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Nearly smooth points and near smoothness in Orlicz spaces

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Abstract. Nearly smooth points and near smoothness in Orlicz spaces are characterized. It is worth to notice that in the nonatomic case smooth points and nearly smooth points are the same, but in the sequence case they differ.

Keywords: Orlicz space, nearly smooth points, near smoothness

Classification: 46E30, 46B20

For a Banach space X, we denote by $S(X)$, $B(X)$ and X^* the unit sphere, unit ball and the dual space of X, respectively. For $x \in X$ we write $\nabla_x = \{f \in S(X^*) : f(x) = \|x\|\}$, i.e. ∇_x is the set of all norm-one supporting functionals f at $x \in X$. In 1991, Banaś [1] introduced the notion of the modulus of near smoothness and the modulus of near convexity. As refinements of the result of [1], in 1995, Banaś and Sadarangani [2] introduced the concept of near smoothness and showed that: For a sequence of Banach spaces $\{E_i\}$, if every E_i is near smooth, then $c_0(E_i)$ and $l^p(E_i)$ $(1 < p < \infty)$ are both near smooth, too.

Definition. $x \in S(X)$ is called a nearly smooth point of X if ∇_x is a compact subset of X^*. X is said to be nearly smooth if every $x \in S(X)$ is a nearly smooth point.

In this note we will characterize nearly smooth points and near smoothness in Orlicz spaces over a nonatomic finite and over the counting measure.

Let $\mathbb{R} = (-\infty, \infty)$ be the set of all reals, $\mathbb{N}$ be the set of all natural numbers and m the set of all real sequences. Further, let (G, Σ, μ) be a measure space with a non-negative, finite, atomless and complete measure defined on a σ-algebra Σ. We denote by L^0 the set of all μ-equivalence classes of real valued Σ-measurable functions defined on G.

A convex even function $M : \mathbb{R} \to [0, +\infty)$ is called an N-function iff

$$M(u) = 0 \Leftrightarrow u = 0, \ \ \frac{M(u)}{u} \to \infty \ \text{ as } \ u \to \infty \ \text{ and } \ \frac{M(u)}{u} \to 0 \ \text{ as } \ u \to 0.$$

For every N-function $M(u)$, we define a complementary function $N : \mathbb{R} \to [0, +\infty)$ by $N(v) = \max_{u>0}[u|v| - M(u)]$, $v \in \mathbb{R}$. The function $N(v)$ is also an

This subject is supported by NSFC and NSFH.

N-function. Moreover, let $p(u)$, $q(v)$ denote the right-hand derivatives of $M(u)$ and $N(v)$, respectively. We write $M(u) \in \overline{\Delta_2}$ ($M(u) \in \Delta_2$) whenever $M(u)$ satisfies the Δ_2-condition for large u (for small u) (cf. [3]). The functionals

$$\varrho_M(x) = \sum_{i=1}^{\infty} M(x_i) \quad \text{for} \quad x \in m$$

and

$$\varrho_M(x) = \int_G M(x(t))\, d\mu \quad \text{for} \quad x \in L^0$$

are modulars on m and L^0, respectively (cf. [4]). The space

$$l_M = \left\{x \in m : \varrho_M(kx) < \infty \quad \text{for some} \quad k > 0\right\}$$

equipped with the so called Luxemburg norm

$$\|x\|_{(M)} = \inf\left\{a > 0 : \varrho_M\left(\frac{x}{a}\right) \le 1\right\}$$

or with the Orlicz norm

$$\|x\|_M = \inf_{k>0} \frac{1}{k}(1 + \varrho_M(kx))$$

is said to be an Orlicz sequence space. A subspace $h_M \subset l_M$ is defined as the set of all $x \in m$ such that $\varrho_M(kx) < \infty$ for any $k > 0$, i.e. $h_M = \{x \in m : \varrho_M(kx) < \infty$ for any $k > 0\}$. To simplify the notation, we put

$$\begin{gathered} l_M = (l_M, \|\cdot\|_M),\ l_{(M)} = (l_{(M)}, \|\cdot\|_{(M)}), \\ h_{(M)} = (h_{(M)}, \|\cdot\|_{(M)}),\ h_M = (h_M, \|\cdot\|_M). \end{gathered}$$

The Orlicz function spaces L_M and $L_{(M)}$ equipped with the Orlicz norm $\|\cdot\|_M$ and the Luxemburg norm $\|\cdot\|_{(M)}$, respectively, and the subspaces E_M and $E_{(M)}$ are defined analogously ([3]).

For $x \in L_M$ or l_M we write

$$\begin{aligned} Q_M(x) &= \inf\left\{c > 0 : \varrho_M\left(\frac{x}{c}\right) < \infty\right\}, \\ k^*(x) &= \inf\left\{k > 0 : \varrho_N(p(k|x|)) \ge 1\right\}, \\ k^{**}(x) &= \sup\left\{k > 0 : \varrho_N(p(k|x|)) \le 1\right\}, \\ K_M(x) &= \left[k^*(x), k^{**}(x)\right]. \end{aligned}$$

A linear functional $\varphi \in L_M^*$ (l_M^*) is called singular if $\varphi(E_M) = \{0\}$ ($\varphi(h_M) = \{0\}$). In this case, we write $\varphi \in F$.

In this paper, the following result from [5] will be used.

(1) (Theorem 1.44 in [5]). For any $x \in L_M$ (or l_M), there exists $\varphi \in F$ such that $\|\varphi\| = 1$ and $\varphi(x) = Q_M(x)$.

(2) (Theorem 1.54 in [5]). $\varphi \in F, A \cap B = \phi \Rightarrow \|\varphi|_{A\cup B}\| = \|\varphi|_A\| + \|\varphi|_B\|$.

(3) (Theorem 1.76). For $x \in S(L_{(M)})$, $y \in L_N$ and $\varphi \in F$, we have $f = y + \varphi \in \nabla_x$ iff: (i) $\varrho_M(x) = 1$; (ii) $\|\varphi\| = \varphi(x)$; (iii) $x(t)y(t) \geq 0$ and $p_-(|x(t)|) \leq k|y(t)| \leq p(|x(t)|)$ a.e. on G for some $k \in K_N(y)$.

(4) (A remark to Theorem 1.76 in [5]). $x \in S(L_{(M)})$ (or $x \in S(l_{(M)})$), $Q_M(x) < 1 \Rightarrow \nabla_x \subset S(L_N)$ (or $S(l_N)$).

(5) (Theorem 1.77 in [5]). For $x \in S(L_M)$, $y \in L_N$ and $\varphi \in F$, we have $f = y + \varphi \in \nabla_x$ iff: (i) $\varrho_M(y) + \|\varphi\| = 1$; (ii) $\|\varphi\| = \varphi(kx)$; (iii) $x(t)y(t) \geq 0$ and $p_-(k|x(t)|) \leq |y(t)| \leq p(k|x(t)|)$ a.e. on G with some $k \in K_M(x)$.

(6) (A remark to Theorem 1.77 in [5]). $x \in S(L_M)$ (or $x \in S(l_M)$), $Q_M(kx) < 1 \Rightarrow \nabla_x \subset S(L_{(N)})$ (or $S(l_{(N)})$), where $k \in K_M(x)$.

Theorem 1. *$x \in S(L_{(M)})$ is a nearly smooth point of $L_{(M)}$ iff:*

(i) $Q_M(x) < 1$;
(ii) $\mu\{(t \in G : p_-(|x(t)|) < p(|x(t)|)\} = 0$.

Proof: Sufficiency. Follows from the fact that any smooth point is nearly smooth and from Theorem 2.49 in [5].

Necessity. Suppose that $x \in S(L_{(M)})$ is a nearly smooth point. We first prove (i) $Q_M(x) < 1$.
Otherwise, $Q_M(x) = 1$. Write $G_0 = G$ and $G(j) = \{t \in G : j - 1 \leq |x(t)| < j\}$, $j = 1, 2, \dots$. Take a partition of $G_0(j) = G_0 \cap G(j)$ into $G_0'(j)$ and $G_0''(j)$ such that $\mu(G_0'(j)) = \mu(G_0''(j))$, $G_0(j) = G_0'(j) \cup G_0''(j)$, $j = 1, 2, \dots$. Denote

$$G_0' = \bigcup_{j=1}^{\infty} G_0'(j), \quad G_0'' = \bigcup_{j=1}^{\infty} G_0''(j).$$

Then $G_0' \cup G_0'' = G_0$, $G_0' \cap G_0'' = \emptyset$. Take $\varphi_0 \in F$ with

$$1 = \|\varphi_0\| = \varphi_0(x) = Q_M(x).$$

Since $\|\varphi_0\| = \|\varphi_0|_{G_0'}\| + \|\varphi_0|_{G_0''}\|$, we may assume without loss of generality that

$$\|\varphi_0|_{G_0''}\| \geq \frac{1}{2}.$$

Put $x_1 = x|_{G_0'}$. In view of $\varrho_M(x_1) < \varrho_M(x) \le 1$ and the fact that for any $\eta > 0$, there exist $j_0 \in N$ such that $\frac{1+2\eta}{1+\eta} \ge \frac{j}{j-1}$ whenever $j > j_0$, we have

$$\begin{aligned}
\varrho_M((1+2\eta)x_1) &= \sum_{j=1}^{\infty} \int_{G_0'(j)} M((1+2\eta)x(t))\,d\mu \\
&\ge \sum_{j=1}^{\infty} \int_{G_0'(j)} M((1+2\eta)(j-1))\,d\mu \ge \frac{1}{2} \sum_{j>j_0}^{\infty} \int_{G_0(j)} M((1+\eta)j)\,d\mu \\
&\ge \frac{1}{2} \sum_{j>j_0}^{\infty} \int_{G_0(j)} M((1+\eta)x(t))\,d\mu \ge \frac{1}{2}(\varrho_M((1+\eta)x) - M((1+\eta)j_0)\mu G) = \infty.
\end{aligned}$$

Hence $Q_M(x_1) = 1$. Furthermore, there exists $\varphi_1 = \varphi_1|_{G_0'} \in F$ such that $1 = \|\varphi_1\| = \varphi_1(x_1) = Q_M(x_1)$. It is easy to verify that $\varphi_1(x) = \varphi_1|_{G_0'}(x) = \varphi_1(x_1) = 1$, i.e. $\varphi_1 \in \nabla_x$.

Write $G_1 = G_0'$, $G_1(j) = G_1 \cap G(j)$, $j = 1, 2, \dots$. Repeating the same argumentation as above, we can get a decomposition $G_1 = G_1' \cup G_2''$ with $G_1' \cap G_2'' = \emptyset$, $\mu G_1' = \mu G_1''$, $\|\varphi_1|_{G_1''}\| \ge \frac{1}{2}$ and $Q_M(x_2) = 1$, where $x_2 = x_1|_{G_1'} = x|_{G_1'}$. Moreover,

$$\varphi_2(x) = \varphi_2(x|_{G_1'}) = \varphi_2(x_1|_{G_1'}) = \varphi_2(x_2) = 1.$$

Now, using induction, repeating the same argumentation, we get sequences $\{\varphi_n\}_{n=1}^{\infty} \subset F$ and $\{G_n\}_{n=0}^{\infty} \subset G$ with $G \supset G_0 \supset G_1 \supset G_2 \supset \dots$, $G_n = G_n' \cup G_n''$, $G_n' \cap G_n'' = \emptyset$, $\mu G_n' = \mu G_n''$, $G_{n+1} = G_n'$, $\varphi_n = \varphi_n|_{G_n}$, $\|\varphi_n|_{G_n''}\| \ge \frac{1}{2}$ and $1 = \|\varphi_n\| = \varphi_n(x)$, $n = 0, 1, 2, \dots$. For $m > n$, noticing $G_m \subset G_{n+1} = G_n'$, we have

$$\|\varphi_n - \varphi_m\| \ge \|(\varphi_n - \varphi_m)|_{G_n''}\| = \|\varphi_n|_{G_n''}\| \ge \frac{1}{2}\,.$$

Since m, n can be arbitrary, this implies that $\{\varphi_n\}$ is not compact. This contradiction shows that $Q_M(x) < 1$.

Let us now prove (ii). Assume, on the contrary, that (ii) does not hold. Since the set of the discontinuous points of $p(u)$ is at most countable, there exist a point r of discontinuity of $p(u)$ such that

$$\mu F^0 = \mu\{t \in G : |x(t)| = r\} > 0.$$

Without loss of generality, we can assume that $x(t) \ge 0$. By (4), $Q_M(x) < 1 \Rightarrow \nabla_x \subset S(L_N)$. Thus, by (3) we have $\varrho_N(p_-(x(t))) < +\infty$. Take a partition of F^0 into disjoint subsets F_1^1, F_2^1 of equal measure. Divide F_1^1 and F_2^1 into disjoint, equi-measure subsets F_1^2, F_2^2 and F_3^2, F_4^2, respectively. Continue this process, generally, divide F_i^{n-1} into disjoint, equi-measure subsets F_{2i-1}^n, F_{2i}^n, ($n = 1, 2, \dots,\ i = 1, 2, \dots 2^{n-1}$). Denote

$$z_n(t) = p_-(x(t))\Big|_{G\setminus F^0} + p_-(r)\Big|_{\bigcup_{i=1}^{2^{n-1}} F_{2i-1}^n} + p(r)\Big|_{\bigcup_{i=1}^{2^{n-1}} F_{2i-1}^n} \quad (n = 1, 2, \dots).$$

Obviously, $z_n(t) \in L_N$ and $\|z_n\|_N \equiv k$ $(n = 1, 2, \dots)$ for some constant k. Denoting

$$y_n = \frac{1}{k} z_n,$$

we have $y_n \in S(L_N)$, $n = 1, 2, \dots$. Moreover,

$$1 \geq \langle x, y_n \rangle = \frac{1}{k} \langle x, z_n \rangle = \frac{1}{k}(\varrho_M(x) + \varrho_N(z_n)) = \frac{1}{k}(1 + \varrho_N(ky_n)) \geq \|y_n\|_N \geq 1.$$

This implies that $\langle x, y_n \rangle = 1$, i.e. $y_n \in \nabla_x$, $n = 1, 2, \dots$. Obviously, we have

$$\|y_n - y_m\|_N = \frac{1}{k}(p(r) - p_-(r)) \frac{\mu F^0}{2} M^{-1} \left(\frac{2}{\mu F^0} \right), \quad m \neq n.$$

Hence $\{y_n\}_{n=1}^{\infty}$ is not compact and this contradiction completes the proof of Theorem 1. $\square$

Corollary 1. *$L_{(M)}$ is nearly smooth iff $M \in \overline{\Delta}_2$ and $p(u)$ is continuous on $[0, +\infty)$.*

Theorem 2. *$x \in S(L_M)$ is a nearly smooth point of $L_{(M)}$ iff $\varrho_N(p_-(kx)) = 1$ or $Q_M(kx) < 1$ and $\varrho_N(p(kx)) = 1$, for some $k \in K_M(x)$.*

Proof: Sufficiency. Follows from Theorem 2.5 in [5].

Necessity. Without loss of generality, assume that $x(t) \geq 0$. Suppose that the necessity condition of the theorem does not hold. Only the following two cases need to be considered.

(I) $\varrho_N(p_-(kx)) < 1, \quad Q_M(kx) = 1.$

From $Q_M(kx) = 1$, repeating the same argumentation as in the proof of the necessity of condition (i) in Theorem 1, we obtain a sequence $\{\varphi_n\}_{n=0}^{\infty} \subset F$ with $1 = \|\varphi_n\| = \varphi_n(kx) = Q_M(kx)$ and

$$\|\varphi_n - \varphi_m\| \geq \frac{1}{2}, \quad m \neq n, \ m, n = 1, 2, \dots .$$

Denote $f_n = (p_-(kx) + (1 - \varrho_N(p_-(kx)))\varphi_n$. Since $\varrho_N(p_-(kx)) + \|(1 - \varrho_N(p_-(kx)))\varphi_n\| = 1$, $(1 - \varrho_N(p_-(kx)))\varphi_n(kx) = \|(1 - \varrho_N(p_-(kx)))\varphi_n\|$ and $p_-(kx(t)) \leq p(kx(t))$. By (5), $\{f_n\} \subset \nabla_x$ and $\|f_m - f_n\| \geq \frac{1}{2}(1 - \varrho_N(p_-(kx)))$, $m \neq n$, $m, n = 1, 2, \dots$. Hence ∇_x is not compact.

(II) $\varrho_N(p_-(kx)) < 1 < \varrho_N(p(kx))$.

Denote $F_i = \{t \in G : kx(t) = r_i\}$, $i = 1, 2, \dots$, where $\{r_i\}$ is the set of all of discontinuity points of $p(u)$. Take a partition of F_i into F_i', F_i'' in such a way that

$$y(t) = p_-(kx(t))\Big|_{G \setminus \bigcup_i F_i} + \sum_{i=1}^{\infty} \left(p_-(r_i)\Big|_{F_i'} + p(r_i)\Big|_{F_i''} \right) \ \text{with} \ \ \varrho_N(y) = 1.$$

Without loss of generality we may assume that $\mu F_1' \geq \mu F_1'' > 0$. Take $E^0 \subset F_1$ with $\mu E^0 = 2F_1''$.

For E^0, repeat the same partition as in the proof of the necessity of condition (ii) in Theorem 1 and define

$$y_n(t) = y(t)\Big|_{G\setminus E^0} + p_-(r_1)\Big|_{\bigcup_{i=1}^{2^{n-1}} E^n_{2i-1}} + p(r_1)\Big|_{\bigcup_{i=1}^{2^{n-1}} E^n_{2i}}.$$

Obviously, $\varrho_N(y_n) = \varrho_N(y) = 1$, whence $\{y_n\} \subset \nabla_x$, $n = 1, 2, 3, \dots$. Moreover, we have

$$\|y_m - y_n\|_{(N)} = (p(r_1) - p_-(r_1))\frac{1}{N^{-1}\left(\frac{2}{\mu E^0}\right)} \quad \text{whenever} \quad m \neq n,\ m, n = 1, 2, \dots .$$

Hence ∇_x is not compact. This contradiction completes the proof. $\square$

Corollary 2. *L_M is nearly smooth iff $M \in \overline{\Delta}_2$ and $p(u)$ is continuous on $[0, +\infty)$.*

For Orlicz function spaces, we know that smoothness and near smoothness are equivalent. But for Orlicz sequence spaces, these properties differ very much.

Theorem 3. *$x \in S(l_{(M)})$ is a nearly smooth point of $l_{(M)}$ iff* (i) $Q_M(x) < 1$, (ii) $Q_N(w) = 0$, *where* $w = (p(|x(j)|) - p_-(|x(j)|))_{j=1}^{\infty} \in m$.

Proof: Necessity. Denote $x^0 = x$. If (i) is not true, then $Q_M(x^0) = 1$. Take $k_0' = 0$. Since $\sum_{i \geq k_0'} M((1+1)x^0(i)) = \infty$, there exists $k_1 > k_0'$ such that $\sum_{i=k_0'+1}^{k_1} M((1+1)x^0(i)) \geq 1$. By $\sum_{i>k_1} M((1+1)x^0(i)) = \infty$, we can choose $k_1' > k_1$ such that $\sum_{i=k_1+1}^{k_1'} M((1+1)x^0(i)) \geq 1$. Since $\sum_{i>k_1'}^{\infty} M((1+\frac{1}{2})x^0(i)) = \infty$, we can find $k_2 > k_1'$ such that $\sum_{i=k_1'+1}^{k_2} M((1+\frac{1}{2})x^0(i)) \geq 1$. By $\sum_{i>k_2}^{\infty} M((1+\frac{1}{2})x^0(i)) = \infty$, we can take $k_2' > k_2$ such that $\sum_{i=k_2+1}^{k_2'} M((1+\frac{1}{2})x^0(i)) \geq 1$. Using induction, we can get a sequence $k_0 < k_1 < k_1' < k_2 < k_2' < \dots$, satisfying

$$\sum_{i=k_{n-1}'+1}^{k_n} M\Big(\Big(1+\frac{1}{n}\Big)x^0(i)\Big) \geq 1, \quad \sum_{i=k_n+1}^{k_n'} M\Big(\Big(1+\frac{1}{n}\Big)x^0(i)\Big) \geq 1, \ (n = 1, 2, \dots).$$

Put

$$x_1^0(i) = \begin{cases} x^0(i), & k_n' < i \leq k_{n+1} \ (n = 0, 1, 2, \dots), \\ 0 & \text{otherwise} \end{cases}$$

and

$$x_2^0(i) = \begin{cases} x^0(i), & k_n' < i \leq k_n' \ (n = 1, 2, \dots), \\ 0 & \text{otherwise.} \end{cases}$$

Then $x^0 = x_1^0 + x_2^0$, supp $x_1^0 \cap$ supp $x_2^0 = \phi$, and for any $\tau > 0$

$$\varrho_M((1+\tau)x_2^0) = \sum_{n=1}^{\infty} \sum_{i=k_n+1}^{k'_n} M\Big(\Big(1+\frac{1}{n}\Big)x(i)\Big) = \infty.$$

So $Q_M(x_2^0) = 1$. For the same reason, $Q_M(x_1^0) = 1$. By **(1)**, we can find $\varphi_0 \in F$ with $1 = \|\varphi_0\| = \varphi_0(x) = Q_M(x)$. Noticing that $1 = \|\varphi_0\| = \|\varphi_0|_{\text{supp } x_1^0}\| + \|\varphi_0|_{\text{supp } x_2^0}\|$, we can assume that $\|\varphi_0|_{\text{supp } x_2^0}\| \geq \frac{1}{2}$.

Denoting $x_1 = x_1^0$, we get $Q_M(x_1) = Q_M(x_1^0) = 1$. Repeating the same argumentation, we can get $x_1^1, x_2^1 \in l_{(M)}$ such that $x_1 = x_1^1 + x_2^1$, supp $x_1^1 \cap$ supp $x_2^1 = \phi$ and $Q_M(x_1^1) = Q_M(x_2^1) = 1$. Take $\varphi_1 \in \nabla_{x_1}$ such that $\varphi_1 = \varphi_1|_{\text{supp } x_1}$, $\varphi_1(x_1) = Q_M(x_1) = 1$. Then $\varphi_1(x) = \varphi_1(x|_{\text{supp } x_1}) = \varphi_1(x_1^0) = \varphi_1(x_1) = 1$, and so $\varphi_1 \in \nabla_x$.

Since $1 = \|\varphi_1\| = \|\varphi_1|_{\text{supp } x_1^1}\| + \|\varphi_1|_{\text{supp } x_2^1}\|$, we can assume that $\|\varphi_1|_{\text{supp } x'_2}\| \geq \frac{1}{2}$. Denoting $x^2 = x_1^1$, and repeating the same argumentation by induction, one can find a sequence $\{\varphi_n\}_{n=0}^{\infty} \subset F$ and a sequence $\{x^n\}_{n=1}^{\infty} \in l_{(M)}$ such that $x^n = x_1^n + x_2^n$, $x^n = x_1^{n-1}$, supp $x_1^n \cap$ supp $x_2^n = \phi$, $\varphi_n = \varphi_n|_{\text{supp } x^n} = \varphi_n|_{\text{supp } x_1^{n-1}}$, $\|\varphi_n|_{\text{supp } x_2^n}\| \geq \frac{1}{2}$, $\varphi_n \in \nabla_x$, $n = 1, 2, \dots$. It is easy to verify that

$$\|\varphi_m - \varphi_n\| \geq \|(\varphi_m - \varphi_n)|_{\text{supp } x_2^n}\| = \|\varphi_n|_{\text{supp } x_2^n}\| \geq \frac{1}{2} \quad (m > n,\ n = 1, 2, \dots),$$

i.e. $\{\varphi_n\}_{n=1}^{\infty}$ is not compact.

Now we prove that (ii) is necessary. Otherwise, there exists $\varepsilon_0 > 0$ such that $Q_N(w) > \varepsilon_0$. Take a sequence $\{m_n\}$ with $m_1 < m_2 < m_3 < \dots$ of positive integers such that

$$\|(0, 0, \dots, 0, w(m_n+1), \dots, w(m_{n+1}), 0, \dots)\|_N > \varepsilon_0, \quad (n = 1, 2, \dots).$$

Put

$$\begin{aligned} z_n &= (p_-(x(1)), \dots, p_-(x(m_n)), p(x(m_n+1)), \dots, \\ &\qquad p(x(m_{n+1})), p_-(x(m_{n+1}+1)), \dots) \\ k_n &= \|z_n\|_N \quad (n =, 2, \dots). \end{aligned}$$

It is easy to prove that $y_n = \frac{z_n}{k_n} \in \nabla_x$ and $k_n = K_N(y_n)$ $n = 1, 2, \dots$. From $Q_M(x) < 1$ we know that there exist $\tau > 0$ such that $\varrho_M((1+\tau)x) < \infty$. By the inequality

$$M((1+\tau)u) \geq \int_u^{(1+\tau)u} p(s)\,ds \geq \tau u p(u) \geq \tau N(p(u)),$$

we have $\varrho_N(p(x)) < \infty$. By $k_n = \|z_n\|_N = 1 + \varrho_N(k_n y_n) \le 1 + \varrho_N(p(x))$, $\{k_n\}$ is a bounded set. So we can assume that $\lim_{n\to\infty} k_n = k$ (if necessary, we can choose a convergent subsequence of $\{k_n\}$ and denote it still by $\{k_n\}$). Therefore, for $n \ne t$,

$$\begin{aligned}
\|y_n - y_t\|_N &\ge \Big\|\Big(0,\dots,0,\frac{p(x(m_n+1))}{k_n} - \frac{p_-(x(m_n+1))}{k_t}\cdots,\\
&\quad \frac{p(x(m_{n+1}))}{k_n} - \frac{p_-(x(m_{n+1}))}{k_t},0,\dots\Big)\Big\|_N\\
&\ge \Big\|\Big(0,0,\dots,0,\frac{[p(x(m_n+1)) - p_-(x(m_n+1))]}{k_n}\cdots,\\
&\quad \frac{[p(x(m_{n+1})) - p_-(x(m_{n+1}))]}{k_n},0,\cdots\Big)\Big\|_N\\
&\quad - \Big|\frac{1}{k_n} - \frac{1}{k_t}\Big|\,\big\|(0,0,\dots,0,p_-(x(m_n+1)),\dots,p_-(x(m_{n+1})),0,\dots)\big\|_N\\
&\ge \frac{\varepsilon_0}{k_n} - \Big|\frac{1}{k_n} - \frac{1}{k_t}\Big|\|p_-(x)\|_N \to \frac{\varepsilon_0}{k} \quad (n,t\to\infty),
\end{aligned}$$

i.e. ∇_x is not compact. This contradiction completes the proof of the necessity.

Sufficiency. From the condition (i) $Q_M(x) < 1$, we have $\nabla_x \subset l_N$. Taking any sequence $\{y_n\} \subset \nabla_x$ with $k_n \in K_N(y_n)$, we have $k_n = 1 + \varrho_N(k_n y_n) \le 1 + \varrho_N(p(x)) < \infty$. So we can assume that $k_n \to k$ (if necessary, we can take a convergent subsequence of $\{y_n\}$). Using the diagonal method, we can get a subsequence of $\{y_n\}$ (still denoted by $\{y_n\}$) such that

$$\lim_{n\to\infty} y_n(j) = y(j) \quad (j = 1,2,\dots).$$

Denote $e_j = (0,\dots,0,\overset{j}{1},0,\dots)$, $j = 1,2,\dots$. By (ii), for any $\varepsilon > 0$, there exists j_0 such that

$$\Big\|\sum_{j>j_0}(p(x(j)) - p_-(x(j)))e_j\Big\|_N < \frac{\varepsilon}{2k}.$$

For m, n large enough, we have

$$\begin{aligned}
\|y_n - y_m\|_N &\le \Big\|\sum_{j=1}^{j_0}(y_n(j) - y_m(j))e_j\Big\|_N + \Big\|\sum_{j>j_0}(y_n(j) - y_m(j))e_j\Big\|_N\\
&< \frac{\varepsilon}{2} + \Big\|\sum_{j>j_0}\Big|\frac{p(x(j))}{k_n} - \frac{p_-(x(j))}{k_m}\Big|e_j\Big\|_N\\
&\to \frac{\varepsilon}{2} + \frac{1}{k}\Big\|\sum_{j>j_0}(p(x(j)) - p_-(x(j)))e_j\Big\|_N < \varepsilon,
\end{aligned}$$

i.e. $\{y_n\}$ is a Cauchy sequence in ∇_x. Hence, ∇_X is compact □

For the Orlicz sequence space $l_{(M)}$, the conditions for the near smoothness and smoothness are different.

Corollary 3. *$l_{(M)}$ is nearly smooth iff:* (i) *$M \in \Delta_2$;*

(ii)

$$\overline{\lim_{u\to 0}} \frac{N(\lambda[p(u) - p_-(u)])}{M(u)} < \infty \quad \textit{for any} \quad \lambda > 1.$$

Proof: Sufficiency. For $x \in S(l_{(M)})$, from $M \in \Delta_2$ we know that $Q_M(x) = 0 < 1$. For any $\lambda > 0$, by condition (ii), $\varrho_N(\lambda(p(x) - p_-(x))) < \infty$. Thus, $Q_N(w) = 0$. Hence, by Theorem 3, we deduce that x is a nearly smooth point.

Necessity. If (i) is not true, there exists $x \in S(l_{(M)})$ with $Q_N(x) = 1$. By Theorem 3, x is not a nearly smooth point. If (ii) is not true, there exists $\lambda > 1$ and a sequence $\{u_n\}$ with $u_n \downarrow 0$ and $N(\lambda(p(u_n) - p_-(u_n))) > 2^{n+1}M(u_n)$. Without loss of generality, we can assume that $M(u_n) < \frac{1}{2^{n+1}}$. Take a sequence $\{m_n\}$ with $\frac{1}{2^{n+1}} < m_n M(u_n) \le \frac{1}{2^n}$ and $u_0 \ge 0$ with $M(u_0) + \sum_{n=1}^{\infty} m_n M(u_n) = 1$. Put

$$x = \Big(u_0, \overbrace{u_1, \dots, u_1}^{m_1}, \overbrace{u_2, \dots, u_2}^{m_2}, \dots\Big).$$

Then $\varrho_M(x) = 1$ and $\|x\|_{(M)} = 1$. On the other hand,

$$\varrho_N(\lambda w) \ge \sum_{n=1}^{\infty} m_n N(\lambda(p(u_n) - p_-(u_n))) \ge \sum_{n=1}^{\infty} m_n 2^{n+1} M(u_n) = \infty.$$

This implies that $Q_N(w) \ge \frac{1}{\lambda} > 0$. By Theorem 3, x is not a nearly smooth point. $\square$

Theorem 4. *$x \in S(l_M)$ is a nearly smooth point iff:* (i) *$\varrho_N(p_-(kx)) = 1$ or* (ii) *$Q_M(kx) < 1$ and $\varrho_N(p(kx)) = 1$ or* (iii) *$Q_M(kx) < 1$ and $Q_N(w) = 0$, where $k \in K_M(x)$ and $w = (p(k|x(1)|) - p_-(k|x(1)|), p(k|x(2)|) - p_-(k|x(2)|), \dots)$.*

Proof: Sufficiency. It is enough to deal with the following three cases:

(I) $\varrho_N(p_-(kx)) = 1$;
(II) $Q_M(kx) < 1$, $\varrho_N(p(kx)) = 1$;
(III) $Q_M(kx) < 1$ and $Q_N(w) = 0$.

For (I) or (II), in view of Theorem 2.55 in [5], x is a smooth point and also a nearly smooth point. For case (III), we can proceed analogously as in the proof of the sufficiency of Theorem 3.

Necessity. If $\varrho_N(p_-(kx)) < 1$ and $Q_M(kx) = 1$, repeating the proof of the necessity of the first case in Theorem 2, we get a contradiction. So, if the condition in theorem were not necessary, then

$$\varrho_N(p_-(kx)) < 1, \ Q_M(kx) < 1, \ Q_N(w) > \varepsilon_0 > 0 \ \text{ and } \ \varrho_N(p(kx)) \ne 1.$$

But $Q(kx) < 1 \Rightarrow \nabla_x \subset l_{(N)}$, hence $\varrho_N(p(kx)) > 1$. Suppose, without loss of generality, that $x(j) \ge 0$ $(j = 1, 2, \dots)$. In view of the fact that $Q_M(kx) < 1 \Rightarrow$

$\varrho_N(p(kx)) < \infty$, we can take $j_0', j_0'' > 0$ such that $\varrho_N(p_-(kx)) + \sum_{j>j_0'} N(p(kx(j))) < 1$, $\sum_{j=1}^{j_0''} N(p(kx(j))) > 1$. Denoting $j_0 = \max\{j_0', j_0''\}$, we have

$$\sum_{j=1}^{j_0} N(p(k|x(j)|)) + \sum_{j>j_0} N(p_-(kx(j))) > 1,$$

$$\sum_{j=1}^{j_0} N(p_-(kx(j))) + \sum_{j>j_0} N(p(kx(j))) < 1.$$

By $Q_N(w) > \varepsilon_0$, we can find a sequence $\{j_n\} \uparrow \infty$ with $j_1 > j_0$, satisfying

$$\Big\| \sum_{j=j_n+1}^{j_{n+1}} w(j)e_j \Big\|_{(N)} > \varepsilon_0 \quad (n = 1, 2, \dots).$$

Define $\{y_n\}$ by:
$y_n(j) = p(kx(j))$ $(j_n < j \le j_{n+1})$, $y_n(j) = p_-(kx(j))$ $(j_0 < j \le j_n$ or $j > j_{n+1})$ and $p_-(kx(j)) \le y_n(j) \le p(kx(j))$ $(1 \le j \le j_0)$, such that $\varrho_N(y_n) = 1$, $n = 1, 2, \dots$. Then obviously $\{y_n\} \subset \nabla_x$. For $m \ne n$, $m, n = 1, 2, \dots$, we have

$$\|y_m - y_n\|_{(N)} \ge \Big\| \sum_{j=j_n+1}^{j_{n+1}} (p(kx(j)) - p_-(kx(j)))e_j \Big\|_{(N)} > \varepsilon_0,$$

i.e. $\{y_n\} \subset \nabla_x$ is not compact. This contradiction completes the proof of the theorem. □

Corollary 4. *l_M is nearly smooth iff:* (i) $M \in \Delta_2$*;*

(ii)
$$\overline{\lim_{u\to 0}} \frac{N(\lambda(p(u) - p_-(u)))}{up(u)} < \infty \quad \textit{for any} \quad \lambda > 0.$$

Proof: Sufficiency. For $x \in S(l_M)$, $M \in \Delta_2$ implies $Q_M(kx) = 0 < 1$ and for any $\lambda > 1$, we have, by the assumption,

$$\begin{aligned}\varrho_N(\lambda w) &= \sum_{i=1}^{\infty} N(\lambda(p(kx(i)) - p_-(kx(i)))) \le D \sum_{i=1}^{\infty} k|x(i)|p(k|x(i)|) \\ &\le D(\varrho_M(kx) + \varrho_N(p(kx))) < \infty.\end{aligned}$$

Hence $Q_N(w) = 0$. By Theorem 4, we conclude that x is a nearly smooth point.

Necessity. If $M \notin \Delta_2$, it is easy to construct $x \in S(l_M)$ with $\varrho_N(p(kx)) < 1$ and $Q_M(kx) = 1$, where $k \in K_M(x)$. By Theorem 4, x is not a nearly smooth point.

If (ii) were not true, there would exist $\lambda > 1$ and $u_n \downarrow 0$ such that

$$N(\lambda(p(u_n) - p_-(u_n))) > 2^{n+1} u_n p(u_n),$$

where u_n are the points of discontinuity of $p(u)$. Without loss of generality, we can assume that

$$u_n p(u_n) < \frac{1}{2^{n+1}}\,.$$

Take $\{m_n\}$ with $\frac{1}{2^{n+1}} < m_n u_n p(u_n) \le \frac{1}{2^n}$, $n = 1, 2, \ldots$. Then

$$\sum_{i=1}^{\infty} m_n M(u_n) < 1, \quad \sum_{i=1}^{\infty} m_n N(p(u_n)) < 1.$$

Denote

$$u_0 = Sup\Big\{s > 0 : N(p(s)) + m_1 N(p_-(u_1)) + \sum_{n\ge 2} m_n N(p(u_n)) \le 1\Big\},$$

$$k = 1 + M(u_0) + \sum_{i=1}^{\infty} m_n M(u_n),$$

$$x = \frac{1}{k}\Big(u_0, \overbrace{u_1, \ldots, u_1}^{m_1}, \overbrace{u_1, \ldots, u_1}^{m_2}, \ldots\Big).$$

Noticing that $\varrho_N(p_-(kx)) < 1$, $\varrho_N(p(kx)) > 1$, we have $k \in K_M(x)$. So

$$\|x\|_M = \frac{1}{k}(1 + \varrho_M(kx)) = 1.$$

On the other hand,

$$\varrho_N(\lambda w) > \sum_{n\ge 2} m_n N(\lambda(p(u_n) - p(u_n))) > \sum_{n\ge 2} m_n 2^{n+1} u_n p(u_n) = \infty,$$

i.e. $Q_N(w) \ge \frac{1}{\lambda} > 0$. By Theorem 4, x is not a nearly smooth point. □

Remark. If $M \in \Delta_2$, then there exists $D > 0$ such that

$$M(u) \le u p(u) \le D M(u).$$

So, the conditions of Corollary 3 and Corollary 4 are equivalent. This shows that the near smoothness of $l_{(M)}$ and l_M are equivalent. But recall that the smoothness of $l_{(M)}$ and l_M are not equivalent (see [5] and [7]).

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