, 0) \rangle\rangle\rangle$ by:

$$\phi(p) = \begin{cases} p & \text{if } p \in \left\langle\left\langle\left\langle (0, \frac{1}{2^{n-2}}), (1, 0) \right\rangle\right\rangle\right\rangle, \\ \sigma_1(p) & \text{if } p \in \left\langle\left\langle\left\langle (0, \frac{1}{2^{n-2}}), \lambda(r) \right\rangle\right\rangle\right\rangle, \\ (\pi_1(\sigma_1(p)), 0) & \text{if } p \in \langle\langle\langle \lambda(r), \lambda(a_n) \rangle\rangle\rangle \cap (\pi_1 \circ \sigma_1)^{-1}([\frac{1}{n}, 1]) \text{ and,} \\ \delta(\pi_1(\sigma_1(p))) & \text{if } p \in \langle\langle\langle \lambda(r), \lambda(a_n) \rangle\rangle\rangle \cap (\pi_1 \circ \sigma_1)^{-1}([0, \frac{1}{n}]). \end{cases}$$

Case 2. $q_0 \in A_n - A_{n-1}$.

In this case define $\phi : B_n \to \langle\langle\langle q, (1, 0) \rangle\rangle\rangle$ by:

$$\phi(p) = \begin{cases} p & \text{if } p \in \left\langle\left\langle\left\langle (0, \frac{1}{2^{n-2}}), (1, 0) \right\rangle\right\rangle\right\rangle \text{ and,} \\ \sigma_1(p) & \text{if } p \in C_n. \end{cases}$$

□

The example

Define $X = \bigcup\{B_n : n \geq 2\} = \langle\Theta, (0,1)\rangle \cup \langle\Theta, (1,0)\rangle \cup (\bigcup\{C_n : n \geq 2\})$. Clearly, X is a non-locally connected dendroid. The continuum X is illustrated in Figure 4.

In order to prove that X is retractible, define $J = \langle\Theta, (1,0)\rangle$ and take a subcontinuum A of X. We may assume that $A \cap J \neq \emptyset$ and $A \not\subset J$. Since A is a retract of $A \cup J$, we only have to prove that there is a retraction $\rho : X \to A \cup J$. Let $N \geq 2$, be such that $A \cap C_n \neq \emptyset$ for every $n \geq N$ and $A \cap C_n = \emptyset$ for every $n \leq N$. Notice that, for each $n \geq 2$, C_n is an arc in R^2 which joins $(0, \frac{1}{2^{n-2}})$ and $(1, \frac{1}{2^{n-2}} - \frac{1}{2^n})$. For each $n \geq N$, let q_n be the last element in C_n, in the ordering from $(0, \frac{1}{2^{n-2}})$ to $(1, \frac{1}{2^{n-2}} - \frac{1}{2^n})$, such that $q_n \in A$. Then there exists a retraction $\phi_n : B_n \to \langle\langle\langle q_n, (1,0)\rangle\rangle\rangle$, such that $|\pi_1(p) - \pi_1(\phi_n(p))| \leq \frac{3}{n}$. Finally, let p_0 be the last point in $\langle\Theta, (0,1)\rangle$, in the ordering from Θ to $(0,1)$, such that $p_0 \in A$. We are ready to define ρ.

Define $\rho : X \to A \cup J$ by:

$$\rho(p) = \begin{cases} p_0 & \text{if } \ p \in \langle p_0, (0,1)\rangle \cup (\bigcup\{C_n : n < N\}), \\ p & \text{if } \ p \in \left\langle (0, \frac{1}{2^{N-2}}), p_0 \right\rangle \text{ and,} \\ \phi_n(p) & \text{if } \ p \in B_n \ \text{ for some } \ n \geq N. \end{cases}$$

Then ρ is a retraction.

Therefore, X is retractible.

A dendroid is called a *fan* provided that it has exactly one ramification point. A continuum is said to be *rational* provided that each of its points has arbitrarily small neighborhoods with countable boundaries. Shrinking in the constructed dendroid X the arc $\langle\Theta, (0,1)\rangle$ to a point, i.e., applying a monotone mapping $\mu : X \to Y$ such that $\mu(\langle\Theta, (0,1)\rangle)$ is a singleton, while the partial mapping $\mu|(X - \langle\Theta, (0,1)\rangle)$ is a homeomorphism, we get a rational plane fan Y that keeps the main property of X of being retractible.

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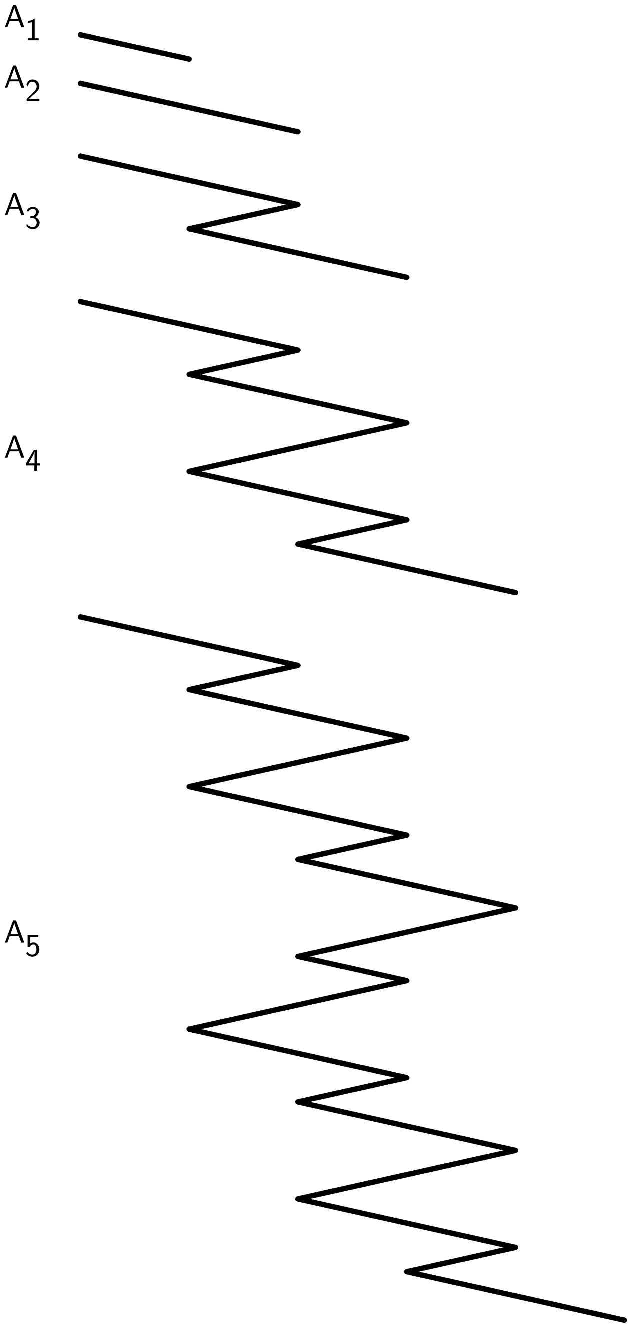
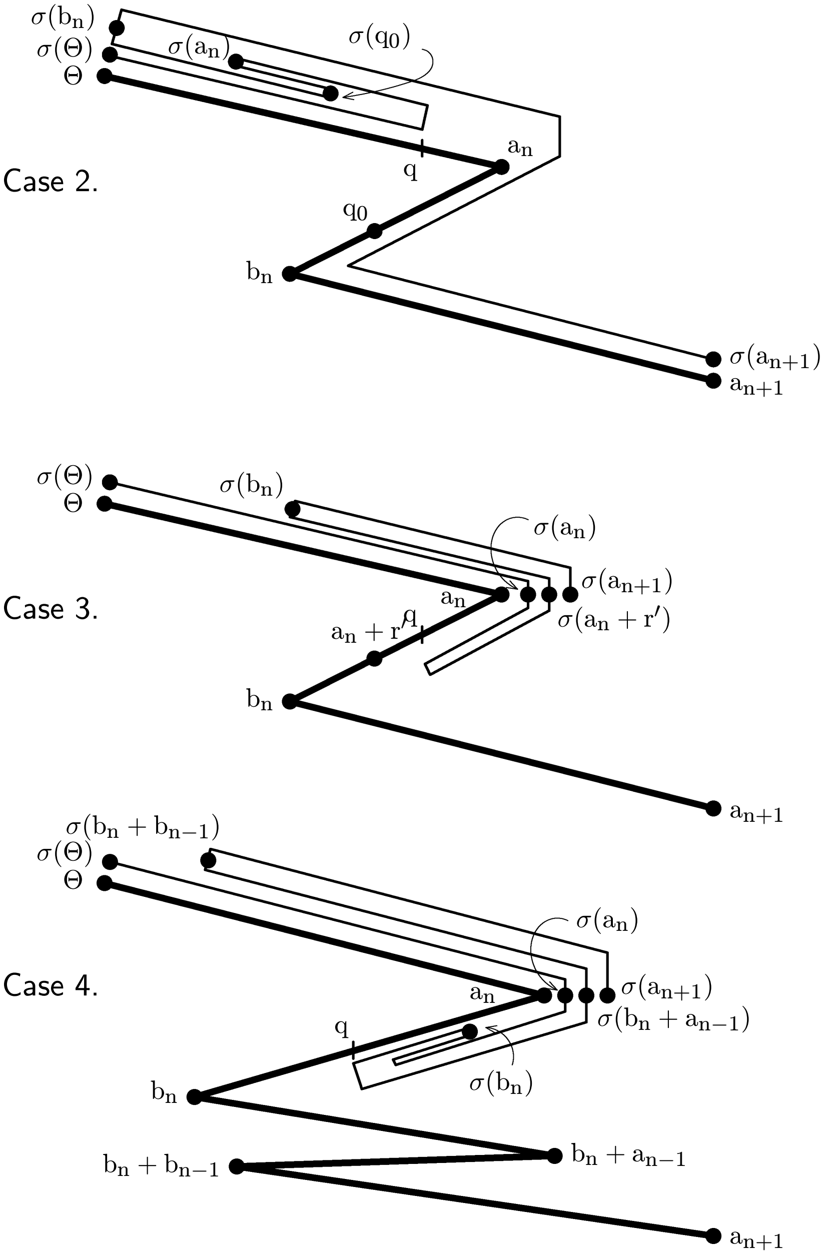
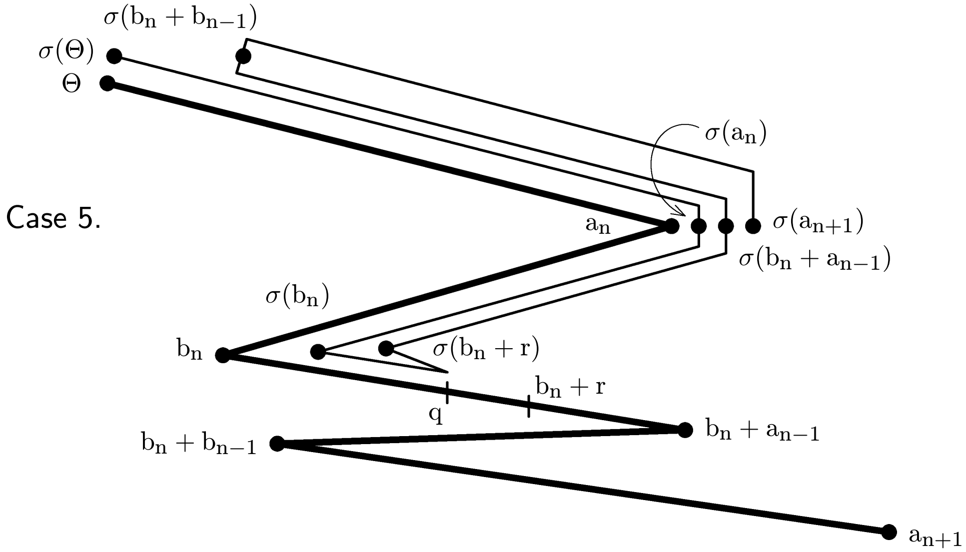

A_1
A_2
A_3
A_4
A_5

Figure 1



Case 2.
$\sigma(b_n)$
$\sigma(\Theta)$
Θ
$\sigma(a_n)$
$\sigma(q_0)$
a_n
q
q_0
b_n
$\sigma(a_{n+1})$
a_{n+1}
Case 3.
$\sigma(\Theta)$
Θ
$\sigma(b_n)$
$\sigma(a_n)$
$\sigma(a_{n+1})$
a_n
q
$a_n + r'$
$\sigma(a_n + r')$
b_n
a_{n+1}
Case 4.
$\sigma(b_n + b_{n-1})$
$\sigma(\Theta)$
Θ
$\sigma(a_n)$
$\sigma(a_{n+1})$
a_n
$\sigma(b_n + a_{n-1})$
q
$\sigma(b_n)$
b_n
$b_n + b_{n-1}$
$b_n + a_{n-1}$
a_{n+1}

Figure 2



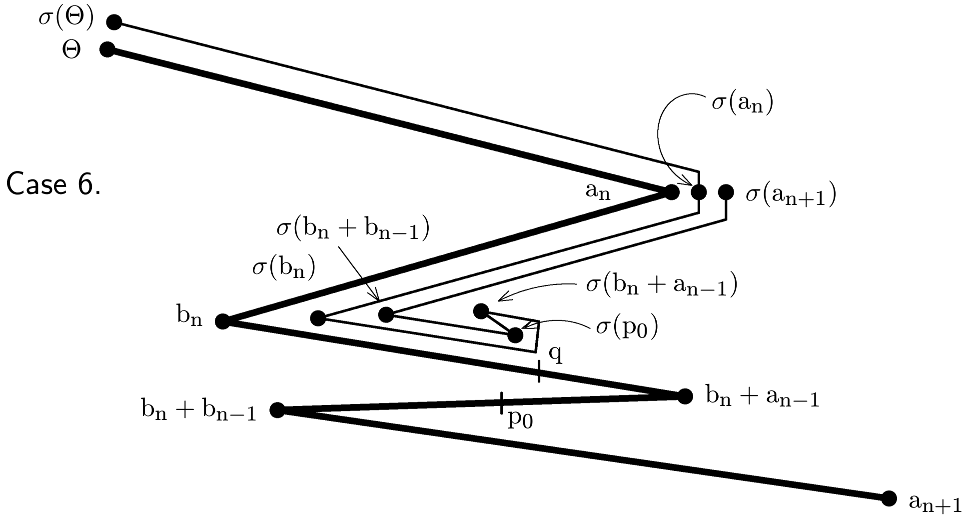


Figure 3

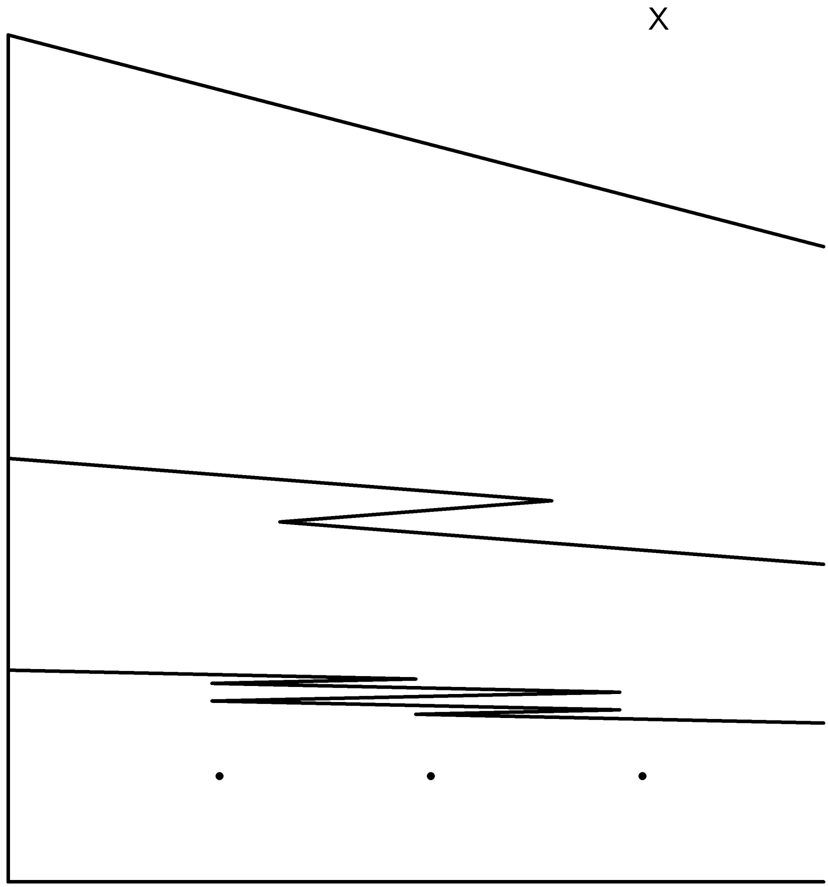


Figure 4

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