Some results on sequentially compact extensions

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Abstract. The class of Hausdorff spaces (or of Tychonoff spaces) which admit a Hausdorff (respectively Tychonoff) sequentially compact extension has not been characterized. We give some new conditions, in particular, we prove that every Tychonoff locally sequentially compact space has a Tychonoff one-point sequentially compact extension. We also give some results about extension of functions and about lattice properties of the family of all minimal sequentially compact extensions of a given space.

Keywords: sequentially compact extension, locally sequentially compact space, extension of functions

Classification: 54D35, 54C20, 54D80

Introduction

It is well known that complete regularity is a necessary and sufficient condition for a space to admit a Hausdorff compact extension, that is, a compactification. However, it is not either necessary or sufficient for the existence of a Hausdorff sequentially compact extension (see [E, Example 3.10.B] and [FV, Proposition 1.2, Example 3]. It is still an open problem to characterize Hausdorff spaces which have a Hausdorff sequentially compact extension. Also, a Hausdorff sequentially compact extension of a Tychonoff space can fail to be Tychonoff.

Some Tychonoff spaces obviously have a sequentially compact compactification, hence a normal sequentially compact extension: (i) spaces of weight less than s; (ii) locally compact spaces of cardinality less than 2^t; (iii) subspaces of LOTS. (The cardinals t and s satisfy $\omega_1 \leq t \leq s \leq c$; see [vD, Sections 3, 6]).

Moreover, it was proved that every metrizable space has a normal first-countable sequentially compact extension ([N]). In [FV] it is proved that every P-space has a Hausdorff (nonregular) sequentially compact extension.

A Hausdorff space is said to be *almost locally sequentially compact* (ALSC) if every point has a closed sequentially compact neighborhood. A regular ALSC space is *locally sequentially compact* (LSC), that is, every point has a local base consisting of closed sequentially compact sets. If X is ALSC, then we can define a Hausdorff one-point sequentially compact extension, denoted by oX in [FV], in which the open neighborhoods of the new point are the complements of the closed sequentially compact subsets of X. In general oX is not the only one-point sequentially compact extension of X.

Partially supported by research funds of the MURST.

A *sequential compactification* of X is a sequentially compact extension of X which is minimal, that is, does not properly contain any sequentially compact extension of X. Every sequentially compact extension of X contains a sequential compactification of X ([FV]). We denote sequential compactifications of X with symbols like aX, $bX, \dots$

We compare sequential compactifications like compactifications, that is, we say that $aX \geq bX$ if there is a continuous map p_{ab} from aX to bX which is the identity on X. The minimality of bX implies that p_{ab} is surjective. In general, p_{ab} is not a quotient map.

The partial ordered set of all sequential compactifications of X (up to equivalence) is denoted by $\mathcal{SK}(X)$. We denote by $\mathcal{SK}_T(X)$ the subfamily of $\mathcal{SK}(X)$ consisting of Tychonoff sequential compactifications of X. (As we have already remarked, those sets can be empty.)

In the first section of this paper we observe that, for Tychonoff LSC spaces, oX can fail to be Tychonoff. However, we prove that every Tychonoff LSC space has a Tychonoff one-point sequentially compact extension. This is obtained as consequence of a general theorem about locally bounded spaces. We also give some results about the existence of *strict* (Hausdorff or Tychonoff) sequentially compact extensions, where "strict" means that closed sequentially compact subsets of X remain closed.

In Section 2 we prove that, for every bounded continuous real-valued function f defined on a Tychonoff LSC space, there exists a Tychonoff sequentially compact extension of X to which f extends. We deduce some lattice results about $\mathcal{SK}_T(X)$ and $\mathcal{SK}(X)$.

In the last section we give some new conditions for a space to have Hausdorff sequentially compact extensions. We also give some more results about extension of functions.

All spaces will be Hausdorff, unless otherwise specified, and "extension of X" always means "Hausdorff extension of X in which X is dense". As usual, we denote by βX the Stone-Čech compactification of X and by $C^*(X)$ the ring of bounded continuous real-valued function on X.

1. One-point Tychonoff extensions

Let X be a Tychonoff LSC space. The following example shows that the one-point extension oX can fail to be Tychonoff.

Example. Let X be the Tychonoff plank $((\omega_1 + 1) \times (\omega + 1)) \setminus \{(\omega_1, \omega)\}$. We know that βX is the only nontrivial Tychonoff extension of X. X is clearly LSC, and the set $\omega_1 \times \{\omega\}$ is closed in X and sequentially compact, then it is closed in oX. This implies that $oX \neq \beta X$, so that oX is not Tychonoff. Note that βX is sequentially compact, hence X has a Tychonoff one-point sequentially compact extension (but no strict Tychonoff sequentially compact extension).

We will prove that every Tychonoff LSC space has a Tychonoff (one-point) sequentially compact extension. This will be obtained as a corollary of a more general result involving the concept of boundedness.

Definition ([H])**.** A *boundedness* on a space X is a family $\mathcal{F}_X$ of subsets of X which is closed with respect to subsets and finite unions. We say that a subset of X is *bounded* (with respect to $\mathcal{F}_X$) if it is in $\mathcal{F}_X$, *unbounded* otherwise. $\mathcal{F}_X$ is said to be *closed* if, for $F \subset X$, $F \in \mathcal{F}_X$ implies $\overline{F} \in \mathcal{F}_X$. We will say that $\mathcal{F}_X$ is *open* if every bounded set is contained in an open bounded set.
We say that X is *locally bounded* with respect to a boundedness $\mathcal{F}_X$ if every $x \in X$ has a neighborhood $V \in \mathcal{F}_X$.

Whenever X is locally bounded and unbounded with respect to a closed boundedness $\mathcal{F}_X$, we can construct a Hausdorff one-point extension such that the open neighborhoods of the new point p are of the form $\{p\} \cup U$, where U is open in X and $X \setminus U$ is in $\mathcal{F}_X$. We denote that extension by $o(\mathcal{F}_X)$. Every one-point Hausdorff extension is of this form.

Clearly, ALSC means locally bounded with respect to the closed boundedness

$$\mathcal{SC}_X = \{A \subset X \mid \overline{A} \text{ is sequentially compact}\}$$

and $oX = o(\mathcal{SC}_X)$.

A one-point extension $o(\mathcal{F}_X)$ is sequentially compact if and only if $\mathcal{F}_X \subset \mathcal{SC}_X$.

Theorem 1.1. *Let $\mathcal{F}_X$ be a closed boundedness on X and let X be locally bounded and unbounded. Let X be T_3 (T_4). Then $o(\mathcal{F}_X) = X \cup \{p\}$ is T_3 (respectively T_4) if and only if $\mathcal{F}_X$ is open.*

Proof: First suppose X is T_3. Then every point of X has a local base of closed members of $\mathcal{F}_X$, hence, in $o(\mathcal{F}_X)$, every point of X has base of closed neighborhoods. Clearly, the condition that every $F = Cl_X F \in \mathcal{F}_X$ is contained in an open $W \in \mathcal{F}_X$ (so that $Cl_X W$ is also in $\mathcal{F}_X$) is equivalent to the condition that every open neighborhood of p in $o(\mathcal{F}_X)$ contains a closed neighborhood. Now, let X be T_4 and suppose $\mathcal{F}_X$ is open. Let A, B be disjoint closed subsets of $o(\mathcal{F}_X)$. Without loss of generality we assume that $p \notin A$. Let V be an open subset of X such that $A \subset V$ and $Cl_X V \cap B = \emptyset$. Since $A \in \mathcal{F}_X$, one has $A \subset U$, where $U \in \mathcal{F}_X$ and U is open. Then $G = Cl_X U \cap V \in \mathcal{F}_X$ and $G \cap B = \emptyset$. Then $U \cap V$ and $o(\mathcal{F}_X) \setminus G$ are disjoint open subsets of $o(\mathcal{F}_X)$ which contain A and B, respectively. The converse follows from the first part. $\square$

Corollary 1.2. *Let X be a T_3 (T_4) LSC space. Then oX is T_3 (respectively T_4) if and only if every closed sequentially compact subset of X has a closed sequentially compact neighborhood.*

Let $\mathcal{F}_X$ be a closed boundedness. Put

$$\mathcal{F}'_X = \{F \subset X \mid \exists f \in C(X, \mathbf{I}), a \in (0,1) : \overline{F} \subset f^{-1}([0,a)) \quad \text{and} \quad f^{-1}([0,1)) \in \mathcal{F}_X\},$$

where $\mathbf{I}$ be the unit interval. Note that $\mathcal{F}'_X \subset \mathcal{F}_X$.

Theorem 1.3. *Let X be a Tychonoff space and suppose X is locally bounded and unbounded with respect to a closed boundedness $\mathcal{F}_X$. Then $\mathcal{F}'_X$ is an open and closed boundedness and X is locally bounded with respect to $\mathcal{F}'_X$. Moreover $o(\mathcal{F}'_X)$ is Tychonoff. If X is T_4, then $o(\mathcal{F}'_X)$ is also T_4.*

PROOF: To prove that $\mathcal{F}'_X$ is a closed boundedness we need only to prove that it is closed with respect to finite unions. If F and G are in $\mathcal{F}'_X$, then the number a in the definition of $\mathcal{F}'_X$ can be supposed to be the same for F and G. Then our claim is proved by the equation

$$f^{-1}([0,a)) \cup g^{-1}([0,a)) = (f \wedge g)^{-1}([0,a)).$$

Notice that all the sets of the form $f^{-1}([0,a))$ and $f^{-1}([0,a])$, where $a < 1$ and $f^{-1}([0,1)) \in \mathcal{F}_X$, are in $\mathcal{F}'_X$. This proves that $\mathcal{F}'_X$ is open. Now let U be a neighborhood of $x \in X$. Then U contains an open neighborhood V of x which is in $\mathcal{F}_X$. Let $f \in C(X,\mathbf{I})$ such that $f(x) = 0$ and $f^{-1}([0,1)) \subset V$. For any $a \in (0,1)$, $f^{-1}([0,a))$ is a neighborhood of x which is in $\mathcal{F}'_X$. Now we prove that $o(\mathcal{F}'_X) = X \cup \{p\}$ is Tychonoff. Let A be a closed subset of $X \cup \{p\}$ and suppose $x \notin A$. First let $x \neq p$, so that x has open neighborhood $U \in \mathcal{F}'_X$ such that $U \subset X \setminus A$. Let $f \in C(X,\mathbf{I})$ be such $f(x) = 0$, and $f(X \setminus U) = 1$. Since $o(\mathcal{F}'_X) \setminus U$ is a (closed) neighborhood of p containing A, then we can define $\tilde{f} : o(\mathcal{F}'_X) \to \mathbf{I}$ which extends f and satisfies $\tilde{f}(A) = 1$. Now, suppose $x = p \notin A$. Then $A \in \mathcal{F}'_X$, hence there is $f \in C(X,\mathbf{I})$ such that, $A \subset f^{-1}([0,a))$ and for $b \in (a,1)$, $f^{-1}([0,b]) \in \mathcal{F}'_X$. Then $(f^{-1}([b,1]) \cup \{p\} \supset o(\mathcal{F}'_X) \setminus f^{-1}([0,b])$ which is a neighborhood of p. Take $g = (f \vee a) \wedge b$. Then $g(A) = a$ and $g(f^{-1}([b,1)) = b$. Then we can extend g to $o(\mathcal{F}'_X)$ and we obtain a map $\tilde{g}$ such that $\tilde{g}(A) = a$ and $\tilde{g}(p) = b$. This proves that $o(\mathcal{F}'_X)$ is Tychonoff. The last claim easily follows, using Theorem 1.1. □

The definition of $\mathcal{F}'_X$ was inspired by a particular boundedness defined in [CG].

Corollary 1.4. *If X is a Tychonoff (T_4) LSC space and X is not sequentially compact, then there exists a one-point sequentially compact extension of X which is Tychonoff (respectively T_4).*

PROOF: We can use the above theorem, taking $\mathcal{F}_X = \mathcal{SC}_X$. □

Using suitable boundednesses, we can deduce analogous results for different kinds of extensions.

We recall that a sequentially compact extension (in particular, a sequential compactification) Y of X is said to be *strict* if every closed sequentially compact subset of X is closed in Y.

Remark. If X is paracompact, then every sequentially compact extension of X is strict. In fact, a closed sequentially compact subset of a paracompact space is both paracompact and countably compact, hence compact.

From Theorem 1.3 we can deduce an equivalent condition for a Tychonoff LSC space to admit a strict Tychonoff sequentially compact extension.

Let $\mathcal{F}_X$ be any closed boundedness in a Tychonoff space X. If X is locally bounded and $o(\mathcal{F}_X)$ is Tychonoff then, clearly, one has $\mathcal{F}'_X = \mathcal{F}_X$. Conversely, if $\mathcal{F}'_X = \mathcal{F}_X$ and $\mathcal{F}_X$ contains all singletons, then X is locally bounded with respect to $\mathcal{F}_X$ and $o(\mathcal{F}_X)$ is Tychonoff, by Theorem 1.3. Then one has

Theorem 1.5. *For a Tychonoff space X the following are equivalent:*

(i) $\mathcal{SC}'_X = \mathcal{SC}_X$;
(ii) *X has a one-point strict Tychonoff sequentially compact extension (equivalently, oX is Tychonoff);*
(iii) *X is LSC and admits a strict sequentially compact extension.*

Proof: The equivalence of (i) and (ii) is proved by the previous remarks; (ii) and (iii) are equivalent by [FV, 1.6 and 2.8]. □

The example of the Tychonoff plank can be generalized as follows:

Proposition 1.6. *If X is almost compact and is not sequentially compact, then X has a strict Tychonoff sequentially compact extension if and only if, for closed subsets of X, compactness is equivalent to sequential compactness.*

For any Tychonoff space, we have the following necessary condition:

Proposition 1.7. *If a Tychonoff space X admits a strict Tychonoff sequentially compact extension, then every infinite subset of X which contains no nontrivial convergent sequence has a countable closed discrete subset D which is functionally separated from every closed sequentially compact subset F of X with $F \cap D = \emptyset$.*

Proof: Let Y be a strict sequential compactification of X and let A be a subset of X which does not contain any nontrivial convergent sequence. Then A contains $D = \{x_n\}$ with $x_n \to y \in Y \setminus X$. Clearly D is closed and discrete in X. Let F be a closed sequentially compact subset of X, disjoint from D. Since F is closed in Y, there exists $f : Y \to \mathbf{I}$ which is 0 on F and greater than 1/2 on a open neighborhood U of y. Let $n \in \omega$ such that, for every $k > n$, $x_k \in U$. For every $h \leq n$, let U_h be an open neighborhood of x_h, in X, which is disjoint from F. Let $f_h : X \to \mathbf{I}$ such that $f_h(X \setminus U_h) = 0$ and $f_h(x_h) = 1$. Then $g = f\mid_X + \sum_h f_h$ is a (bounded) real-valued function on X which separates D from F. □

We do not know a characterization of Tychonoff spaces which admit a Tychonoff strict sequentially compact extension. An analogous problem for countably compact extensions was studied by several authors (see [M], [K] and [vD, Section 7]). They mostly studied possible *countably-compactifications* (that is, “strict” countably compact extensions) of X contained in βX. In fact, it was proved that, if X admits a countably-compactification, then there is one between X and βX ([M]). This is not true for sequentially compact extensions.

The following proposition shows that a Hausdorff space admits a strict (Hausdorff) sequentially compact extension if and only if it admits a sequentially compact extension.

Proposition 1.8. *For every sequentially compact extension Y of X, there exists a strict sequentially compact extension Z of X such that $Z \setminus X \cong Y \setminus X$.*

Proof: Let Z be the space with the same underlying set as Y and with the topology generated by the topology of Y and by the family of the complements of the closed sequentially compact subsets of X. Clearly Z is Hausdorff and $Z \setminus X \cong Y \setminus X$. Since we can suppose X is not sequentially compact, X is dense in Z. It is easy to prove that Z is sequentially compact. $\square$

With the same notation of the above theorem, if Y is a sequential compactification of X, then Z is also a sequential compactification, hence one has:

Corollary 1.9. *For every sequential compactification aX of X there is a strict sequential compactification bX such that $bX \geq aX$ and $bX \setminus X \cong aX \setminus X$. Furthermore, bX is the least strict sequential compactification greater than or equal to aX.*

It is easy to see that a sequential compactification of X, which is greater than a strict sequential compactification, or is a quotient of a strict sequential compactification, is also strict. Then, by [FV, 2.3 and 2.8], we obtain:

Proposition 1.10. *Let $\mathcal{SK}(X)$ be nonempty. Then the set of strict sequential compactifications of X is a (nonempty) upper semilattice. Moreover, it is complete lower semilattice if and only if X is ALSC.*

2. Extension of functions

If X is Tychonoff and $\mathcal{G} \subset C^*(X)$ separates points from closed sets, we can use $\mathcal{G}$ to embed X into a Tychonoff cube of weight $|\mathcal{G}|$. The closure of the image of X is a compactification of X to which every member of $\mathcal{G}$ continuously extends. Since Tychonoff cubes of weight greater than or equal to s are not sequentially compact, we cannot create sequentially compact extensions of X in this way, when $w(X) \geq s$.

However, it is known that, if X is ALSC, then every continuous function from X to a sequentially compact space Y can be extended to a sequentially compact extension of X [FV, Theorem 3.1]. We want to prove that, if we also suppose that X and Y are Tychonoff, then we can extend f to a Tychonoff sequentially compact extension of X.

A construction equivalent to Whyburn's unified space ([W]) was used in [L] and [CFV] to find the smallest compactification of a locally compact space to which a given function extends. We will imitate that construction.

Let X, Y be (Hausdorff) spaces and let $f : X \to Y$ be a continuous mapping. Suppose X is locally bounded with respect to a closed boundedness $\mathcal{F}_X$. Put, on the disjoint union of X and Y, the topology $\mathcal{T}_f$ generated by the open subsets of

X and by the subsets of the form $V \cup (f^{-1}(V) \setminus G)$, where V is open in Y and G is a closed member of $\mathcal{F}_X$.

Lemma 2.1. *$(X \cup Y, \mathcal{T}_f)$ is a Hausdorff space to which f extends. For a closed subset F of X, $F \in \mathcal{F}_X$ implies that F is closed in $(X \cup Y, \mathcal{T}_f)$. If Y is compact, then the converse is also true.*

Proof: The proof of the first statement is straightforward. A map $\hat{f}$ extending f is defined by $\hat{f}\mid_X = f$ and $\hat{f}\mid_Y = 1_Y$. It is also clear, by definition, that elements of $\mathcal{F}_X$ are closed in $X \cup Y$.
Now, suppose Y is compact and let $F \subset X$ be a closed subset of $X \cup Y$. For every $y \in Y$ there is an open subset U_y of Y and a closed member G_y of $\mathcal{F}_X$ such that $y \in U_y \cup (f^{-1}(U_y) \setminus G_y) \subset (X \cup Y) \setminus F$. Let $\{U_{y_i}\}$ be a finite subcover of Y. One has $\bigcup_i (f^{-1}(U_{y_i}) \setminus G_{y_i}) \subset X \setminus F$. Since $\bigcup_i f^{-1}(U_{y_i}) = X$, if an element x of X is not in $\bigcup_i (f^{-1}(U_{y_i}) \setminus G_{y_i})$, then x must belong to some G_{y_i}. Therefore we obtain

$$F \subset X \setminus (\bigcup_i (f^{-1}(U_{y_i}) \setminus G_{y_i})) \subset \bigcup_i G_{y_i},$$

which implies $F \in \mathcal{F}_X$. $\square$

Let us denote by $E(f, \mathcal{F}_X)$ the closure of X in $X \cup Y$, so that $E(f, \mathcal{F}_X)$ is a Hausdorff extension of X to which f extends. We denote by $\tilde{f}$ the (unique) extension of f, obtained as restriction of $\hat{f}$. Note that X is open in $E(f, \mathcal{F}_X)$.

Suppose Y is sequentially compact, X is ALSC (non-sequentially compact) and $\mathcal{F}_X \subset \mathcal{SC}_X$. Then $E(f, \mathcal{F}_X)$ is a sequentially compact extension of X where closed elements of $\mathcal{F}_X$ are closed. In particular, $E(f, \mathcal{SC}_X)$, is a strict sequentially compact extension.

$E(f, \mathcal{SC}_X)$ can fail to be Tychonoff even if X and Y are Tychonoff. In fact, $oX = E(f, \mathcal{SC}_X)$ when f is constant (see Section 1). However, one has:

Theorem 2.2. *Let X be a Tychonoff LSC space. For every continuous $f : X \to Y$, where Y is Tychonoff and sequentially compact, there exists a Tychonoff sequentially compact extension of X to which f extends.*

Proof: Let K be any compactification of Y. We denote by f_1 the composition of f with the embedding of Y into K. We will prove that $E(f_1, \mathcal{SC}'_X)$ is Tychonoff. Put $S = E(f_1, \mathcal{SC}'_X) \setminus X$. Let $q : E(f_1, \mathcal{SC}'_X) \to o(\mathcal{SC}'_X) = X \cup \{p\}$ be the natural map which collapses S to the point p. By the above lemma, q is a quotient map. Let now A be a closed subset of $E(f_1, \mathcal{SC}'_X)$ and suppose $z \notin A$. First let $z \in X$ and put $B = A \cup S$. Then $q(B)$ is a closed subset of $o(\mathcal{SC}'_X)$ which does not contain z. Therefore z and $q(B)$ are separated by a continuous function g from $o(\mathcal{SC}'_X)$ to $\mathbf{I}$. Clearly $g \circ q$ separates z and A. Now, let $z \in S$ and $A \subset X$. Then $A = q(A)$ is closed in $o(\mathcal{SC}'_X)$. Let $g_1 : o(\mathcal{SC}'_X) \to \mathbf{I}$ be a function which separates p from A. Then $g_1 \circ q$ separates z from A. Note that $g_1 \circ q$ is constant on S. Finally let $F = A \cap S \neq \emptyset$ and $z \in S$. Take a map $v : S \to \mathbf{I}$ such that

$v(F) = 0$ and $v(z) = 1$. Let $h = v \circ \tilde{f}_1$ and let $U = h^{-1}([0, 1/2))$. Then $A \setminus U$ is a closed subset of $E(f_1, \mathcal{SC}'_X)$ contained in X. We can take, as before, a function $u : E(f_1, \mathcal{SC}'_X) \to \mathbf{I}$ such that $u(S) = 1$ and $u(A \setminus U) = 0$. Then the map $h \wedge u$ is less than $1/2$ in $U \cup A$ and maps z to 1. We have proved that $E(f_1, \mathcal{SC}'_X)$ is Tychonoff. It is easy to see that $E(f, \mathcal{SC}'_X)$ is a subspace of $E(f_1, \mathcal{SC}'_X)$, so it is also Tychonoff. This completes the proof. □

Let

$$h = \min\{\kappa \mid \exists \text{ a product of } \kappa \text{ sequentially compact spaces which is not sequentially compact}\}.$$

One has $\omega_1 \leq h \leq c$ and, more precisely, $t \leq h \leq s$. A set theoretic definition of h was given in [S].

Corollary 2.3. *Let X be a Tychonoff LSC space. For every family $\{f_l : X \to Y_l\}_{l<\kappa}$, where $\kappa < h$ and Y_l is sequentially compact for every l, there exists a Tychonoff sequentially compact extension of X to which every f_l extends.*

PROOF: We can use the above theorem, putting $Y = \prod Y_l$ and $f = \Delta\{f_l\}$, the diagonal map. □

Corollary 2.4. *Let X be a Tychonoff LSC space. For every family $\{f_l\}_{l<\kappa} \subset C^*(X)$, where $\kappa < s$, there exists a Tychonoff sequentially compact extension of X to which every f_l extends.*

If X is not pseudocompact, then a space Y with $X \subset Y \subset \beta X$ cannot be sequentially compact. In fact, let N be a C-embedded copy of ω contained in X. Then $Cl_{\beta X} N \cong \beta\omega$, hence N contains no nontrivial sequence which converges in βX. Since $N \subset Y \subset \beta X$, Y is not sequentially compact.

Obviously, if Y is a Tychonoff extension of X such that every element of $C^*(X)$ extends to Y, then $\beta Y = \beta X$. Therefore one has:

Proposition 2.5. *If X is LSC and it is not pseudocompact, then there is no Tychonoff sequentially compact extension of X to which every member of $C^*(X)$ extends.*

Let X be a Tychonoff space and suppose $\mathcal{SK}_T(X) \neq \emptyset$ (see Introduction).

Any nonempty subfamily of $\mathcal{SK}_T(X)$ of cardinality $\kappa < h$ has a supremum. Also, whenever a subfamily of $\mathcal{SK}_T(X)$ has an upper bound (lower bound) in $\mathcal{SK}(X)$, then it has a supremum (respectively, an infimum) which is in $\mathcal{SK}_T(X)$. The proofs are analogous to [FV, 2.3, 2.4, 2.5]

By [FV, Prop. 2.4] we also deduce that, if $\mathcal{SK}_T(X)$ is nonempty and does not have a supremum, then $\mathcal{SK}(X)$ cannot have a supremum.

Proposition 2.6. *Let X be a Tychonoff non-sequentially compact space.*

(i) *If X has a sequential compactification $tX \subset \beta X$, then $tX = \sup \mathcal{SK}_T(X)$.*

(ii) *If X is LSC and* $\sup \mathcal{SK}_T(X)$ *exists, then it is contained in* βX.
(iii) *If X is LSC and it is not pseudocompact, then* $\mathcal{SK}_T(X)$ *does not have a supremum.*
(iv) *If X is second countable, then* $\mathcal{SK}_T(X)$ *does not have a supremum.*

Proof: (i). Let aX be a Tychonoff sequential compactification , with $aX \subset \beta X$, and suppose that aX is not the supremum of $\mathcal{SK}_T(X)$. Then there is $Y = bX \in \mathcal{SK}_T(X)$ with $bX > aX$. The canonical map $p_{ba} : bX \to aX$, considered as a map into βX, extends to βY. Let $q : \beta Y \to \beta X$ be the extension. Since $q\mid_X$ is the identity of X and βY is a compactification of X, q must be a homeomorphism, so in particular, p_{ba} is a homeomorphism.

(ii). If $sX = \sup \mathcal{SK}_T(X)$, then, by Theorem 2.2, every $f \in C^*(X)$ extends to sX.

(iii) directly follows from Proposition 2.5.

(iv). If X is second countable, then, for every $f \in C^*(X)$, there is a metrizable compactification, hence a Tychonoff sequentially compact extension, to which f extends. Using the same argument as before, we deduce that, if $\mathcal{SK}_T(X)$ had a supremum, then it would be a subspace of βX containing X. But this is impossible, since X is not pseudocompact. □

Example. If X is the Tychonoff plank, then $\mathcal{SK}_T(X)$ is trivially a complete lattice.

We do not know any space X such that $\mathcal{SK}(X)$ has a supremum.

3. Ψ-systems

Let X be any Hausdorff space. It is known that, if X has a sequentially compact extension, then one has:

(I) every infinite subset of X which does not contain any (nontrivial) convergent sequence has an infinite closed discrete subset;
(II) every closed countable discrete subset of X contains an infinite set D which is separated from every point of $X \setminus D$ by disjoint open sets ([FV, Proposition 1.1]).

Proposition 3.1. *If X is sequential, then X satisfies* (I).

Proof: Let S be an infinite subset of X which does not contain any convergent sequence. Then S is closed. Furthermore, S is not sequentially compact, hence it is not countably compact. Then S contains an infinite discrete subset D which is closed in S, hence in X. □

Remark. Note that ω-collectionwise Hausdorff spaces, in particular, functionally Hausdorff spaces, satisfy (II).

We recall that X is said to be *weakly ω-collectionwise Hausdorff* if for every closed countable discrete subset E of X, there exists an infinite $F = \{x_n\} \subset E$ and a family $\{U_n\}$ of pairwise disjoint open sets such that $x_n \in U_n$ for each n.

Proposition 3.2. *If X satisfies* (II), *then X is weakly ω-collectionwise Hausdorff.*

Proof: Suppose E is a closed countable discrete subset of X. Let x_1 be any point of E and let E_1 be an infinite subset of $E \setminus \{x_1\}$ which is separated, by open sets, from every point of its complement. Let U_1, V_1 be disjoint open sets which separate x_1 from E_1. Let now $x_2 \in E_1$, and let E_2 be an infinite subset of $E_1 \setminus \{x_2\}$ which is separated from any point of its complement. Let U and V be disjoint open sets which separate x_2 from E_2. Then we put $U_2 = U \cap V_1$ and $V_2 = V \cap V_1$, so that one U_1, U_2 and V_2 are pairwise disjoint. In this way we can construct, inductively, an infinite subset $F = \{x_n\}$ of E and a family $\{U_n\}$ of pairwise disjoint open subsets of X with $x_n \in U_n$ for each n. □

Corollary 3.3. *If X has a sequentially compact extension, then X is weakly ω-collectionwise Hausdorff.*

The space in the Example 2 in [FV] is weakly ω-collectionwise Hausdorff, but does not satisfy (II), hence the converse of Lemma 3.2 is false.

In the following definitions we do not suppose that X is Hausdorff. Let $\mathcal{A}$ be an (infinite) maximal almost disjoint family (MADF) of countable closed discrete subsets of X. For each $A \in \mathcal{A}$ we take a point $x_A \notin X$ ($x_A \neq x_B$ when $A \neq B$). Following [NV], we put

$$\Psi(X, \mathcal{A}) = X \cup \{x_A\}_{A \in \mathcal{A}},$$

endowed with the topology generated by the open subsets of X and by the sets of the form $\{x_A\} \cup U$, where U is open in X and $A \setminus U$ is finite. For $X = \omega$ we obtain the well known example by Mrówka.

The following construction was done in [J] for $X = \omega$ and generalized in [NV].

A Ψ*-system on* X is a family $\{(X_\alpha, \mathcal{A}_\alpha)\}_{\alpha \leq \omega_1}$, with $X_0 = X$, such that: $\mathcal{A}_\alpha$ is a MADF of countable closed discrete subsets of the space X_α; $X_{\alpha+1} = \Psi(X_\alpha, \mathcal{A}_\alpha)$ for each α; if α is a limit ordinal, then $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ with the topology generated by the topologies of the X_β's.

It is easy to see that X_{ω_1} does not have infinite closed discrete subsets, that is, $\mathcal{A}_{\omega_1} = \emptyset$. If X is T_1, then X_{ω_1} is T_1. For $X = \omega$, X_{ω_1} is Hausdorff [J], but this is not true in general for Hausdorff spaces.

It was proved that, if a (Hausdorff) space X satisfies (I), then X_{ω_1} is (T_1 and) sequentially compact for every Ψ-system on X. Moreover, for every sequential compactification aX of X, there is a Ψ-system $\{(X_\alpha, \mathcal{A}_\alpha)\}$ on X and a continuous map from X_{ω_1} onto aX which is the identity on X [FV, 1.8 and 2.10].

It is easy to prove that X is ω-collectionwise Hausdorff if and only if $\Psi(X, \mathcal{A})$ is Hausdorff for each MADF $\mathcal{A}$ of countable closed discrete subsets of X.

However, the condition that X is ω-collectionwise Hausdorff does not imply that, for every Ψ-system on X, X_{ω_1} is Hausdorff. In fact, this is not true even for normal zero-dimensional spaces, as the following example shows.

Example. Since $\mathbf{R} \cup \{-\infty, +\infty\}$ is a sequential compactification of $\mathbf{Q}$, there exists a Ψ-system $\{X_\alpha, \mathcal{A}_\alpha\}$ on $\mathbf{Q}$ and a continuous surjective map $f : X_{\omega_1} \to \mathbf{R} \cup \{-\infty, +\infty\}$ which is the identity on $\mathbf{Q}$. Let $q \in \mathbf{Q}$ and let $\{y_n\}$ be a sequence in $\mathbf{R} \setminus \mathbf{Q}$ converging to q. For every n, let z_n be a point of $X_{\omega_1} \setminus \mathbf{Q}$ such that $f(z_n) = y_n$. The sequence $\{z_n\}$ has an accumulation point $z \in X_{\omega_1} \setminus \mathbf{Q}$ and one has $f(z) = q$. Then X_{ω_1} is not Hausdorff, since the identity of $\mathbf{Q}$ cannot be extended to any Hausdorff space.

If a space X has a weak base consisting of countable compact sets, then X is sequential, by [NV, Lemma 2.8]. Therefore, in view of Proposition 3.1, we can remove one of the hypotheses of Theorem 1.10 in [FV] and we obtain:

If X has a weak base consisting of countable compact sets, then for every Ψ-system on X, X_{ω_1} is a Hausdorff, hence it is a sequentially compact extension of X.

There exists a space which satisfies the hypothesis of the above theorem and is not ALSC or first countable [FV, Example 6].

For every continuous map $f : X \to Z$ with Z sequentially compact, we will denote by $\mathcal{E}(f)$ the family of countable closed discrete subsets E of X with the following property: there is $z \in Z$ such that, for every open neighborhood U of z, $E \setminus f^{-1}(U)$ is finite. This is equivalent to the following: for every surjective $u : \omega \to E$, the sequence $\{f(u(n))\}_{n\in\omega}$ is convergent.

Lemma 3.4. *If $f : X \to Z$ is a continuous map and Z is sequentially compact, then there exists a MADF $\mathcal{A}$ of countable closed discrete subsets of X and a unique continuous map $\tilde{f} : \Psi(X, \mathcal{A}) \to Z$ which extends f.*

Proof: Let $\mathcal{A}$ be a subfamily of $\mathcal{E}(f)$ which is almost disjoint and is maximal among the almost disjoint subfamilies of $\mathcal{E}(f)$. Then $\mathcal{A}$ is a MADF of countable closed discrete subsets of X. In fact, if B is a countable closed discrete subset of X, then either $f(B)$ is finite or contains a nontrivial convergent sequence. In any case, B contains an element of $\mathcal{E}(f)$, and this implies that there exists $A \in \mathcal{A}$ such that $B \cap A$ is infinite. Since, for every $A \in \mathcal{A}$, $f(A)$, viewed as a sequence, converges to a point z_A of Z, we can extend f to $\Psi(X, \mathcal{A})$ putting $\tilde{f}(x_A) = z_A$. To prove that $\tilde{f}$ is continuous, let U be an open neighborhood of $\tilde{f}(x_A)$. Then $A \setminus \tilde{f}^{-1}(U)$ is finite. Therefore $\{x_A\} \cup f^{-1}(U)$ is a basic neighborhood of x_A whose image is contained in U. The extension $\tilde{f}$ is clearly unique. □

Theorem 3.5. *If $f : X \to Z$ is a continuous map and Z is sequentially compact, then there exists a Ψ-system $\{X_\alpha, \mathcal{A}_\alpha\}$ on X and a continuous map $f_{\omega_1} : X_{\omega_1} \to Z$ such that $f_{\omega_1}|_X = f$ (X_{ω_1} may not be Hausdorff).*

Proof: The proof can be done by induction, using the above proposition. If α is a limit ordinal, and we have extended f to X_β, for each $\beta < \alpha$, then it is easy to define a continuous extension of f to X_α. □

The above theorem generalizes Theorem 2.10 in [FV]. The proof here is simplified.

Suppose X is a space such that, for every Ψ-system $\{X_\alpha, \mathcal{A}_\alpha\}$ on X, X_{ω_1} is Hausdorff. We have proved that, in this case, for every mapping from X to a sequentially compact space, there exists a sequentially compact extension of X to which f extends. We recall that an analogous result was known for ALSC spaces. Using the diagonal map, we also obtain:

Corollary 3.6. *For every family $\{f_l : X \to Y_l\}_{l<\kappa}$, where $\kappa < h$, and Y_l is sequentially compact for every l, there exists a Ψ-system $\{X_\alpha, \mathcal{A}_\alpha\}$ such that every f_l extends to X_{ω_1}.*

Acknowledgment. The author is grateful to Professor Mikhail Matveev for the useful conversations on this subject.

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(*Received* November 25, 1997)