Weakly uniformly rotund Banach spaces

A. Moltó, V. Montesinos, J. Orihuela, S. Troyanski

Abstract. The dual space of a WUR Banach space is weakly K-analytic.

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A Banach space is said to be *weakly uniformly rotund* (WUR for short) if given sequences (x_n) and (y_n) in the unit sphere with $\|x_n + y_n\| \to 2$ we have weak-$\lim_n(x_n - y_n) = 0$. This notion has become more important since Hájek proved that every WUR Banach space must be Asplund ([8]). To obtain this result he uses ideas of Stegall for the equivalence between being an Asplund space and having the Radon-Nikodym property on its dual. Using this result and the Fabian-Godefroy ([4]) projectional resolution of the identity in the dual of an Asplund space, Fabian, Hájek and Zizler have recently showed that for a WUR Banach space E the dual space E^* is a subspace of a WCG Banach space. Indeed they proved that for a Banach space E to have an equivalent WUR norm is equivalent to the fact that the bidual unit ball $B_{E^{**}}$, endowed with the weak-* topology, will be a uniform Eberlein compact ([5]). Consequently they obtain that E must be LUR renormable, too ([7]). The aim of this note is to provide a direct proof of the fact that every WUR Banach space E has a dual space E^* which is weakly K-analytic. This provides a topological approach to Hájek's result on the Asplundness of the space E as well as the LUR renorming consequence on E after ([6]).

In this paper, E will denote a Banach space, E^* its dual, B_E its closed unit ball, S_E its unit sphere.

Definition 1. *A Banach space* $(E, \|\cdot\|)$ *is said to be uniformly Gâteaux differentiable* (*UGD for short*) *if for every* $0 \neq x \in E$,

$$\lim_{t\to 0} \sup_{\|y\|=1} \frac{\|y + tx\| + \|y - tx\| - 2}{t} = 0.$$

The following theorem is the main result of this note:

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Theorem 1. *Let E be a Banach space such that E^* has an equivalent (not necessarily dual) UGD norm (in particular, let E be WUR Banach space). Then E^* is weakly K-analytic.*

The proof is based on the following assertions.

Fact 1 (Šmulyan, see [3, Theorem II.6.7]). *The Banach space E is WUR if and only if E^* is UGD.*

Theorem 2 (Talagrand [9]). *Let K be a compact space. The following assertions are equivalent:*

1. *$C(K)$ is weakly K-analytic;*
2. *there is an increasing mapping $\sigma \to S_\sigma$ from $\mathbb{N}^\mathbb{N}$ (endowed with the product order) in the family of compact subsets of $C(K)$ endowed with the topology of pointwise convergence, such that $\bigcup\{S_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$ separates points of K.*

Remark 1. In [1] the validity of the previous theorem for an arbitrary topological space is studied. In particular, for every subset W of a Banach space E it follows that (W, weak) is K-analytic if and only if $W = \bigcup\{S_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$ and every S_σ is weakly compact with $S_\sigma \subset S_\gamma$ whenever $\sigma \le \gamma$ in the product order. This will be the only tool necessary here from the theory of K-analytic spaces.

Remark 2. From Theorem 1 and [6], see also [3, p. 296], we get that every WUR Banach space admits an equivalent LUR norm.

Remark 3. From Theorem 1 it follows the Hájek's ([8]) result asserting that every WUR Banach space is Asplund. Indeed, if we assume that E is also separable the K-analytic structure of (E^*, weak) should imply that E^* is separable too. Let us explain here an easy argument following ideas from [2]: Assume (E^*, weak) is K-analytic. Let T be an usco mapping from $\mathbb{N}^\mathbb{N}$ into the set of subsets of (E^*, weak) with $T(\mathbb{N}^\mathbb{N}) = E^*$ (T can be assumed to be increasing by Remark 1). Let P be the natural projection from $(E^*, \text{weak}^*) \times \mathbb{N}^\mathbb{N}$ onto (E^*, weak). Consider the restriction Q of P to $\Sigma := \{(x, \alpha) : (x, \alpha) \in E^* \times \mathbb{N}^\mathbb{N},\ x \in T(\alpha)\}$. It is easy to prove that Q is continuous: let (x_i^*, α_i) be a net in Σ such that $(x_i^*, \alpha_i) \to (x, \alpha) \in \Sigma$. As $\alpha_i \to \alpha$ we can find $\beta \in \mathbb{N}^\mathbb{N}$ such that $\alpha \le \beta$ and $\alpha_i \le \beta$ for all $i \in \mathbb{N}$. Then $x_i \in T(\beta)$, $x \in T(\beta)$, and $x_i \xrightarrow{\text{weak}^*} x$, hence $x_i \xrightarrow{\text{weak}} x$. Therefore E^* is separable too. See also Theorem 2.4 in [9]. With more generality, any submetrizable topological space X is analytic if and only if there is a family of compact sets $\{S_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$ in X, $S_\sigma \subset S_\gamma$ whenever $\sigma \le \gamma$ in the product order and $X = \bigcup\{S_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$, [2, Theorem 7].

Proof of Theorem 1: It is well known that E admits an equivalent WUR norm. Then E^* has an equivalent dual UGD norm. Then given $x^* \in S_{E^*}$ and $\epsilon > 0$, there exists $\delta_\epsilon(x^*) > 0$ such that

$$\|y^* + tx^*\| + \|y^* - tx^*\| \le 2 + \epsilon|t|, \text{ if } |t| < \delta_\epsilon(x^*) \text{ and } y^* \in S_{E^*}.$$

Given a positive integer p define

$$S_p(\epsilon) := \Big\{x^* \in S_{E^*} :\ \delta_\epsilon(x^*) > \frac{1}{p}\Big\}.$$

Obviously,

$$S_1(\epsilon) \subset S_2(\epsilon) \subset \ldots \subset S_p(\epsilon) \subset S_{p+1}(\epsilon) \subset \ldots$$

and $\bigcup_{p=1}^{\infty} S_p(\epsilon) = S_{E^*}$. Let $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$. Define

$$S_\alpha := \bigcap_{n=1}^{\infty} S_{a_n}\left(\frac{1}{n}\right).$$

We have

$$S_{E^*} = \bigcup \Big\{S_\alpha :\ \alpha \in \mathbb{N}^{\mathbb{N}}\Big\},$$

and

$$S_\alpha \subset S_\beta, \text{ whenever } \alpha = (a_n) \leq \beta = (b_n) \text{ (i.e., } a_n \leq b_n,\ \forall n).$$

This sets will give us the K-analytic structure of E^* in the weak topology. Indeed, we have the following

Claim 1. *Given $x^{**} \in B_{E^{**}}$, $\epsilon > 0$ and $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$, there is $x \in B_E$ such that*

$$|\langle x^{**} - x, x^*\rangle| < \epsilon,\ \forall\, x^* \in S_\alpha.$$

Proof of the claim: Find $n \in \mathbb{N}$ such that $\frac{3}{n} < \epsilon$. Pick $y^* \in S_{E^*}$ such that

$$\langle x^{**}, y^*\rangle > 1 - \frac{1}{na_n}.$$

Find $x \in B_E$ such that

$$\langle x, y^*\rangle > 1 - \frac{1}{na_n}.$$

Let $x^* \in S_\alpha$. Since $x^* \in S_{a_n}(\frac{1}{n})$

$$\|y^* + \frac{1}{a_n}x^*\| + \|y^* - \frac{1}{a_n}x^*\| \leq 2 + \frac{1}{na_n}.$$

In particular we have

$$\langle x^{**}, y^* + \frac{1}{a_n}x^*\rangle + \langle x, y^* - \frac{1}{a_n}x^*\rangle \leq 2 + \frac{1}{na_n} \tag{1}$$

hence

$$\frac{1}{a_n}\langle x^{**} - x, x^*\rangle \leq 2 + \frac{1}{na_n} - \langle x^{**}, y^*\rangle - \langle x, y^*\rangle < \frac{3}{na_n} < \frac{\epsilon}{a_n}$$

and so

$$\langle x^{**} - x, x^* \rangle < \epsilon, \ \forall\, x^* \in S_\alpha.$$

By interchanging x^{**} and x in (1), we get

$$|\langle x^{**} - x, x^* \rangle| < \epsilon, \ \forall\, x^* \in S_\alpha$$

and this proves the claim.

To finish the proof of the Theorem, observe that, by the claim, each S_α is weakly relatively compact since it is weak*-relatively compact. Thus, we have

$$S_{E^*} \subset \bigcup \{\overline{S_\alpha}^{\text{weak}} : \ \alpha \in \mathbb{N}^{\mathbb{N}}\} := W$$

and W is weakly K-analytic in E^* (Theorem 2 and Remark 1).

Consider the map

$$(W, \text{weak}) \times [0, +\infty[\xrightarrow{\Psi} (E^*, \text{weak})$$

given by $\Psi(x^*, t) := t.x^*$. Ψ is continuous, $[0, +\infty[$ is a Polish space, $(W, \text{weak}) \times [0, +\infty[$ is K-analytic and $\Psi(W \times [0, +\infty[) = E^*$, so (E^*, weak) is itself K-analytic. $\square$

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Departament d'Anàlisi Matemàtica, Universitat de València, Dr. Moliner 50, 46100 Burjassot (València), Spain

Departamento de Matemática Aplicada, E.T.S.I. Telecomunicación, Universidad Politécnica de Valencia, C/ Vera, s/n. 46071-Valencia, Spain

Departamento de Análisis Matemático, Universidad de Murcia, Campus de Espinardo, Murcia, Spain

Faculty of Mathematics and Informatics, Sofia University, 5, James Bourchier blvd., 1126 Sofia, Bulgaria

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