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On the homology of free Lie algebras

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Abstract. Given a principal ideal domain R of characteristic zero, containing $1/2$, and a connected differential non-negatively graded free finite type R-module V, we prove that the natural arrow $\mathbb{L}FH(V) \to FH\mathbb{L}(V)$ is an isomorphism of graded Lie algebras over R, and deduce thereby that the natural arrow $UFH\mathbb{L}(V) \to FHU\mathbb{L}(V)$ is an isomorphism of graded cocommutative Hopf algebras over R; as usual, F stands for free part, H for homology, $\mathbb{L}$ for free Lie algebra, and U for universal enveloping algebra. Related facts and examples are also considered.

Keywords: differential graded Lie algebra, free Lie algebra on a differential graded module, universal enveloping algebra

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Letting H, $\mathbb{L}$ and U respectively denote the homology functor, the free Lie algebra functor and the universal enveloping algebra functor, Quillen [10] has shown *inter alia* that, given a field K of characteristic zero, if V is a differential graded K-vector space, then the natural arrow $\mathbb{L}H(V) \to H\mathbb{L}(V)$ is an isomorphism of graded K-Lie algebras, and if L is a differential graded K-Lie algebra, then the natural arrow $UH(L) \to HU(L)$ is an isomorphism of graded cocommutative K-Hopf algebras. This is no longer the case in non-zero characteristic ([2], [8]) or if the ground field is replaced by a commutative ring of characteristic zero, containing $1/2$ ([1], [9], [11]). However, in this latter situation, under suitable reasonable hypotheses, the natural arrow $UH(L) \to HU(L)$ is still an isomorphism up to a certain dimension, which depends on the first non-invertible prime in the ground ring and the connectivity of L ([1], [9], [11]). Within an appropriate framework, factoring torsion out in homology seems to be a first step towards recovering Quillen-like results, *i.e.*, corresponding induced isomorphisms in *all* dimensions: if, for instance, R is a principal ideal domain of characteristic zero, containing $1/2$, and (L, d) is a connected differential non-negatively graded Lie algebra over R, with a free finite type underlying module, then the natural arrow $UFH(L) \to FHU(L)$ is an isomorphism of graded cocommutative Hopf algebras, provided that $\mathrm{ad}^{\varrho-1}(x)(dx) = 0$, for homogeneous x in L_{even} ([9]); here F stands for free part, $\mathrm{ad}(u)(v)$ is the Lie bracket $[u, v]$ of the elements u and v of L (so $\mathrm{ad}^n(u)(v) = [u, [u, \dots [u, v] \dots]]$, n nested Lie brackets), and $\varrho = \varrho(R)$ denotes the least non-invertible prime in R (of course, if $\mathbb{Q} \subseteq R$, we set $\varrho = \infty$ and agree that $\mathrm{ad}^{\varrho-1} = \mathrm{ad}^\infty = 0$). If the “nilpotency” condition is removed away in the preceding, then the natural arrow $UFH(L) \to FHU(L)$ might no longer be an isomorphism, though it still is a monomorphism in *all* dimensions ([9]). This failure can, however, be remedied and Quillen-like results recovered by considering suitably generated free Lie algebras:

1. Theorem. *If R is a principal ideal domain of characteristic zero, containing $1/2$, and V is a connected differential non-negatively graded R-free module of finite type, then the natural morphisms $\mathbb{L}FH(V) \to FH\mathbb{L}(V)$, of graded R-Lie algebras, and $UFH\mathbb{L}(V) \to FHU\mathbb{L}(V)$, of graded cocommutative R-Hopf algebras, are both isomorphisms.*

Consequently, $FH(V)$ embeds into $FH\mathbb{L}(V)$, as a submodule, via $\mathbb{L}FH(V)$, and $FH\mathbb{L}(V)$ embeds into $FHU\mathbb{L}(V)$, as a sub Lie algebra, via $UFH\mathbb{L}(V)$.

Before proceeding, let us make some remarks upon the ingredients.

2. Remarks. (1) As already noticed in the introduction, within the context under consideration, the natural morphisms $UFH(\_) \to FHU(\_)$ are always monic ([9]), so, for such an arrow to be an isomorphism it is sufficient that it be epic.

(2) The importance of factoring torsion out in homology, in order to obtain Quillen-like results, is easily shown by considering $\mathbb{L}(x, dx)$ over $R = \mathbb{Z}[1/2]$, with x of degree 2: the natural morphisms $\mathbb{L}H(x, dx) \to H\mathbb{L}(x, dx)$ and $UH\mathbb{L}(x, dx) \to HU\mathbb{L}(x, dx)$ are both trivial in positive dimensions, though $H\mathbb{L}(x, dx)$ and $UH\mathbb{L}(x, dx)$ are not, their first relevant components arising in dimension 4, where both equal $(R/3)[dx, [x, dx]]$.

(3) Let L be a connected differential non-negatively graded R-Lie algebra, with a free finite type underlying module. If Q denotes the quotient field of R, then, by standard identifications and Künneth isomorphisms, the "rationalized" natural arrow $Q \otimes_R UFH(L) \to Q \otimes_R FHU(L)$ is essentially Quillen's isomorphism $UH(Q \otimes_R L) \xrightarrow{\cong} HU(Q \otimes_R L)$ ([9]), so $FHU(L) = R$ if and only if $UFH(L) = R$, and this is obviously the case, if and only if $FH(L) = 0$ (in $UFH(L) = R$, take primitives both sides).

Back to our theorem, let $L = \mathbb{L}(V)$ in the preceding and deduce that $FH\mathbb{L}(V) = 0$ if and only if $FH(V) = 0$; this is easily seen by identifying, as usual, the cocommutative Hopf algebras $U\mathbb{L}(V)$ and $T(V)$, where T denotes the tensor algebra functor, and noting that $TFH(V) \xrightarrow{\cong} FHT(V)$, as cocommutative Hopf algebras, by the Künneth theorem. Our theorem is thus quite straightforward for V with $FH(V) = 0$ (*e.g.*, for acyclic V).

(4) Under the hypotheses in the theorem, we can therefore exhibit $FH\mathbb{L}(V)$ as a free graded Lie algebra (on $FH(V)$), a remarkable fact that does not necessarily hold for free graded Lie algebras equipped with a *decomposable* differential (*i.e.*, a differential sending the generating module W into $\mathbb{L}^{\geq 2}(W)$). This is easily seen by considering the minimal model of Quillen for the complex projective plane ([12]): $(\mathbb{L}(x, y), d)$ over the rationals, with x of degree 1, y of degree 3, $dx = 0$ and $dy = [x, x]$.

(5) Finally, if R is a field of characteristic zero, we obviously recover Quillen's cited result on the natural morphism $\mathbb{L}H(V) \to H\mathbb{L}(V)$. $\square$

The remainder of the paper is almost entirely devoted to the proof of the theorem, of which the following lemma, adapted from [1], is an important step.

3. Lemma. *Let R be a principal ideal domain of characteristic zero, containing $1/2$; let further (L,∂) be a connected differential non-negatively graded Lie algebra over R, with a free finite type underlying module; and let, finally, (W,δ) be a connected differential non-negatively graded free finite type R-module, with $FH(W,\delta)=0$.*

If either $\partial = 0$ or (W,δ) is acyclic, then the canonical injection $(L,\partial) \to (L,\partial) \amalg \mathbb{L}(W,\delta)$ induces an isomorphism $FH(L,\partial) \xrightarrow{\cong} FH((L,\partial) \amalg \mathbb{L}(W,\delta))$ of graded Lie algebras.

If, in addition, the natural arrow $UFH(L,\partial) \to FHU(L,\partial)$ is an isomorphism of graded cocommutative Hopf algebras, then so is the natural morphism $UFH((L,\partial) \amalg \mathbb{L}(W,\delta)) \to FHU((L,\partial) \amalg \mathbb{L}(W,\delta))$.

4. Remark. Letting, as usual, $\varrho = \varrho(R)$ denote the least prime (or ∞) not invertible in R (see the introduction), the condition on $UFH(L,\partial) \to FHU(L,\partial)$ in the second half of the lemma is satisfied if, for instance, $\mathrm{ad}^{\varrho-1}(x)(\partial x) = 0$, for homogeneous x in L_{even} ([9]), this latter being automatically fulfilled for $\partial = 0$ on L_{even} or on all of L, or for ν-nilpotent L with $\nu \leq \varrho$, or for $\mathbb{Q} \subseteq R$. $\square$

Proof of the Lemma: The canonical injection $\iota : (L,\partial) \to (L,\partial) \amalg \mathbb{L}(W,\delta)$ is clearly a right inverse for the surjection $\pi : (L,\partial) \amalg \mathbb{L}(W,\delta) \to (L,\partial)$, sending (L,∂) identically onto itself and $\mathbb{L}(W,\delta)$ to zero. Then $H(\iota)$ is a right inverse for $H(\pi)$ and there results a trivial connecting morphism in the long exact homology sequence associated to the (right split) short exact sequence of differential graded Lie algebras

$$0 \to (L',\partial') \xrightarrow{\varkappa} (L,\partial) \amalg \mathbb{L}(W,\delta) \xrightarrow{\pi} (L,\partial) \to 0,$$

in which, of course, (L',∂') is the kernel of π. Consequently,

$$0 \to H(L',\partial') \xrightarrow{H(\varkappa)} H((L,\partial) \amalg \mathbb{L}(W,\delta)) \xrightarrow{H(\pi)} H(L,\partial) \to 0$$

is a short exact sequence of graded Lie algebras with right splitting $H(\iota)$, yielding another short exact sequence of graded Lie algebras

$$(1) \quad 0 \to FH(L',\partial') \xrightarrow{FH(\varkappa)} FH((L,\partial) \amalg \mathbb{L}(W,\delta)) \xrightarrow{FH(\pi)} FH(L,\partial) \to 0,$$

with a right R-splitting induced by $H(\iota)$.

On the other hand, under the given hypotheses, (W,δ) is free on a finite type positively graded R-basis $\{u_\alpha, v_\alpha\}$, with $\delta u_\alpha = 0$ and $\delta v_\alpha = a_\alpha u_\alpha$, $a_\alpha \in R \setminus \{0\}$; of course, for acyclic (W,δ), we may (and will) assume all $a_\alpha = 1$. It then follows that $(L', \partial') = (\mathbb{L}(\mathrm{ad}(w)(u_\alpha), \mathrm{ad}(w)(v_\alpha)), \partial')$, with w running through an R-basis for $U(L)$, and ∂' satisfying

$$\partial' \ \mathrm{ad}(w)(u_\alpha) = \mathrm{ad}(U(\partial)w)(u_\alpha),$$

$$\partial' \ \mathrm{ad}(w)(v_\alpha) = \mathrm{ad}(U(\partial)w)(v_\alpha) + (-1)^{|w|} a_\alpha \ \mathrm{ad}(w)(u_\alpha),$$

where $|w|$ denotes the modulo 2 reduction of the degree of w. Letting $\{x_\beta\}$ and $\{y_\beta\}$ respectively equal the sets $\{\mathrm{ad}(w)(u_\alpha)\}$ and $\{\mathrm{ad}(w)(v_\alpha)\}$, we see easily that:

(a) for $\partial = 0$, $(L', \partial') = \mathbb{L}(W', \partial')$, where (W', ∂') is free on the finite type positively graded R-basis $\{x_\beta, y_\beta\}$, with $\partial' x_\beta = 0$ and $\partial' y_\beta = b_\beta x_\beta$, $b_\beta \in R \setminus \{0\}$; and

(b) for acyclic (W, δ), $(L', \partial') = \mathbb{L}(y_\beta, \partial' y_\beta)$.

In either case, $FH(L', \partial') = 0$ (*cf.* Remark 2(3)), and the first half of the lemma follows by (1).

To prove the second half, consider the commutative diagram

$$\begin{array}{ccc} UFH(L,\partial) & \longrightarrow & UFH((L,\partial) \amalg \mathbb{L}(W,\delta)) \\ \downarrow & & \downarrow \\ FHU(L,\partial) & \longrightarrow & FHU((L,\partial) \amalg \mathbb{L}(W,\delta)) \end{array}$$

in which the vertical arrows are the respective natural morphisms $UFH(\_) \to FHU(\_)$, while the horizontal arrows are both induced by ι. The top arrow is an isomorphism, by the first half of the lemma, the left one is an isomorphism, by hypothesis, so the proof will be complete if the bottom arrow is shown an isomorphism. To this end, observe, by the preceding and [2], [7], that $U(L', \partial') \otimes_R U(L, \partial) \xrightarrow{\cong} U((L, \partial) \amalg \mathbb{L}(W, \delta))$, as left $U(L', \partial')$-modules and right $U(L, \partial)$-comodules, under $U(\varkappa) \otimes_R U(\iota)$ followed by multiplication. A simple glance at the Künneth theorem yields $FHU(L', \partial') \otimes_R FHU(L, \partial) \xrightarrow{\cong} FHU((L, \partial) \amalg \mathbb{L}(W, \delta))$, and since $FHU(L', \partial') = R$ (*cf.* Remark 2(3)), the lemma follows. □

The proof of the theorem is now straightforward.

Proof of the Theorem: Let d denote the differential on V and note that, under the assumed hypotheses, (V, d) splits as $(V, d) = (V', d') \oplus (FH(V, d), 0)$, where V' is a d-stable submodule of V, d' is the restriction of d to V', and $FH(V', d') = 0$. Observing further that $\mathbb{L}(V, d) = \mathbb{L}(V', d') \amalg \mathbb{L}(FH(V, d), 0)$, nothing remains but apply the lemma with $L = \mathbb{L}FH(V, d)$, $\partial = 0$ and $(W, \delta) = (V', d')$. □

5. Remark. There is another argument proving that the natural arrow $UFH\mathbb{L}(V) \to FHU\mathbb{L}(V)$ is an isomorphism: by Remark 2(1), it suffices to show it epic, which amounts to prove the corresponding induced morphism between the respective indecomposables epic ([7]); and this is easily seen by noting that this latter morphism is essentially the identity on $FH(V)$ — of course, the first part of the theorem along with the Künneth theorem play a key rôle in the argument. □

The theorem now provides the following completion of the lemma.

6. Corollary. *If R is a principal domain of characteristic zero, containing $1/2$, if (L, ∂) is a connected differential non-negatively graded Lie algebra over R, with a free finite type underlying module, and if (W, δ) is a connected differential non-negatively graded free finite type R-module, then the canonical projection $(L, \partial) \amalg \mathbb{L}(W, \delta) \to (L, \partial)$, sending $\mathbb{L}(W, \delta)$ to zero, yields an R-spilt short exact sequence*

$$0 \to \mathbb{L}(\{\mathrm{ad}(u)(w)\}) \to FH((L, \partial) \amalg \mathbb{L}(W, \delta)) \to FH(L, \partial) \to 0$$

of graded Lie algebras, with u and w running through R-bases of $FHU(L, \partial)$ and $FH(W, \delta)$, respectively.

And if, in addition, the natural arrow $UFH(L, \partial) \to FHU(L, \partial)$ is an isomorphism of graded cocommutative Hopf algebras, then so is the natural morphism $UFH((L, \partial) \amalg \mathbb{L}(W, \delta)) \to FHU((L, \partial) \amalg \mathbb{L}(W, \delta))$.

The proof goes along the lines in the proof of the lemma and is hence omitted.

We now consider some examples. In all cases, the ground ring R, of course, is a principal ideal domain of characteristic zero, containing $1/2$; as usual, $\varrho = \varrho(R)$ denotes the least prime (or ∞) not invertible in R, and, for integer $k \geq 1$, we set $N(k, \varrho) = k'\varrho - 2$, where $k' = 2\lfloor k/2 + 1 \rfloor$ is the smallest even integer exceeding k.

Our first example is inspired by the surjective model of Quillen for the Hopf fibration $S^3 \to S^7 \to S^4$ ([12]); as usual, S^n denotes the n-sphere.

7. Example. Consider a finite type graded set of generators $\{x_\alpha, y_\alpha, z_\alpha\}$, with x_α of degree $2n_\alpha$, y_α of degree $2n_\alpha + 1$, and z_α of degree $4n_\alpha + 2$, all n_α being positive integers. Let further $L = \mathbb{L}(x_\alpha, y_\alpha, z_\alpha)$ be endowed with the differential d given by $dx_\alpha = 0$, $dy_\alpha = a_\alpha x_\alpha$, $a_\alpha \in R \setminus \{0\}$, and $dz_\alpha = 2a_\alpha[x_\alpha, y_\alpha]$. A mere change of generators, $z'_\alpha = z_\alpha - [y_\alpha, y_\alpha]$, renders d "indecomposable", so $FH(L, d) = \mathbb{L}(z'_\alpha)$ and $FHU(L, d) = T(z'_\alpha)$. □

Next, we deal with a relatively general pattern that can also be related to Quillen minimal models, as examples in the sequel will show.

To begin with, suppose we are given the following short exact sequence of differential graded Lie algebras

$$0 \to \mathbb{L}(V', d') \xrightarrow{\varkappa} (\mathbb{L}(V), d) \xrightarrow{\pi} \mathbb{L}(V'', d'') \to 0, \tag{2}$$

with right splitting $\iota : \mathbb{L}(V'', d'') \to (\mathbb{L}(V), d)$, and free finite type connected non-negatively graded R-modules V, V' and V''; we do *not* require that $dV \subseteq V$. As in the proof of the lemma, we derive the short exact sequences of graded Lie algebras below:

$$0 \to H\mathbb{L}(V', d') \xrightarrow{H(\varkappa)} H(\mathbb{L}(V), d) \xrightarrow{H(\pi)} H\mathbb{L}(V'', d'') \to 0, \tag{3}$$

with right splitting $H(\iota)$; and

$$0 \to FH\mathbb{L}(V', d') \xrightarrow{FH(\varkappa)} FH(\mathbb{L}(V), d) \xrightarrow{FH(\pi)} FH\mathbb{L}(V'', d'') \to 0, \tag{4}$$

with a right splitting induced by $H(\iota)$. Under obvious identifications, the theorem allows us to rewrite (4) as

$$(4') \qquad 0 \to \mathbb{L}FH(V', d') \xrightarrow{FH(\varkappa)} FH(\mathbb{L}(V), d) \xrightarrow{FH(\pi)} \mathbb{L}FH(V'', d'') \to 0,$$

with a corresponding right splitting, so $FH(\mathbb{L}(V), d) = \mathbb{L}FH(V', d') \oplus \mathbb{L}FH(V'', d'')$, as R-modules.

On the other hand, since both (2) and (4′) involve R-free objects of finite type, the corresponding universal enveloping algebras form, respectively, short exact sequences of homology Hopf algebras ([2]). Thus, under obvious identifications, (2) yields $(T(V), d) = T(V', d') \otimes_R T(V'', d'')$, as left $T(V', d')$-modules and $T(V'', d'')$-comodules, so $FH(T(V), d) = FHT(V', d') \otimes_R FHT(V'', d'') = TFH(V', d') \otimes_R TFH(V'', d'')$, by the Künneth theorem; similarly, (4′) yields $UFH(\mathbb{L}(V), d) = TFH(V', d') \otimes_R TFH(V'', d'')$, as left $TFH(V', d')$-modules and $TFH(V'', d'')$-comodules, and it should now be clear that the natural arrow $UFH(\mathbb{L}(V), d) \to FHU(\mathbb{L}(V), d)$ is an isomorphism of graded cocommutative Hopf algebras.

If either $H(V', d')$ or $H(V'', d'')$ is R-flat (*e.g.*, if either of them is R-free), then ([2]) $H(T(V), d) = HT(V', d') \otimes_R HT(V'', d'')$; and if it happens that $H(V', d')$ and $H(V'', d'')$ are *both* R-flat, then ([2]) $H(T(V), d) = TH(V', d') \otimes_R TH(V'', d'')$. Thus, if (V', d') and (V'', d'') both have R-free homologies, then the homology of $U(\mathbb{L}(V), d) = (T(V), d)$ is R-free, as well: the natural arrow $UFH(\mathbb{L}(V), d) \to HU(\mathbb{L}(V), d)$ is an isomorphism of graded cocommutative Hopf algebras and it follows ([4]) that any minimal model $(\Lambda W, D)$ for the Cartan-Chevalley-Eilenberg complex $C^*(\mathbb{L}(V), d)$ is *decomposable*, and $FH(\mathbb{L}(V), d)$ may be regarded as the homotopy Lie algebra of $(\Lambda W, D)$; as usual, Λ denotes the free commutative algebra functor. Furthermore, for k-connected V, with integer $k \geq 1$, in dimensions below $N(k, \varrho)$, $H(\mathbb{L}(V), d)$ is R-free, and the natural arrow $UH(\mathbb{L}(V), d) \to HU(\mathbb{L}(V), d)$ is an isomorphism compatible with the Hopf algebra structures ([1], [9]). However, $H(\mathbb{L}(V), d)$ might have R-torsion in dimension $N(k, \varrho)$ or beyond (*cf.* Remark 2 (2)).

We now proceed to specialize the preceding to several particular situations. A first such instance is related to the minimal model of Quillen for the product of two spheres ([12]).

8. Example. Given integers $p \geq 1$, $q \geq 1$ and $r \geq 0$, consider $\mathbb{L}(x, y, z)$ over R, with x of degree p, y of degree q, and z of degree $pr + q + 1$, equipped with the differential d given by $dx = 0$, $dy = 0$ and $dz = a\ \mathrm{ad}^r(x)(y)$, $a \in R \setminus \{0\}$. Of course, for $r = 0$, d is "indecomposable" ($dz = ay$), and we fall right in the context of the theorem. On the other hand, for $r = 1$, $R = \mathbb{Q}$ and $a = 1$, we recover Quillen's minimal model for the product $S^{p+1} \times S^{q+1}$ of the spheres S^{p+1} and S^{q+1} ([12]). A direct computation of the Euler-Poincaré series ([2]) reveals that the kernel of the natural projection $(\mathbb{L}(x, y, z), d) \to (\mathbb{L}(x), 0)$, with obvious right inverse $(\mathbb{L}(x), 0) \hookrightarrow (\mathbb{L}(x, y, z), d)$, is the free differential graded Lie algebra on $\{\mathrm{ad}^n(x)(y), \mathrm{ad}^n(x)(z)\}$, $n \in \mathbb{N}$, whose differential d' satisfies $d'\ \mathrm{ad}^n(x)(y) = 0$ and $d'\ \mathrm{ad}^n(x)(z) = (-1)^{np}\, a\ \mathrm{ad}^{n+r}(x)(y)$, whatever n in $\mathbb{N}$. Thus, d' is "indecomposable" (*i.e.*, the module of generators is d'-stable, so d' preserves wordlength) and the preceding pattern applies: the graded cocommutative Hopf algebras $UFH(\mathbb{L}(x,y,z),d)$ and $FHU(\mathbb{L}(x,y,z),d)$ are isomorphic under the natural arrow, both being correspondingly identified to $T(\{\mathrm{ad}^k(x)(y)\}_{k=0,\dots,r-1}) \otimes_R T(x)$.

As for $FH(\mathbb{L}(x,y,z),d)$, it fits in the R-split short exact sequence of graded Lie algebras

$$0 \to \mathbb{L}(\{\mathrm{ad}^k(x)(y)\}_{k=0,\dots,r-1}) \to FH(\mathbb{L}(x,y,z),d) \to \mathbb{L}(x) \to 0,$$

corresponding to $(4')$. Consequently,

$$FH(\mathbb{L}(x,y,z),d) = \mathbb{L}(\{\mathrm{ad}^k(x)(y)\}_{k=0,\dots,r-1}) \oplus \mathbb{L}(x),$$

as R-modules, the Lie algebra structure for $FH(\mathbb{L}(x,y,z),d)$ being subject to $\mathrm{ad}^r(x)(y) = 0$ (*e.g.*, $\mathrm{ad}^{\geq r}(x)(y) = 0$, $[[x,x], \mathrm{ad}^{\geq r-2}(x)(y)] = 0$ etc.). Thus,

$$FH(\mathbb{L}(x,y,z),d) = \begin{cases} \mathbb{L}(x), & \text{for } r = 0, \\ \mathbb{L}(x) \oplus \mathbb{L}(y), & \text{for } r = 1, \end{cases}$$

as Lie algebras, as well.

It seems worth remarking that

$$H(T(x,y,z),d) = HT(\{\mathrm{ad}^n(x)(y), \mathrm{ad}^n(x)(z)\}_{n\in\mathbb{N}}, d') \otimes_R T(x),$$

so, if a is a unit in R, then $H(T(x,y,z),d) = T(\{\mathrm{ad}^k(x)(y)\}_{k=0,\dots,r-1}) \otimes_R T(x) = UFH(\mathbb{L}(x,y,z),d)$, which is R-free, and it follows that any minimal model $(\Lambda W, D)$ for $C^*(\mathbb{L}(x,y,z),d)$ is decomposable, and $FH(\mathbb{L}(x,y,z),d)$ may be regarded as the homotopy Lie algebra of $(\Lambda W, D)$; also, R-torsion cannot occur in $H(\mathbb{L}(x,y,z),d)$ in dimensions less than $N(\min\{p,q\}, \varrho)$. $\square$

The situation described below is another instance related to Quillen models.

9. Example. Given integers $p \geq 1$, $q \geq 1$, $r \geq 1$ and $s \geq 0$, consider $\mathbb{L}(u,v,x,y,z)$ over R, with u of degree p, v of degree $p+1$, x of degree q, y of degree r and z of degree $qs+r+1$, equipped with the differential d given by $du = 0$, $dv = au$, $dx = 0$, $dy = 0$ and $dz = b\ \mathrm{ad}^s(x)(y)$, where $a, b \in R \setminus \{0\}$. For $p = 2$, $q = 1$, $r = 4$, $s = 1$, $R = \mathbb{Q}$ and $a = b = 1$, we recover the main ingredient of the surjective model of Quillen for the composition $S^2 \times S^5 \to S^7 \to S^4$, where the first arrow is the projection smashing the 6-skeleton down, and the second is the Hopf fibration ([12]). As in Example 7, a direct computation of the Euler-Poincaré series reveals that the kernel of the natural projection $(\mathbb{L}(u,v,x,y,z),d) \to (\mathbb{L}(x),0)$, with obvious right inverse $(\mathbb{L}(x),0) \hookrightarrow (\mathbb{L}(u,v,x,y,z),d)$, is the free differential graded Lie algebra on $\{\mathrm{ad}^n(x)(u), \mathrm{ad}^n(x)(v), \mathrm{ad}^n(x)(y), \mathrm{ad}^n(x)(z)\}$, $n \in \mathbb{N}$, whose differential d' satisfies $d' \operatorname{ad}^n(x)(u) = 0$, $d' \operatorname{ad}^n(x)(v) = (-1)^{nq}\, a \operatorname{ad}^n(x)(u)$, $d' \operatorname{ad}^n(x)(y) = 0$ and $d' \operatorname{ad}^n(x)(z) = (-1)^{nq}\, b \operatorname{ad}^{n+s}(x)(y)$, whatever n in $\mathbb{N}$. *Mutatis mutandis*, the considerations in the preceding example now repeat *verbatim.* $\square$

We end with two more examples inspired by Quillen minimal models. Recall that the abelian R-Lie algebra on a graded set $\{x_\alpha\}$, denoted by $\langle x_\alpha \rangle$, is the free graded R-module generated by $\{x_\alpha\}$.

10. Example. Fix integers $n \geq 2$ and $p \geq 1$ and consider $\mathbb{L}(x_1, \dots, x_n)$ over R, with x_i of degree $2ip - 1$, $i = 1, \dots, n$, equipped with the differential d given by $dx_i = (-a/2) \sum_{j+k=i} [x_j, x_k]$, $i = 1, \dots, n$, where $a \in R \setminus \{0\}$. For $p = 1$, $R = \mathbb{Q}$ and $a = 1$, we recover the minimal model of Quillen for the complex projective n-space, $\mathbb{CP}^n$ ([12]). A direct computation of the Euler-Poincaré series ([2]) reveals that the kernel of the natural projection $(\mathbb{L}(x_1, \dots, x_n), d) \to (\langle x_1 \rangle, 0)$, with obvious right inverse $(\langle x_1 \rangle, 0) \hookrightarrow (\mathbb{L}(x_1, \dots, x_n), d)$, has the form $\mathbb{L}(V, d)$, with $V = \langle x_2, x_2', \dots, x_n, x_n', x \rangle$, where $x_i' = (-1/2) \sum_{j+k=i} [x_j, x_k]$, $i = 2, \dots, n$, $x = (-1/2) \sum_{i+j=n+1} [x_i, x_j]$, and, of course, $dx_i = ax_i'$, $dx_i' = 0$, $i = 2, \dots, n$, and $dx = 0$. Thus, $FH(V, d) = \langle x \rangle$, so $FH\mathbb{L}(V, d) = \mathbb{L}(x)$ and $FHT(V, d) = T(x)$, by the theorem. On the other hand, it should now be clear that we can, *mutatis mutandis*, go through the previous general pattern to obtain: $FH(\mathbb{L}(x_1, \dots, x_n), d) = \mathbb{L}(x) \oplus \langle x_1 \rangle$, as Lie algebras — note that $a[x_1, x] = d((-1/2) \sum_{i+j=n} [x_{i+1}, x_{j+1}])$; $UFH(\mathbb{L}(x_1, \dots, x_n), d) = T(x) \otimes_R \Lambda(x_1)$, as left $T(x)$-modules and right $\Lambda(x_1)$-comodules; and $(T(x_1, \dots, x_n), d) = T(V, d) \otimes_R (\Lambda(x_1), 0)$, as left $T(V, d)$-modules and right $(\Lambda(x_1), 0)$-comodules. Consequently, $H(T(x_1, \dots, x_n), d) = HT(V, d) \otimes_R \Lambda(x_1)$, so $FH(T(x_1, \dots, x_n), d) = T(x) \otimes_R \Lambda(x_1)$, which shows that the natural arrow $UFH(\mathbb{L}(x_1, \dots, x_n), d) \to FHU(\mathbb{L}(x_1, \dots, x_n), d)$ is an isomorphism of graded cocommutative Hopf algebras. And if, in addition, a is invertible in R, then $H(T(x_1, \dots, x_n), d) = T(x) \otimes_R \Lambda(x_1)$ is R-free, so ([4]) any minimal model $(\Lambda W, D)$ for $C^*(\mathbb{L}(x_1, \dots, x_n), d)$ is decomposable, and $FH(\mathbb{L}(x_1, \dots, x_n), d) = \mathbb{L}(x) \oplus \langle x_1 \rangle$ may be regarded as the homotopy Lie algebra of $(\Lambda W, D)$; it might once again be worth remarking that $H(\mathbb{L}(x_1, \dots, x_n), d)$ is not necessarily R-torsion free: *e.g.*, for $n = 2$, letting $y = [x_1, [x_1, x_2]]$ and $z = [x_2, x_2]$, $(R/3)[y, [y, z]]$ is a direct summand in $H(\mathbb{L}(x_1, x_2), d)_{24p+16}$. $\square$

11. Example. Fix integer $p \geq 1$ and consider $\mathbb{L}(\{x_n\}_{n=1,2,\dots})$ over R, with x_n of degree $2np - 1$, $n = 1, 2, \dots$, equipped with the differential d given by $dx_n = (-a/2) \sum_{i+j=n} [x_i, x_j]$, $n = 1, 2, \dots$, where $a \in R \setminus \{0\}$. For $p = 1$, $R = \mathbb{Q}$ and $a = 1$, we recover the minimal model of Quillen for $\mathbb{CP}^\infty$ ([12]). The kernel of the natural projection $(\mathbb{L}(\{x_n\}_{n=1,2,\dots}), d) \to (\langle x_1 \rangle, 0)$, with obvious right inverse $(\langle x_1 \rangle, 0) \hookrightarrow (\mathbb{L}(\{x_n\}_{n=1,2,\dots}), d)$, has again the form $\mathbb{L}(V, d)$, this time V being $\langle \{x_n, x_n'\}_{n=2,3,\dots} \rangle$, where $x_n' = (-1/2) \sum_{i+j=n} [x_i, x_j]$, $n = 2, 3, \dots$, and, of course, $dx_n = ax_n'$ and $dx_n' = 0$, $n = 2, 3, \dots$. Clearly, $FH(V, d) = 0$, so $FH\mathbb{L}(V, d) = 0$ and $FHT(V, d) = R$ (*cf.* Remark 2. (3)). As in the previous example, we now obtain successively: $FH(\mathbb{L}(\{x_n\}_{n=1,2,\dots}),d) = \langle x_1\rangle$; $UFH(\mathbb{L}(\{x_n\}_{n=1,2,\dots}),d) = \Lambda(x_1)$; $(T(\{x_n\}_{n=1,2,\dots}),d) = T(V,d)\otimes_R (\Lambda(x_1),0)$, as left $T(V,d)$-modules and right $(\Lambda(x_1),0)$-comodules, so $H(T(\{x_n\}_{n=1,2,\dots}),d) = HT(V,d)\otimes_R \Lambda(x_1)$, and $FH(T(\{x_n\}_{n=1,2,\dots}),d) = \Lambda(x_1)$. The natural arrow

$$UFH(\mathbb{L}(\{x_n\}_{n=1,2,\dots}),d) \to FHU(\mathbb{L}(\{x_n\}_{n=1,2,\dots}),d)$$

is thus an isomorphism of graded cocommutative Hopf algebras. As before, if a is a unit in R, then $H(T(\{x_n\}_{n=1,2,\dots}),d) = \Lambda(x_1)$ is R-free, so ([4]) any minimal model $(\Lambda W, D)$ for $C^*(\mathbb{L}(\{x_n\}_{n=1,2,\dots}),d)$ is decomposable, and $FH(\mathbb{L}(\{x_n\}_{n=1,2,\dots}),d) = \langle x_1\rangle$ may be regarded as the homotopy Lie algebra of $(\Lambda W, D)$. □

12. Remark. Quite similar examples can analogously be derived from other Quillen minimal models, *e.g.*, from that for the reduced product $\mathbb{CP}^2 \wedge \mathbb{CP}^3$, or from free models for total spaces of fibrations, *e.g.*, from that for the total space E of the fibration $\mathbb{CP}^2 \to E \to S^6$ ([12]). □

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