Classifications and characterizations of Baire-1 functions

P. Kiriakouli

Abstract. Kechris and Louveau in [5] classified the bounded Baire-1 functions, which are defined on a compact metric space K, to the subclasses $\mathcal{B}_1^\xi(K)$, $\xi < \omega_1$. In [8], for every ordinal $\xi < \omega_1$ we define a new type of convergence for sequences of real-valued functions (ξ-uniformly pointwise) which is between uniform and pointwise convergence. In this paper using this type of convergence we obtain a classification of pointwise convergent sequences of continuous real-valued functions defined on a compact metric space K, and also we give a characterization of the classes $\mathcal{B}_1^\xi(K)$, $1 \leq \xi < \omega_1$.

Keywords: Baire-1 functions, convergence index, oscillation index, trees
Classification: 46E99, 54C30, 54C35, 54C50

1. Introduction

By N we mean the set of all natural numbers (i.e. $\mathsf{N} = \{1, 2, \dots, \}$), by ω we mean the first infinite ordinal (i.e. $\omega = \{0, 1, 2, \dots\}$) and by ω_1 we mean the first uncountable ordinal. Mercourakis in [10] introduced a new type of pointwise convergence (uniformly pointwise convergence) which is weaker than uniform convergence and stronger than pointwise convergence. Also Mercourakis in [11] extended this convergence with the definition of m-uniformly pointwise convergence for every $1 \leq m < \omega$. For $m = 1$ this convergence coincides with uniformly pointwise convergence and $(m+1)$-uniformly pointwise convergence is weaker than m-uniformly pointwise convergence and stronger than pointwise convergence. Also in [11] it has been proved that if a sequence (f_k) of continuous real-valued functions converges m-uniformly pointwise to some function f then f is also continuous.

In [8], for every ordinal $\xi < \omega_1$ we define a new type of pointwise convergence (ξ-uniformly pointwise) which extends the definition of the above convergence. Using this convergence we obtain a complete classification of all pointwise convergent sequences of continuous real-valued functions defined on a countably compact space for which the limit function is continuous. An equivalent definition of ξ-uniformly pointwise convergence is given in [9].

In this paper, by the aid of this convergence we obtain a classification of pointwise convergent sequences of continuous real-valued functions defined on a compact metric space for which the limit function is a general function not necessarily continuous and also we obtain a characterization of some subclasses of bounded Baire-1 functions. These results are described in the following way.

Kechris and Louveau in [5] defined the convergence index "γ" of a sequence of continuous real-valued functions. We prove that if K is a compact metric space, (f_k) a sequence of continuous real-valued functions on K and $1 \leq \xi < \omega_1$ then the following are equivalent: (i) for every strictly increasing sequence (k_n) of natural numbers there exists a subsequence $(k_n^{'})$ of (k_n) so that $\gamma((f_{k_n^{'}})) \leq \omega^{\xi}$; (ii) for every strictly increasing sequence (k_n) of natural numbers there exists a subsequence $(k_n^{'})$ of (k_n) so that the sequence $(f_{k_{n+1}^{'}} - f_{k_n^{'}})$ converges ξ-uniformly pointwise to zero; (iii) for every strictly increasing sequence (k_n) of natural numbers there exists a subsequence $(k_n^{'})$ of (k_n) so that the sequence $(f_{k_{2n+1}^{'}} - f_{k_{2n}^{'}})$ converges ξ-uniformly pointwise to zero (cf. Theorem 3.5).

Also Kechris and Louveau in [5] classified the bounded Baire-1 functions, which are defined on a compact metric space K, to the subclasses $\mathcal{B}_1^{\xi}(K)$, $\xi < \omega_1$. Using Theorem 3.5 we get the following: If K is a compact metric space, $1 \leq \xi < \omega_1$ and f is a bounded real-valued function on K then the next conditions are equivalent: (i) $f \in \mathcal{B}_1^{\xi}(K)$; (ii) there exists a sequence (f_k) of continuous real-valued functions defined on K which converges pointwise to f and for every strictly increasing sequence (k_n) of natural numbers there is a subsequence $(k_n^{'})$ of (k_n) so that the sequence $(f_{k_{n+1}^{'}} - f_{k_n^{'}})$ converges ξ-uniformly pointwise to zero; (iii) there exists a sequence (f_k) of continuous real-valued functions defined on K which converges pointwise to f and for every strictly increasing sequence (k_n) of natural numbers there is a subsequence $(k_n^{'})$ of (k_n) so that the sequence $(f_{k_{2n+1}^{'}} - f_{k_{2n}^{'}})$ converges ξ-uniformly pointwise to zero (cf. Theorem 3.6).

2. Preliminaries

A real-valued function f defined on a set X is bounded if $\|f\|_\infty := \sup_{x \in X} |f(x)| < +\infty$. A sequence (f_k) of real-valued functions defined on a set X is uniformly bounded if $\sup_k \|f_k\|_\infty < +\infty$.

Let K be a compact metric space and $C(K)$ the set of continuous real-valued functions on K. By $\mathbf{R}$ we mean the set of all real numbers. A function $f : K \to \mathbf{R}$ is Baire-1 if there exists a sequence (f_k) in $C(K)$ that converges pointwise to f. Let $\mathcal{B}_1(K)$ be the set of all bounded Baire-1 real-valued functions on K. Haydon, Odell and Rosenthal in [4], Kechris and Louveau in [5] defined the oscillation index $\beta(f)$ of a general function $f : K \to \mathbf{R}$ and proved that f is Baire-1 iff $\beta(f) < \omega_1$.

Definition 2.1 (cf. [4], [5]). Let K be a compact metric space, $f : K \to \mathbf{R}$, $P \subseteq K$ and $\epsilon > 0$. Let $P_{\epsilon,f}^0 = P$ and for any ordinal α let $P_{\epsilon,f}^{\alpha+1}$ be the set of those $x \in P_{\epsilon,f}^{\alpha}$ such that for every open set U around x there are two points x_1 and x_2 in $P_{\epsilon,f}^{\alpha} \cap U$ such that $|f(x_1) - f(x_2)| \geq \epsilon$.

At a limit ordinal α we set $P_{\epsilon,f}^{\alpha} = \bigcap_{\beta<\alpha} P_{\epsilon,f}^{\beta}$.

Let $\beta(f,\epsilon)$ be the least α with $K^{\alpha}_{\epsilon,f}=\emptyset$ if such an α exists, and $\beta(f,\epsilon)=\omega_1$, otherwise. Define the oscillation index $\beta(f)$ of f by

$$\beta(f)=\sup\{\beta(f,\epsilon):\epsilon>0\}.$$

For every $\xi<\omega_1$ we define $\mathcal{B}_1^{\xi}(K)=\{f\in\mathcal{B}_1(K):\beta(f)\leq\omega^{\xi}\}$.

(The class $\mathcal{B}_1^1(K)$ has been studied also by Haydon, Odell and Rosenthal in [4], where it is denoted by $\mathcal{B}_{1/2}(K)$.)

The complexity of pointwise convergent sequences of continuous real-valued functions defined on a compact metric space is described by a countable ordinal index "γ" which is defined in the following way.

Definition 2.2 (cf. [5]). Let K be a compact metric space, (f_k) a sequence of continuous real-valued functions defined on K, $P\subseteq K$ and $\epsilon>0$. Let $P^0_{\epsilon,(f_k)}=P$ and for any ordinal α let $P^{\alpha+1}_{\epsilon,(f_k)}$ be the set of those $x\in P^{\alpha}_{\epsilon,(f_k)}$ such that for every open set U around x and for every $p\in\mathsf{N}$ there are $m,n\in\mathsf{N}$ with $m>n>p$ and a point x' in $P^{\alpha}_{\epsilon,(f_k)}\cap U$ such that $|f_m(x')-f_n(x')|\geq\epsilon$.

At a limit ordinal α we set $P^{\alpha}_{\epsilon,(f_k)}=\bigcap_{\beta<\alpha}P^{\beta}_{\epsilon,(f_k)}$. (It can be noticed that $P^{\alpha}_{\epsilon,(f_k)}$ is a closed subset of P with the relative topology in P.) Let $\gamma((f_k),\epsilon)$ be the least α with $K^{\alpha}_{\epsilon,(f_k)}=\emptyset$ if such an α exists, and $\gamma((f_k),\epsilon)=\omega_1$, otherwise. (Notice that if $\gamma((f_k),\epsilon)<\omega_1$ then it is a successor ordinal.) Define the convergence index $\gamma((f_k))$ of (f_k) by

$$\gamma((f_k))=\sup\{\gamma((f_k),\epsilon):\epsilon>0\}.$$

Also in [5] it is proved that $\gamma((f_k))<\omega_1$ iff (f_k) is pointwise converging.

Generalized Schreier families.

Definition 2.3 (cf. [1]). If F and H are finite non-empty subsets of N and $n\in\mathsf{N}$, then we define $F<H$ iff $\max F<\min H$, $n\leq F$ iff $n\leq\min F$. Let $\mathcal{F}_0=\{\{n\}:n\in\mathsf{N}\}\cup\{\emptyset\}$ and $\mathcal{F}_1$ be the usual Schreier family, i.e. $\mathcal{F}_1=\{A\subseteq\mathsf{N}:|A|\leq\min A\}$. If $\mathcal{F}_\xi$ has been defined then we set

$$\mathcal{F}_{\xi+1}=\bigcup_{k=1}^{\infty}\left\{\cup_{i=1}^{k}F_i:F_1,\ldots,F_k\in\mathcal{F}_\xi\ \text{ with }\ k\leq F_1<\ldots<F_k\right\}.$$

If ξ is a limit ordinal with $\mathcal{F}_\zeta$ defined for each $\zeta<\xi$, choose and fix a strictly increasing sequence of ordinals (ξ_k) with $\xi=\sup_k\xi_k$ and let

$$\mathcal{F}_\xi=\bigcup_{k=1}^{\infty}\left\{F\in\mathcal{F}_{\xi_k}:\min F\geq k\right\}.$$

It can be noticed that the families $\mathcal{F}_m$, $1\leq m<\omega$ appeared for the first time in an example constructed by Alspach and Odell ([2]).

Notice that if (k_j) is a strictly increasing sequence of natural numbers, $\xi<\omega_1$ and $F\in\mathcal{F}_\xi$, then $\{k_j:j\in F\}\in\mathcal{F}_\xi$ (see also [12, Lemma 3.5]).

Trees.

Definition 2.4 (cf. [3])**.** Let X be a set. For every $n \in \mathsf{N}$ we set $X^n := \{(x_1, \dots, x_n) : x_1, \dots, x_n \in X\}$.

(i) A *tree* T on X will be a subset of $\bigcup_{n=1}^{\infty} X^n$ with the property that $(x_1, \dots, x_n) \in T$ whenever $(x_1, \dots, x_n, x_{n+1}) \in T$.

(ii) A tree T on X is *well-founded* if there is no sequence (x_n) in X satisfying $(x_1, \dots, x_n) \in T$ for each $n \in \mathsf{N}$.

(iii) Proceeding by induction we associate to each ordinal α a new tree T^α as follows: We set $T^0 = T$. If T^α is obtained, let $T^{\alpha+1} = \bigcup_{n=1}^{\infty} \{(x_1, \dots, x_n) \in T^\alpha : (x_1, \dots, x_n, x) \in T^\alpha \text{ for some } x \in X\}$.

If γ is a limit ordinal, define $T^\gamma = \bigcap_{\alpha<\gamma} T^\alpha$. If there exists an ordinal α so that $T^\alpha = \emptyset$ then we denote by $o(T)$ the smallest such ordinal α. This is the order of the tree T.

Definition 2.5 (see also [8])**.** Let X be a set, (f_k) a sequence of real-valued functions defined on X and δ a positive real number. We define the tree $T((f_k), \delta)$ on N as follows:

$T((f_k), \delta) = \{(1)\} \cup \bigcup_{n=1}^{\infty} \{(1, k_1, \dots, k_n) \in \mathsf{N}^{n+1} : 1 < k_1 < \dots < k_n$ and there exists $x \in X$ so that $|f_{k_i}(x)| > \delta$ for all $i = 1, 2, \dots, n\}$.

The δ-index $i((f_k), \delta)$ of the sequence (f_k) is the order of the tree $T((f_k), \delta)$, i.e. $i((f_k), \delta) = o(T((f_k), \delta))$. We notice that $i((f_k), \delta)$ is a successor ordinal.

The following result is included in [8]. For completeness we give an outline of the proof.

Lemma 2.6. *If $\xi < \omega_1$, (f_k) a sequence of real-valued functions on a set X and $\delta > 0$ such that for every $F \in \mathcal{F}_\xi$ there is $x \in X$ with $|f_k(x)| > \delta$ for every $k \in F$, then $(T((f_k), \delta))^{\omega^\xi} \neq \emptyset$.*

Proof: We use a technique developed by Prof. S. Negrepontis and the author (cf. [6] or [12, Definition 3.6, Lemma 3.7]). We apply this technique as follows: For any $n \in \mathsf{N}$, $\xi_1, \dots, \xi_n < \omega_1$ we say that the n-tuple $(\xi_1, \dots, \xi_n)$ has the property (I) if whenever (f_k) is a sequence of real-valued functions on a set X and $\delta > 0$ such that for every $F_1 \in \mathcal{F}_{\xi_1}, \dots, F_n \in \mathcal{F}_{\xi_n}$ with $F_1 < \dots < F_n$ there exists $x \in X$ with $|f_k(x)| > \delta$ for every $k \in \cup_{i=1}^n F_i$, then $(T((f_k), \delta))^{\omega^{\xi_n} + \dots + \omega^{\xi_1}} \neq \emptyset$.

(a) By induction on $\xi < \omega_1$ we prove that if $(\xi_1, \dots, \xi_n)$ has the property (I), then $(\xi, \xi_1, \dots, \xi_n)$ has the property (I).

For $\xi = 0$, it is proved by using the definition of the property (I) for the set $Y = \{x \in X : |f_2(x)| > \delta\}$ and for the functions $g_k := f_{2k+1}$, $k \in \mathsf{N}$.

(b) By induction on $\xi < \omega_1$ we prove that (ξ) has property (I) for every $\xi < \omega_1$.

[Indeed, it is obvious for $\xi = 0$ by using Definitions 2.4 (iii) and 2.5. Let $1 \leq \xi < \omega_1$ and assume that it is true for every $\zeta < \xi$. If $\xi = \zeta + 1$, we use that $(\underbrace{\zeta, \dots, \zeta}_{l-times})$ has the property (I) for all $l \in \mathsf{N}$, and we use the definition of the property (I) and Definitions 2.4 (iii) and 2.5. If ξ is a limit ordinal and (ξ_k) is the strictly increasing sequence with $\sup_k \xi_k = \xi$ that defines the family $\mathcal{F}_\xi$, we use that (ξ_l) has the property (I) for all $l \in \mathsf{N}$, and we use the definition of the property (I) and Definitions 2.4 (iii) and 2.5.] $\square$

Definition 2.7 (see [8])**.** Let (f_k) be a sequence of real-valued functions defined on a set X and $\xi < \omega_1$. We say that the sequence (f_k) converges **ξ-uniformly pointwise** on X to the function f if $i((f_k - f), \delta) \leq \omega^\xi$ for every positive real number δ.

We notice that for any $1 \leq m < \omega$ the above definition is equivalent to the definition of m-uniformly pointwise convergence which has been introduced by Mercourakis in [11] (cf. [8]). Also in [8] we prove the following: (i) If X is a topological space, $\xi < \omega_1$ and (f_k) a uniformly bounded sequence of continuous real-valued functions on X which converges ξ-uniformly pointwise to f, then f is also continuous. (ii) If X is a countably compact space (i.e. every infinite sequence (x_k) in X has an accumulation point in X) and (f_k) a sequence of continuous real-valued functions pointwise converging to some continuous function f on X, then there exists $\xi < \omega_1$ such that (f_k) converges ξ-uniformly pointwise to f.

3. Main results

In this section we shall study the complexity of pointwise converging sequences of continuous real-valued functions defined on a compact metric space (cf. Theorems 3.3 and 3.5) and also we shall prove a characterization of those bounded Baire-1 functions which have the oscillation index less than or equal to ω^ξ, where $1 \leq \xi < \omega_1$ (cf. Theorem 3.6).

Before we proceed to the proof of these results we need a few propositions which are proved by using the same technique, developed by Prof. S. Negrepontis and the author, which is used in the proof of Lemma 2.6. We start with the proposition:

Proposition 3.1. *Let K be a compact metric space, $\xi < \omega_1$ and (f_k) a sequence of continuous real-valued functions on K. Assume that there is $\epsilon > 0$ such that for every strictly increasing sequence (n_k) of natural numbers there exists a subsequence (n_k') of (n_k) so that for every $E = \{k_1 < \ldots < k_\lambda\} \in \mathcal{F}_\xi$, there exists $x \in K$ with $|f_{n'_{2k_j+1}}(x) - f_{n'_{2k_j}}(x)| > \epsilon$ for all $1 \leq j \leq \lambda$. Then $\gamma((f_{n_k}), \epsilon) > \omega^\xi$ for every strictly increasing sequence (n_k) of natural numbers.*

For the proof of this proposition we need the next definition and Lemmas 3.1.2 and 3.1.3.

Definition 3.1.1. For $n \in \mathsf{N}$ and $\xi_1, \ldots, \xi_n < \omega_1$ we say that the n-tuple $(\xi_1, \ldots, \xi_n)$ has **property** (Γ) if whenever K is a compact metric space, (f_k) a sequence of continuous real-valued functions on K, $\epsilon > 0$ and $m \in \mathsf{N}$ such that for all $E_1 \in \mathcal{F}_{\xi_1}, \ldots, E_n \in \mathcal{F}_{\xi_n}$ with $m \leq E_1 < \ldots < E_n$ and $\cup_{i=1}^n E_i = \{k_1 < \ldots < k_\lambda\}$

(where $\lambda \in \mathsf{N}$) there exists $x_{E_1\dots E_n} \in K$ with $|f_{2k_j+1}(x_{E_1\dots E_n}) - f_{2k_j}(x_{E_1\dots E_n})| > \epsilon$ for all $j = 1, \dots, \lambda$, then there exists a limit point x of the set $\{x_{E_1\dots E_n} : E_1 \in \mathcal{F}_{\xi_1}, \dots, E_n \in \mathcal{F}_{\xi_n}$ with $m \le E_1 < \dots < E_n\}$ in K such that $x \in K^{\omega^{\xi_n}+\dots+\omega^{\xi_1}}_{\epsilon,(f_k)}$.

Lemma 3.1.2. *If* $(\xi_1, \dots, \xi_n)$ *has property* (Γ) *then* $(\xi, \xi_1, \dots, \xi_n)$ *has property* (Γ) *for every* $\xi < \omega_1$.

Proof: We proceed by induction on $\xi < \omega_1$.

Case 1 ($\xi = 0$). Let $(\xi_1, \dots, \xi_n)$ have property (Γ), let K be a compact metric space, (f_j) a sequence of continuous real-valued functions on K, $\epsilon > 0$ and $m \in \mathsf{N}$ such that for each $k \in \mathsf{N}$, $E_1 \in \mathcal{F}_{\xi_1}, \dots, E_n \in \mathcal{F}_{\xi_n}$ with $m \le k < E_1 < \dots < E_n$ and $\{k\} \cup \cup_{i=1}^n E_i = \{k_1 < \dots < k_\lambda\}$ (where $\lambda \in \mathsf{N}$) there is $x_{kE_1\dots E_n} \in K$ with $|f_{2k_j+1}(x_{kE_1\dots E_n}) - f_{2k_j}(x_{kE_1\dots E_n})| > \epsilon$ for all $j = 1, \dots, \lambda$. We shall show that there exists a limit point x of the set $\{x_{kE_1\dots E_n} : k \in \mathsf{N}, E_1 \in \mathcal{F}_{\xi_1}, \dots, E_n \in \mathcal{F}_{\xi_n}$ with $m \le k < E_1 < \dots < E_n\}$ in K such that $x \in K^{\omega^{\xi_n}+\dots+\omega^{\xi_1}+1}_{\epsilon,(f_j)}$. Because $(\xi_1, \dots, \xi_n)$ has property (Γ) we have that for every $k \in \mathsf{N}$ with $k \ge m$, there is a limit point x_k of the set $A_k = \{x_{kE_1\dots E_n} : E_1 \in \mathcal{F}_{\xi_1}, \dots, E_n \in \mathcal{F}_{\xi_n}$ with $k < E_1 < \dots < E_n\}$ such that $x_k \in K^{\omega^{\xi_n}+\dots+\omega^{\xi_1}}_{\epsilon,(f_j)}$. Since $|f_{2k+1}(x_{kE_1\dots E_n}) - f_{2k}(x_{kE_1\dots E_n})| > \epsilon$ for all $x_{kE_1\dots E_n} \in A_k$ and $k \in \mathsf{N}$ with $k \ge m$, f_{2k+1}, f_{2k} are continuous and x_k is a limit point of A_k, we get that $|f_{2k+1}(x_k) - f_{2k}(x_k)| \ge \epsilon$ for all $k \in \mathsf{N}$ with $k \ge m$. Since K is a compact metric space there is a sequence $m \le k_1 < k_2 < \dots < k_i < k_{i+1} < \dots$ such that the sequence (x_{k_i}) converges to some x in $K^{\omega^{\xi_n}+\dots+\omega^{\xi_1}}_{\epsilon,(f_j)}$. Then $x \in K^{\omega^{\xi_n}+\dots+\omega^{\xi_1}+1}_{\epsilon,(f_j)}$.

[Indeed, if not, then there are an open neighborhood U of x in $K^{\omega^{\xi_n}+\dots+\omega^{\xi_1}}_{\epsilon,(f_j)}$ and $k_0 \in \mathsf{N}$ such that $|f_k(y) - f_\lambda(y)| < \epsilon$ for all $y \in U$ and $k, \lambda \in \mathsf{N}$ with $k, \lambda \ge k_0$. Since $\lim_{i\to\infty} x_{k_i} = x$ there exists $i \in \mathsf{N}$ so that $k_i > k_0$ and $x_{k_i} \in U$. Then $|f_{2k_i+1}(x_{k_i}) - f_{2k_i}(x_{k_i})| < \epsilon$, a contradiction.]

Thus the proof of Case 1 is complete.

Case 2 (isolated ordinals). Assume that the conclusion of our lemma is true for ξ; we shall show that it is true for $\xi+1$. Suppose that $(\xi_1, \dots, \xi_n)$ has property (Γ); then, for every k, the sequence $(\underbrace{\xi, \dots, \xi}_{k}, \xi_1, \dots, \xi_n)$ has property (Γ). We show that $(\xi+1, \xi_1, \dots, \xi_n)$ has property (Γ). Let K be a compact metric space, (f_k) a sequence of continuous real-valued functions on K, $\epsilon > 0$ and $m \in \mathsf{N}$ such that for each $E \in \mathcal{F}_{\xi+1}, E_1 \in \mathcal{F}_{\xi_1}, \dots, E_n \in \mathcal{F}_{\xi_n}$ with $m \le E < E_1 < \dots < E_n$ and $E \cup \cup_{i=1}^n E_i = \{k_1 < \dots < k_\lambda\}$ (where $\lambda \in \mathsf{N}$) there is a point $x_{EE_1\dots E_n}$ of K with

$$|f_{2k_j+1}(x_{EE_1\dots E_n}) - f_{2k_j}(x_{EE_1\dots E_n})| > \epsilon \quad \text{for all} \quad j = 1, 2, \dots, \lambda.$$

We shall show that there is a limit point x of the set

$$A = \{x_{EE_1\dots E_n} : E \in \mathcal{F}_{\xi+1}, E_1 \in \mathcal{F}_{\xi_1}, \dots, E_n \in \mathcal{F}_{\xi_n} \text{ with}$$

$$m \leq E < E_1 < \ldots < E_n\},$$

such that $x \in K^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+\omega^{\xi+1}}_{\epsilon,(f_j)}$. By Definition 2.3, $\mathcal{F}_{\xi+1} = \bigcup_{k=1}^{\infty} \mathcal{F}^{(k)}_{\xi}$ where $\mathcal{F}^{(k)}_{\xi} = \{\cup_{i=1}^{k} E_i : E_1, \ldots E_k \in \mathcal{F}_{\xi}$ with $k \leq E_1 < \ldots < E_k\}$ for all $k \in \mathsf{N}$. Then by inductive hypothesis, it follows that, for all $k \in \mathsf{N}$ with $k \geq m$, there is a limit point x_k of the set
$A_k = \{x_{EE_1\ldots E_n} : E \in \mathcal{F}^{(k)}_{\xi}, E_1 \in \mathcal{F}_{\xi_1}, \ldots , E_n \in \mathcal{F}_{\xi_n}$ with $E < E_1 < \ldots < E_n\}$, such that $x_k \in K^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+k\omega^{\xi}}_{\epsilon,(f_j)}$. Now, let x be a limit point of the set $\{x_k : k \in \mathsf{N}$ with $k \geq m\}$. Then $x \in \bigcap_{k=1}^{\infty} K^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+k\omega^{\xi}}_{\epsilon,(f_j)} \equiv K^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+\omega^{\xi+1}}_{\epsilon,(f_j)}$. Clearly x is a limit point of the set A. Thus the proof of Case 2 is complete.

Case 3 (limit ordinal ξ). Assume that the conclusion of our lemma is true for every $\zeta < \xi$. Let (ζ_k) be the strictly increasing sequence of ordinals with $\sup_k \zeta_k = \xi$ that defines the family $\mathcal{F}_{\xi}$. Let $(\xi_1, \ldots , \xi_n)$ have property (Γ). Let K be a compact metric space, (f_k) a sequence of continuous real-valued functions on K, $\epsilon > 0$ and $m \in \mathsf{N}$ such that for every $E \in \mathcal{F}_{\xi}, E_1 \in \mathcal{F}_{\xi_1}, \ldots , E_n \in \mathcal{F}_{\xi_n}$ with $m \leq E < E_1 < \ldots < E_n$ and $E \cup \cup_{i=1}^{n} E_i = \{k_1 < \ldots < k_\lambda\}$ (where $\lambda \in \mathsf{N}$) there exists $x_{EE_1\ldots E_n} \in K$ with $|f_{2k_j+1}(x_{EE_1,\ldots,E_n}) - f_{2k_j}(x_{EE_1\ldots E_n})| > \epsilon$ for all $j = 1, \ldots , \lambda - 1$. Since $\zeta_k < \xi$, also $(\zeta_k, \xi_1, \ldots , \xi_n)$ has property (Γ) for any $k \in \mathsf{N}$. Then for any $k \in \mathsf{N}$ with $k \geq m$, there is a limit point x_k of the set

$$A_k = \{x_{EE_1\ldots E_n} : E \in \mathcal{F}_{\zeta_k}, E_1 \in \mathcal{F}_{\xi_1}, \ldots , E_n \in \mathcal{F}_{\xi_n} \text{ with } k \leq E < E_1 < \ldots < E_n\},$$

such that $x_k \in K^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+\omega^{\zeta_k}}_{\epsilon,(f_j)}$. By compactness of K, it follows that the set $\{x_k : k \in \mathsf{N}$ with $k \geq m\}$ has a limit point x in K. Then $x \in K^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+\omega^{\xi}}_{\epsilon,(f_j)}$ and, also x is a limit point of the set
$\{x_{EE_1\ldots E_n} : E \in \mathcal{F}_{\xi}, E_1 \in \mathcal{F}_{\xi_1}, \ldots , E_n \in \mathcal{F}_{\xi_n}$ with $m \leq E < E_1 < \ldots < E_n\}$. The proof of Case 3 is complete. □

Lemma 3.1.3. *For every $\xi < \omega_1$, (ξ) has property* (Γ).

Proof: We shall use induction on ξ. Let $\xi = 0$, (f_k) be a sequence of continuous real-valued functions defined on a compact metric space K, $\epsilon > 0$ and $m \in \mathsf{N}$ such that for all $\{n\} \in \mathcal{F}_0$ with $m \leq n$ there exists $x_n \in K$ with $|f_{2n+1}(x_n) - f_{2n}(x_n)| > \epsilon$. Let x be a limit point of the set $\{x_n : n \geq m\}$. Then $x \in K^1_{\epsilon,(f_k)}$. Let $1 \leq \xi < \omega_1$ and assume that (ζ) has property (Γ) for every $\zeta < \xi$. If $\xi = \zeta + 1$ then $(\underbrace{\zeta, \ldots , \zeta}_{l-times})$ has property (Γ) for all $l \in \mathsf{N}$, by Lemma 3.1.2. So, using the definition of the property (Γ) and Definition 2.2 we prove that (ξ) has property (Γ). If ξ is limit and (ξ_k) is the strictly increasing sequence with $\sup_k \xi_k = \xi$ that defines the family $\mathcal{F}_{\xi}$ then using that the 1-tuple (ξ_n) has property (Γ) and Definition 2.2 we prove that (ξ) has property (Γ). □

Proof of Proposition 3.1: Let (n_k) be a strictly increasing sequence of natural numbers and (n'_k) a subsequence of (n_k) so that for every $E = \{k_1 < \ldots < k_\lambda\} \in \mathcal{F}_\xi$, there is $x \in K$ with $|f_{n'_{2k_j+1}}(x) - f_{n'_{2k_j}}(x)| > \epsilon$ for all $1 \le j \le \lambda$. By Lemma 3.1.3, (ξ) has property (Γ) and so $K^{\omega^\xi}_{\epsilon,(f_{n'_k})} \neq \emptyset$. Therefore $K^{\omega^\xi}_{\epsilon,(f_{n_k})} \neq \emptyset$ i.e. $\gamma((f_{n_k}),\epsilon) > \omega^\xi$. □

Proposition 3.2. *Let K be a compact metric space, $(f_k) \subseteq C(K)$, $\epsilon > 0$ and $1 \le \xi < \omega_1$ such that $K^{\omega^\xi}_{\epsilon,(f_{n_k})}$ is non-empty for every strictly increasing sequence (n_k) of natural numbers. Then there exists a subsequence (n_k) of N such that for every $F = \{k_1 < \ldots < k_\lambda\} \in \mathcal{F}_\xi$ (where $\lambda \ge 2$) there is $x_F \in K$ such that $|f_{n_{k_{j+1}}}(x_F) - f_{n_{k_j}}(x_F)| > \epsilon/4$ for all $1 \le j \le \lambda - 1$.*

Before we proceed to the proof of our proposition we need the next definitions and Lemma 3.2.3.

Definition 3.2.1. Let X be a set, (f_k) a sequence of real-valued functions defined on X, $N = (n_k)$ a strictly increasing sequence of natural numbers and $\epsilon > 0$. We define the tree $S((f_{n_k}),\epsilon)$ on N as follows:

$S((f_{n_k}),\epsilon) = \{(n_1)\} \cup \bigcup_{m=1}^{\infty}\{(n_1, n_{k_1}, \ldots, n_{k_m}) \in N^{m+1} : n_1 < n_{k_1} < \ldots < n_{k_m}$ and there exists $x \in X$ such that $|f_{n_{k_1}}(x) - f_{n_1}(x)| > \epsilon$ and $|f_{n_{k_{j+1}}}(x) - f_{n_{k_j}}(x)| > \epsilon$ for all $j = 1, 2, \ldots, m-1\}$.

Definition 3.2.2. For any $\alpha < \omega_1$ we say that α has **property** (Γ') if whenever K is a compact metric space, $(f_k) \subseteq C(K)$ and $\epsilon > 0$ such that $K^{\alpha}_{\epsilon,(f_k)}$ is non-empty, then $(S((f_k),\epsilon/3))^\alpha$ is non-empty.

Lemma 3.2.3. *Every $\alpha < \omega_1$ has property (Γ').*

Proof: We proceed by induction on $\alpha < \omega_1$. For $\alpha = 0$, it is trivial (cf. Definitions 2.2, 2.4 (iii) and 3.2.1).

Case 1 ($\alpha = \beta + 1$). Assume that the ordinal $\beta < \omega_1$ has property (Γ') and we shall show that $\beta + 1$ has property (Γ'). Let K be a compact metric space, $(f_k) \subseteq C(K)$ and $\epsilon > 0$ such that $K^{\beta+1}_{\epsilon,(f_k)}$ is non-empty. Then it is easy to see that there is $j \in \mathsf{N}$ with $j > 1$ and $x_0 \in K^{\beta}_{\epsilon,(f_k)}$ such that $|f_j(x_0) - f_1(x_0)| > \epsilon/3$. Since f_1 and f_j are continuous it follows that the set $U = \{x \in K : |f_j(x) - f_1(x)| > \epsilon/3\}$ is an open neighborhood of x_0. We choose an open neighborhood V of x_0 such that $clV \subseteq U$. We set $Q = clV$. Clearly Q is a compact metric subspace of K and

$$x_0 \in V \cap K^{\beta}_{\epsilon,(f_k)} \subseteq Q^{\beta}_{\epsilon,(f_k)_{k \ge j}}.$$

Since β has the property (Γ') it follows that $(S((f_{k|Q})_{k \ge j}, \epsilon/3))^\beta$ is non-empty, where $f_{k|Q}$ denotes the restriction of f_k on Q. The proof of Case 1 can be finished by using the definition of Q.

Case 2 (α is a limit ordinal). Assume that the conclusion of our lemma is true for every $\beta < \alpha$ and we shall show that it is true for α. Indeed, let K be a compact metric space, $(f_k) \subseteq C(K)$ and $\epsilon > 0$ such that $K^{\alpha}_{\epsilon,(f_k)}$ is non-empty. Then $K^{\beta}_{\epsilon,(f_k)}$ is non-empty for every $\beta < \alpha$. Since every $\beta < \alpha$ has the property (Γ') it follows that $(S((f_k), \epsilon/3))^{\beta}$ is non-empty for every $\beta < \alpha$. Then from Definitions 2.4 (iii) and 3.2.1 we get $(1) \in (S((f_k), \epsilon/3))^{\alpha}$ which finishes the proof of Case 2. $\square$

PROOF OF PROPOSITION 3.2: By Lemma 3.2.3 the ordinal ω^{ξ} has the property (Γ') and hence for every strictly increasing sequence (n_j) of natural numbers it holds $(S((f_{n_j}), \epsilon/3))^{\omega^{\xi}} \neq \emptyset$.

For $n \in \mathsf{N}$ and $\zeta_1, \ldots, \zeta_n < \omega_1$, we say that the n-tuple $(\zeta_1, \ldots, \zeta_n)$ has property (A) if whenever P is a closed subset of K and N an infinite subset of N such that $(S((f_{n_j|P}), \epsilon/3))^{\omega^{\zeta_n}+\ldots+\omega^{\zeta_1}} \neq \emptyset$ for every strictly increasing sequence (n_j) of elements of N with $n_1 = \min N$, then there exists a strictly increasing sequence (l_j) of elements of $N \setminus \{\min N\}$ such that for every $F_1 \in \mathcal{F}_{\zeta_1}, \ldots, F_n \in \mathcal{F}_{\zeta_n}$ with $F_1 < \ldots < F_n$ and $\cup_{i=1}^{n} F_i = \{k_1 < \ldots < k_\lambda\}$ (where $\lambda \in \mathsf{N}$), there exists $x \in P$ such that $|f_{\min N}(x) - f_{l_{k_1}}(x)| > \epsilon/4$ and $|f_{l_{k_{j+1}}}(x) - f_{l_{k_j}}(x)| > \epsilon/4$ for all $j = 1, \ldots, \lambda-1$. It is enough to show that the 1-tuple (ξ) has property (A). We divide this proof into two steps:

Step 1. For every $\zeta < \omega_1$, whenever $(\zeta_1, \ldots, \zeta_n)$ has property (A) then $(\zeta, \zeta_1, \ldots, \zeta_n)$ has also property (A).

We shall prove it by induction on $\zeta < \omega_1$.

Case 1 ($\zeta = 0$). Assume that $(\zeta_1, \ldots, \zeta_n)$ has property (A) and we shall show that $(0, \zeta_1, \ldots, \zeta_n)$ has property (A). Indeed, let $P \subseteq K$ be closed and N an infinite subset of N such that $(S((f_{n_j|P}), \epsilon/3))^{\omega^{\zeta_n}+\ldots+\omega^{\zeta_1}+1} \neq \emptyset$ for every strictly increasing sequence (n_j) of elements of N with $n_1 = \min N$. We set $n_0 = \min N$. For every $m \in N \setminus \{n_0\}$ we set $Q_m = \{x \in P : |f_{n_0}(x) - f_m(x)| \geq \epsilon/3\}$ which is a closed subset of K.

Claim. There is an infinite subset M of $N \setminus \{n_0\}$ such that for each infinite subset M' of M there is $m \in M'$ so that for each strictly increasing sequence (n_j) of elements of M' with $n_1 = m$ we have $(m) \in (S((f_{n_j|Q_m}), \epsilon/3))^{\omega^{\zeta_n}+\ldots+\omega^{\zeta_1}}$.

Proof of the claim. Assume the contrary. Then there exists a decreasing sequence (M_λ) of infinite subsets of $N \setminus \{n_0\}$ so that if $m_\lambda = \min M_\lambda$, then $m_\lambda < m_{\lambda+1}$ and $(m_\lambda) \notin (S((f_{k|Q_{m_\lambda}})_{k \in \{m_\lambda\} \cup M_{\lambda+1}}, \epsilon/3))^{\omega^{\zeta_n}+\ldots+\omega^{\zeta_1}}$ for all $\lambda \in \mathsf{N}$. We get the sequence (k_j) with $k_1 = n_0$ and $k_{j+1} = m_j$ for all $j \in \mathsf{N}$.

Then from the assumption we have that $(S((f_{k_j|P}), \epsilon/3))^{\omega^{\zeta_n}+\ldots+\omega^{\zeta_1}+1}$ is non-empty.

Hence there exists $\lambda \in \mathsf{N}$ such that $(n_0, m_\lambda) \in (S((f_{k_j|P}), \epsilon/3))^{\omega^{\zeta_n}+\ldots+\omega^{\zeta_1}}$.

Then $(m_\lambda) \in (S((f_{k|Q_{m_\lambda}})_{k \in \{m_\lambda\} \cup M_{\lambda+1}}, \epsilon/3))^{\omega^{\zeta_n}+\ldots+\omega^{\zeta_1}}$, a contradiction. This completes the proof of the claim.

By repeated application of the claim and using that the n-tuple $(\zeta_1, \dots, \zeta_n)$ has property (A), we find strictly increasing sequences $M_\lambda = (m_j^\lambda)$, $\lambda \in \mathsf{N}$, of $N \setminus \{n_0\}$ and a strictly increasing sequence (m_λ) of elements of $N \setminus \{n_0\}$ such that for every $\lambda \in \mathsf{N}$ it holds $m_\lambda \in M_\lambda$, $m_\lambda^\lambda \le m_\lambda < \min M_{\lambda+1}$ and for every $F_1 \in \mathcal{F}_{\zeta_1}, \dots, F_n \in \mathcal{F}_{\zeta_n}$ with $F_1 < \dots < F_n$ and $\cup_{i=1}^n F_i = \{j_1 < \dots < j_\nu\}$ (where $\nu \in \mathsf{N}$), there exists $x \in Q_{m_\lambda}$ such that $|f_{m_\lambda}(x) - f_{m_{j_1}^{\lambda+1}}(x)| > \epsilon/4$ and $|f_{m_{j_{l+1}}^{\lambda+1}}(x) - f_{m_{j_l}^{\lambda+1}}(x)| > \epsilon/4$ for every $l = 1, \dots, \nu - 1$. The proof of Case 1 can be finished by taking the sequence (m_λ) and using the fact that if $\zeta < \omega_1$, (k_j) a strictly increasing sequence of natural numbers then for every $F \in \mathcal{F}_\zeta$ it follows that $\{k_j : j \in F\} \in \mathcal{F}_\zeta$.

Case 2 ($\zeta = \eta + 1$). Assume that the n-tuple $(\zeta_1, \dots, \zeta_n)$ has property (A) and we shall show that $(\zeta, \zeta_1, \dots, \zeta_n)$ has property (A). Indeed, let P be a closed subset of K and N an infinite subset of N such that $(S((f_{n_j|P}), \epsilon/3))^{\omega^{\zeta_n} + \dots + \omega^{\zeta_1} + \omega^\zeta} \neq \emptyset$ for every strictly increasing sequence (n_j) of elements on N with $n_1 = \min N$.

Then $(S((f_{n_j|P}), \epsilon/3))^{\omega^{\zeta_n} + \dots + \omega^{\zeta_1} + k\omega^\eta} \neq \emptyset$ for every $k \in \mathsf{N}$ and for every strictly increasing sequence (n_j) of elements of N with $n_1 = \min N$. From the induction assumption we get that $(\underbrace{\eta, \dots, \eta}_{k}, \zeta_1, \dots, \zeta_n)$ has property (A) for every $k \in \mathsf{N}$. We set $n_0 = \min N$ and $N_0 = N \setminus \{n_0\}$. Then, by induction on $k \ge 1$, there exists a subsequence $N_k = (n_j^k)$ of N_{k-1} such that for every $E_1, \dots, E_k \in \mathcal{F}_\eta, F_1 \in \mathcal{F}_{\zeta_1}, \dots, F_n \in \mathcal{F}_{\zeta_n}$ with $E_1 < \dots < E_k < F_1 < \dots < F_n$ and $\cup_{j=1}^k E_j \cup \cup_{i=1}^n F_i = \{j_1 < \dots < j_\lambda\}$ (where $\lambda \in \mathsf{N}$) there exists $x \in P$ such that $|f_{n_0}(x) - f_{n_{j_1}^k}(x)| > \epsilon/4$ and $|f_{n_{j_{l+1}}^k}(x) - f_{n_{j_l}^k}(x)| > \epsilon/4$ for every $l = 1, \dots \lambda - 1$. The proof of Case 2 can be finished by taking the diagonal sequence (n_k^k).

Case 3 (ζ is a limit ordinal). Let (η_k) be the strictly increasing sequence of ordinals with $\sup_k \eta_k = \zeta$ that defines the family $\mathcal{F}_\zeta$. Assume that $(\zeta_1, \dots, \zeta_n)$ has property (A) and we shall show that $(\zeta, \zeta_1, \dots, \zeta_n)$ has property (A). Indeed, let P be a closed subset of K and N an infinite subset of N such that $(S((f_{n_j|P}), \epsilon/3))^{\omega^{\zeta_n} + \dots + \omega^{\zeta_1} + \omega^\zeta} \neq \emptyset$ for every strictly increasing sequence (n_j) of elements of N with $n_1 = \min N$. Then

$$(S((f_{n_j|P}), \epsilon/3))^{\omega^{\zeta_n} + \dots + \omega^{\zeta_1} + \omega^{\eta_k}} \neq \emptyset$$

for every $k \in \mathsf{N}$ and for every strictly increasing sequence (n_j) of elements of N with $n_1 = \min N$. By induction assumption we get that $(\eta_k, \zeta_1, \dots, \zeta_n)$ has the property (A) for every $k \in \mathsf{N}$. We set $n_0 = \min N$ and $N_0 = N \setminus \{n_0\}$. Then by induction on $k \ge 1$ and using the definition of the property (A), there exists a subsequence $N_k = (n_j^k)$ of N_{k-1} such that for every $F \in \mathcal{F}_{\eta_k}, F_1 \in \mathcal{F}_{\zeta_1}, \dots, F_n \in \mathcal{F}_{\zeta_n}$ with $F < F_1 < \dots < F_n$ and $F \cup \cup_{i=1}^n F_i = \{j_1 < \dots < j_\lambda\}$ (where $\lambda \in \mathsf{N}$)

there exists $x \in P$ so that $|f_{n_0}(x) - f_{n^k_{j_1}}(x)| > \epsilon/4$ and $|f_{n^k_{j_{l+1}}}(x) - f_{n^k_{j_l}}(x)| > \epsilon/4$ for every $l = 1, \dots, \lambda - 1$. The proof of Case 3 can be again finished by taking the diagonal sequence (n^k_k).

Step 2. The 1-tuple (ζ) has property (A) for each $\zeta < \omega_1$.

We use induction on ζ. For $\zeta = 0$, it is proved easily by using the definition of the property (A) and Definitions 2.4 (iii) and 3.2.4. Let $1 \leq \zeta < \omega_1$ and assume that it is true for every $\eta < \zeta$. If $\zeta = \eta + 1$, then by using Step 1 we get that for each $l \in \mathsf{N}$, the l-tuple $(\underbrace{\eta, \dots, \eta}_{l-times})$ has property (A), and using the definition of the property (A) and a diagonal argument (as in Case 2 of Step 1) we get that (ζ) has property (A). Let ζ is a limit ordinal and let (η_k) be the strictly increasing sequence of ordinals with $\sup_k \eta_k = \zeta$ that defines the family $\mathcal{F}_\zeta$. By using that the 1-tuple (η_l) has property (A) for all $l \in \mathsf{N}$ and working as in Case 3 of Step 1, we get the conclusion. This finishes the proof of Step 2.

The proof of the proposition is complete. $\square$

Combining the above results we get the following characterization for the ordinal index "$\gamma((f_k), \epsilon)$" where (f_k) is a sequence of continuous real-valued functions defined on a compact metric space and $\epsilon > 0$.

Theorem 3.3. *Let K be a compact metric space, $1 \leq \xi < \omega_1$ and (f_k) a sequence of continuous real-valued functions on K. Then the following are equivalent:*

(i) *there are $\epsilon > 0$ and a strictly increasing sequence (k_n) of natural numbers such that $\gamma((f_{k'_n}), \epsilon) > \omega^\xi$ for every strictly increasing sequence (k'_n) of (k_n);*

(ii) *there are $\epsilon > 0$ and a strictly increasing sequence (k_n) of natural numbers so that for every $E = \{n_1 < \dots < n_\lambda\} \in \mathcal{F}_\xi$ (where $\lambda \geq 2$) there is $x_E \in K$ with $|f_{k_{n_{j+1}}}(x_E) - f_{k_{n_j}}(x_E)| > \epsilon$ for all $1 \leq j \leq \lambda - 1$;*

(iii) *there are $\epsilon > 0$ and a strictly increasing sequence (k_n) of natural numbers such that for every subsequence (k'_n) of (k_n) and for every $E = \{n_1 < \dots < n_\lambda\} \in \mathcal{F}_\xi$ (where $\lambda \in \mathsf{N}$) there is $x_E \in K$ with $|f_{k'_{2n_j+1}}(x_E) - f_{k'_{2n_j}}(x_E)| > \epsilon$ for all $1 \leq j \leq \lambda$.*

Proof: The implication (iii) $\Rightarrow$ (i) follows from Proposition 3.1 and the implication (i) $\Rightarrow$ (ii) follows from Proposition 3.2. Finally, the implication (ii) $\Rightarrow$ (iii) is proved by using that (a) if $E = \{n_1 < n_2 < \dots < n_\lambda\} \in \mathcal{F}_\xi$ (where $\lambda \in \mathsf{N}$) then the set $F = \{2n_1 < 2n_1 + 1 < 2n_2 < 2n_2 + 1 < \dots < 2n_\lambda < 2n_\lambda + 1\}$ belongs to $\mathcal{F}_\xi$, and (b) if (n_j) is a strictly increasing sequence of natural numbers, then $\{n_j : j \in F\} \in \mathcal{F}_\xi$ for all $F \in \mathcal{F}_\xi$. $\square$

Proposition 3.4. *Let K be a compact metric space, $1 \leq \xi < \omega_1$, $\epsilon > 0$ and (f_k) a sequence of continuous real-valued functions on K such that for every strictly increasing sequence (k_n) of natural numbers there is a subsequence (k'_n) of (k_n) so* *that* $i((f_{k'_{n+1}} - f_{k'_n}), \epsilon) > \omega^\xi$. *Then* $\gamma((f_{k_n}), \epsilon) > \omega^\xi$ *for every strictly increasing sequence* (k_n) *of natural numbers.*

PROOF: We give the following definition: For any $\alpha < \omega_1$, we say that α has property (B) if whenever P is a non-empty closed subset of K and $N = (k_j)$ a strictly increasing sequence of natural numbers such that for every subsequence (k'_j) of N, there exists a subsequence (k''_j) of (k'_j) so that $(T((f_{k''_{j+1}|P} - f_{k''_j|P}), \epsilon))^\alpha \neq \emptyset$, then $P^\alpha_{\epsilon,(f_{k_j})} \neq \emptyset$.

For the proof of our proposition, it is enough to show that every $\alpha < \omega_1$ has property (B). We shall prove it by induction on $\alpha < \omega_1$. For $\alpha = 0$, it is trivial (cf. Definitions 2.2, 2.4 (iii) and 2.5).

Case 1 ($\alpha = \beta+1$). Assume that β has property (B) and we shall show that $\beta+1$ has property (B). Indeed, let $P \subseteq K$ be non-empty closed and $N = (k_j)$ a strictly increasing sequence of natural numbers such that for every subsequence (k'_j) of N, there exists a subsequence (k''_j) of (k'_j) so that $(T((f_{k''_{j+1}|P} - f_{k''_j|P}), \epsilon))^{\beta+1} \neq \emptyset$. For every $k, l \in N$ we set $Q_{k,l} = \{x \in P : |f_k(x) - f_l(x)| \geq \epsilon\}$ which is a closed subset of K.

Claim. For every subsequence M of N there exist a subsequence M' of M and $m, m' \in M'$ with $m < m'$ and $Q_{m,m'} \neq \emptyset$, such that for every subsequence (k'_j) of M' with $k'_1 = m'$ there exists a subsequence (k''_j) of (k'_j) so that $(1) \in (T((f_{k''_{j+1}|Q_{m,m'}} - f_{k''_j|Q_{m,m'}}), \epsilon))^\beta$.

Proof of the claim. Assume the contrary and let M_0 a subsequence of N so that the claim fails. Then, by induction on $\lambda \geq 1$, we find a subsequence $M_\lambda = (k^\lambda_j)$ of $M_{\lambda-1}$ and we find a strictly increasing sequence (m_λ) of elements of M_0 such that $m_{\lambda+1} = \min M_\lambda$ and $(1) \notin (T((f_{k'_{j+1}|Q_{m,m'}} - f_{k'_j|Q_{m,m'}}), \epsilon))^\beta$ for every subsequence (k'_j) of M_λ and for every $m, m' \in \{m_1, \dots, m_\lambda\}$ with $m < m'$ and $Q_{m,m'} \neq \emptyset$ for all $\lambda = 1, 2, \dots$. We get the sequence (k'_j) with $k'_j = m_j$ for every $j = 1, 2, \dots$. Then, from the assumption, there exists a subsequence $(k''_j) = (m_{p_j})$ of (k'_j) so that $(1) \in (T((f_{k''_{j+1}|P} - f_{k''_j|P}), \epsilon))^{\beta+1}$. Then there exists $\lambda \in \mathsf{N}$, $\lambda > 1$ such that $(1, \lambda) \in (T((f_{k''_{j+1}|P} - f_{k''_j|P}), \epsilon))^\beta$. We set $m = k''_\lambda = m_{p_\lambda}$ and $m' = k''_{\lambda+1} = m_{p_{\lambda+1}}$. Also we get the sequence (l_j) with $l_j = k''_{\lambda+j}$ for all $j = 1, 2, \dots$. Clearly (l_j) is a subsequence of $M_{p_{\lambda+1}-1}$. Also it is obvious that $(1) \in (T((f_{l_{j+1}|Q_{m,m'}} - f_{l_j|Q_{m,m'}}), \epsilon))^\beta$, a contradiction. This completes the proof of the claim.

By repeated application of the claim and using that the n-tuple $(\zeta_1, \dots, \zeta_n)$ has property (B), we find a strictly increasing sequence (m_λ) of elements of N such that for every $\lambda \in \mathsf{N}$ there exists $x_\lambda \in P$ with $x_\lambda \in (Q_{m_\lambda, m_{\lambda+1}})^\beta_{\epsilon,(f_{k_j})}$. By compactness of K, the sequence (x_λ) has a limit point x. Then it is easy to show that $x \in P^{\beta+1}_{\epsilon,(f_{k_j})}$.

Case 2 (α is a limit ordinal). Assume that every $\beta < \alpha$ has property (B) and we shall show that α has property (B). Indeed, let P be a non-empty closed subset of K and $N = (k_j)$ a strictly increasing sequence of natural numbers such that for every subsequence $(k^{'}_j)$ of N, there exists a subsequence $(k^{"}_j)$ of $(k^{'}_j)$ so that $(T((f_{k^{"}_{j+1}}|P - f_{k^{"}_j}|_P), \epsilon))^\alpha \neq \emptyset$. Then $(T((f_{k^{"}_{j+1}}|P - f_{k^{"}_j}|_P), \epsilon))^\beta \neq \emptyset$ for every $\beta < \alpha$. From induction assumption every $\beta < \alpha$ has property (B) and hence $P^\beta_{\epsilon,(f_{k_j})} \neq \emptyset$ for every $\beta < \alpha$. Then, by compactness of K, we get $P^\alpha_{\epsilon,(f_{k_j})} \neq \emptyset$. This completes the proof of our proposition. □

Theorem 3.5. *Let K be a compact metric space, $1 \leq \xi < \omega_1$ and (f_k) a sequence of continuous real-valued functions on K. Then the following are equivalent:*

(i) *for every strictly increasing sequence (k_n) of natural numbers there exists a subsequence $(k^{'}_n)$ of (k_n) such that $\gamma((f_{k^{'}_n})) \leq \omega^\xi$;*

(ii) *for every strictly increasing sequence (k_n) of natural numbers there exists a subsequence $(k^{'}_n)$ of (k_n) such that the sequence $(f_{k^{'}_{n+1}} - f_{k^{'}_n})$ converges ξ-uniformly pointwise to zero;*

(iii) *for every strictly increasing sequence (k_n) of natural numbers there exists a subsequence $(k^{'}_n)$ of (k_n) such that the sequence $(f_{k^{'}_{2n+1}} - f_{k^{'}_{2n}})$ converges ξ-uniformly pointwise to zero.*

Proof: (i) ⇒ (ii). Assume that for every strictly increasing sequence (k_n) of natural numbers there is a subsequence $(k^{'}_n)$ of (k_n) such that $\gamma((f_{k^{'}_n})) \leq \omega^\xi$.

Claim 1. For every $\epsilon > 0$ and for every strictly increasing sequence $M = (k_n)$ of natural numbers there is a subsequence $M^{'} = (k^{'}_n)$ of M such that $i((f_{k^{"}_{n+1}} - f_{k^{"}_n}), \epsilon) < \omega^\xi$ for every subsequence $(k^{"}_n)$ of $(k^{'}_n)$.

[*Proof of Claim* 1. Assume the contrary. Then there are $\epsilon > 0$ and $M_\epsilon = (k_n)$ a strictly increasing sequence of natural numbers such that the claim fails. Then by using Proposition 3.4 for the sequence $g_n = f_{k_n}$, $n \in \mathsf{N}$, we get $\gamma((f_{k^{'}_n}), \epsilon) > \omega^\xi$ for every subsequence $(k^{'}_n)$ of (k_n) which is a contradiction.]

Let (k_n) be a strictly increasing sequence of natural numbers. We set $M_0 = (k_n)$. For any $m \in \mathsf{N}$ and by repeated application of the claim for $\epsilon = \frac{1}{m}$, we get sequences $M_m = (k^m_n)$, $m \in \mathsf{N}$, so that M_m is a subsequence of M_{m-1} and $i((f_{l_{n+1}} - f_{l_n}), \frac{1}{m}) < \omega^\xi$ for each subsequence (l_n) of M_m. We set $(k^{'}_n) = (k^n_n)$ and we get the conclusion.

(ii) ⇒ (iii). It is obvious since for every subsequence $(k^{'}_n)$ of (k_n) the sequence $(f_{k^{'}_{2n+1}} - f_{k^{'}_{2n}})$ is a subsequence of the sequence $(f_{k^{'}_{n+1}} - f_{k^{'}_n})$.

(iii) $\Rightarrow$ (i). Assume that for every strictly increasing sequence (k_n) of natural numbers there is a subsequence $(k_n^{'})$ of (k_n) so that the sequence $(f_{k'_{2n+1}} - f_{k'_{2n}})$ converges ξ-uniformly pointwise to zero, i.e. $i((f_{k'_{2n+1}} - f_{k'_{2n}}), \epsilon) < \omega^\xi$ for every $\epsilon > 0$.

Claim 2. For each $\epsilon > 0$ and for each strictly increasing sequence $M = (k_n)$ of natural numbers there is a subsequence $M^{'} = (k_n^{'})$ of M such that $\gamma((f_{k'_n}), \epsilon) < \omega^\xi$.

[*Proof of Claim* 2. Assume the contrary. Then there are $\epsilon > 0$ and a strictly increasing sequence $M = (k_n)$ of natural numbers so that for every subsequence $M^{'} = (k_n^{'})$ of M implies $\gamma((f_{k'_n}), \epsilon) > \omega^\xi$. Then, by using Proposition 3.2 for the sequence (f_{k_n}), there exists a subsequence $(k_n^{'})$ of (k_n) so that for each $E = \{n_1 < \ldots < n_\lambda\} \in \mathcal{F}_\xi$ $(\lambda \geq 2)$ there is $x_E \in K$ so that $|f_{k'_{n_{j+1}}}(x_E) - f_{k'_{n_j}}(x_E)| > \epsilon/4$ for $j = 1, \ldots, \lambda - 1$. Then for each subsequence $(k_n^{"})$ of $(k_n^{'})$ and for each $E = \{n_1 < \ldots < n_\lambda\} \in \mathcal{F}_\xi$ there is $x_E \in K$ so that $|f_{k^{"}_{2n_j+1}}(x_E) - f_{k^{"}_{2n_j}}(x_E)| > \epsilon/4$ for all $j = 1, \ldots, \lambda$. Thus by using Definition 2.5 and Lemma 2.6, we get $i((f_{k^{"}_{2n+1}} - f_{k^{"}_{2n}}), \epsilon/4) > \omega^\xi$ for each subsequence $(k_n^{"})$ of $(k_n^{'})$, a contradiction.]

Let (k_n) be a strictly increasing sequence of natural numbers. We set $M_0 = (k_n)$. For every $m \in \mathsf{N}$ and by repeated application of Claim 2 for $\epsilon = \frac{1}{m}$, we get a strictly increasing sequence $M_m = (k_n^m)$ of M_{m-1} so that $\gamma((f_{k_n^m}), \frac{1}{m}) < \omega^\xi$. We set $(k_n^{'}) = (k_n^n)$ and we get the conclusion. $\square$

Theorem 3.6. *Let K be a compact metric space, f a bounded real-valued function on K and $1 \leq \xi < \omega_1$. The following are equivalent:*

(i) $f \in \mathcal{B}_1^\xi(K)$;

(ii) *there exists a sequence $(f_k) \subseteq C(K)$ which converges pointwise to f and for every strictly increasing sequence (k_n) of natural numbers there exists a subsequence $(k_n^{'})$ of (k_n) such that the sequence $(f_{k'_{n+1}} - f_{k'_n})$ converges ξ-uniformly pointwise to zero;*

(iii) *there exists a sequence $(f_k) \subseteq C(K)$ which converges pointwise to f and for every strictly increasing sequence (k_n) of natural numbers there exists a subsequence $(k_n^{'})$ of (k_n) so that the sequence $(f_{k'_{2n+1}} - f_{k'_{2n}})$ converges ξ-uniformly pointwise to zero.*

Proof: Using [5, Theorem 1.3] or the proof of [7, Theorem 17] we get that if $1 \leq \xi < \omega_1$ and $f \in \mathcal{B}_1^\xi(K)$ then there exists a sequence $(f_k) \subseteq C(K)$ pointwise converging to f so that $\gamma((f_k)) \leq \omega^\xi$. Also it is known that if $(f_k) \subseteq C(K)$ is pointwise converging to f then $\beta(f) \leq \gamma((f_k))$ (cf. [5, Proposition 1.1]). The proof can be finished by using these results, the definition of the set $\mathcal{B}_1^\xi(K)$ (cf. Definition 2.1) and Theorem 3.5. $\square$

Remarks 3.7. (1) We note that using [5, Theorem 1.3] or the proof of [7, Theorem 17] we prove that if $1 \leq \xi < \omega_1$, $f \in \mathcal{B}_1^{\xi}(K)$ and (f_k) is a (uniformly bounded) sequence of continuous real-valued functions on a compact metric space K pointwise converging to f, then there exists a sequence (g_k) of convex blocks of (f_k) (i.e. $g_k \in conv((f_p)_{p \geq k})$, for all k) such that $\gamma((g_k)) \leq \omega^{\xi}$. (Here $conv((h_k))$ denotes the set of convex combinations of the h_k's.) Combining this result and Theorem 3.5 we get that the conditions (ii) and (iii) of Theorem 3.6 are equivalent respectively with the following conditions:

(ii)$'$ For every (uniformly bounded) sequence $(f_k) \subseteq C(K)$ pointwise converging to f, there exists a sequence (g_k) of convex blocks of (f_k) such that for every strictly increasing sequence (k_n) of natural numbers there exists a subsequence $(k_n^{'})$ of (k_n) so that the sequence $(g_{k_{n+1}^{'}} - g_{k_n^{'}})$ converges ξ-uniformly pointwise to zero.

(iii)$'$ For every (uniformly bounded) sequence $(f_k) \subseteq C(K)$ pointwise converging to f, there exists a sequence (g_k) of convex blocks of (f_k) such that for every strictly increasing sequence (k_n) of natural numbers there exists a subsequence $(k_n^{'})$ of (k_n) so that the sequence $(g_{k_{2n+1}^{'}} - g_{k_{2n}^{'}})$ converges ξ-uniformly pointwise to zero.

(2) Prof. S. Argyros asked me the following: If $\gamma((f_k)) \leq \omega$, does it hold that for each strictly increasing sequence (k_n) of natural numbers the sequence $(f_{k_{n+1}} - f_{k_n})$ converges 1-uniformly pointwise to zero? The next example shows that the answer of this question is negative. Let $K = \mathsf{N} \cup \{\infty\}$ be the Alexandroff's compactification of N, (k_n) a strictly increasing sequence of natural numbers and (A_n) a sequence of finite subsets of N with $A_1 < A_2 < \ldots < A_n < A_{n+1} < \ldots$ so that the cardinality of each A_n is equal to n. For every $k \in \mathsf{N}$ we define the function $f_k : K \to \mathbf{R}$ as follows: For any $m \in \mathsf{N}$ we define $f_{2m}(t) = 1$ if $t = k_n$ with $m \in A_n$ and $f_{2m}(t) = 0$ otherwise. Also we define $f_{2m+1}(t) = 0$ for all $m \in \mathsf{N}$ and $t \in K$. Then it is easy to show that $\gamma((f_k)) = 2$, but $i((f_{2m+1} - f_{2m}), \frac{1}{2}) = \omega + 1$. I thank Prof. S. Argyros for the above question.

Acknowledgment. I am grateful to referee for his (her) kind corrections and helpful comments.

References

[1] Alspach D., Argyros S., *Complexity of weakly null sequences*, Dissertationes Math. **CCCXXI** (1992), 1–44.

[2] Alspach D., Odell E., *Averaging null sequences*, Lectures Notes in Math. 1332, Springer, Berlin, 1988.

[3] Bourgain J., *On convergent sequences of continuous functions*, Bull. Soc. Math. Belg. Ser. B **32** (1980), 235–249.

[4] Haydon R., Odell E., Rosenthal H., *On certain classes of Baire-1 functions with applications to Banach space theory*, Longhorn Notes, The University of Texas at Austin, Functional Analysis Seminar 1987–89.

[5] Kechris A.S., Louveau A., *A classification of Baire class 1 functions*, Trans. Amer. Math. Soc. **318** (1990), 209–236.
[6] Kiriakouli P., *Namioka spaces, Baire-1 functions, Combinatorial principles of the type of Ramsey and their applications in Banach spaces theory* (in Greek), Doctoral Dissertation, Athens Univ., 1994.
[7] Kiriakouli P., *A classification of Baire-1 functions*, Trans. Amer. Math. Soc., to appear.
[8] Kiriakouli P., *On pointwise convergent sequences of continuous functions with continuous limits*, preprint.
[9] Kiriakouli P., Papanastassiou N., *Convergence for sequences of functions and an Egorov type theorem*, preprint.
[10] Mercourakis S., *On Cesaro summable sequences of continuous functions*, Mathematika **42** (1995), 87–104.
[11] Mercourakis S., *On some generalizations of the classical Banach-Saks properties*, preprint, 1994.
[12] Mercourakis S., Negrepontis S., *Banach spaces and topology II*, Recent Progress in General Topology, M. Hušek and J. van Mill, eds., Elsevier Science Publishers B.V., 1992, pp. 495-536.

Sogdianis 6, Ano Ilissia, 15771 Athens, Greece

(*Received* July 8, 1997, *revised* February 3, 1998)