On a theorem of W.W. Comfort and K.A. Ross

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Abstract. A well known theorem of W.W. Comfort and K.A. Ross, stating that every pseudocompact group is C-embedded in its Weil completion [5] (which is a compact group), is extended to some new classes of topological groups, and the proofs are very transparent, short and elementary (the key role in the proofs belongs to Lemmas 1.1 and 4.1). In particular, we introduce a new notion of canonical uniform tightness of a topological group G and prove that every G_δ-dense subspace Y of a topological group G, such that the canonical uniform tightness of G is countable, is C-embedded in G.

Keywords: topological group, pseudocompact, Frechet-Urysohn, G_δ-dense, C-embedded, Moscow space, canonical uniform tightness, Hewitt completion, Rajkov completion, bounded set, extremally disconnected, normal space, k_1-space

Classification: 54A05, 54D55

§1. A lemma and some classical results

All spaces considered in this paper are assumed to be Tychonoff, all functions are real-valued functions. Terminology and notation are as in [7] and [3]. If G is a group, e stands for its neutral element.

In their 1966 paper [5] on pseudocompact topological groups W.W. Comfort and K.A. Ross established a series of most interesting results on properties of such groups. It is well known that every pseudocompact topological group G is a (dense) subgroup of a compact topological group $b(G)$. Among other things, Comfort and Ross proved that G is C-embedded in $b(G)$, that is, every continuous function f on G can be extended to a continuous function on $b(G)$, which implies that $b(G)$ is, in fact, the Stone-Čech compactification of G. This provided a basis for their characterization of pseudocompactness of a totally bounded topological group G by its G_δ-denseness in the Weil completion $b(G)$ of G. This, in its turn, had lead to the important theorem on preservation of pseudocompactness in the class of topological groups by (even uncountable) products.

A more elementary approach to this topic was found by J. de Vries [19]. A far reaching generalization of the product theorem of Comfort and Ross has been obtained by M.G. Tkachenko [16]. He proved it for bounded (not necessarily dense) subsets of topological groups. An original approach to results of Comfort and Ross, and of Tkachenko, has been discovered by M. Hušek [11]. Tkachenko's approach was further developed by V.V. Uspenskij in [18]. More contributions to this subject can be found in [6] and [9].

In this section we exhibit a simple lemma, allowing us to give a very transparent proof of the above mentioned results of Comfort and Ross, and to extend some of them to certain new classes of spaces and topological groups. We also demonstrate that a few other well known results can be easily obtained under this approach.

Recall that a subspace Y of a space X is said to be *C-embedded in X*, if every continuous function on Y can be extended to a continuous function on X.

If A is a subset of a space X, and $x \in X$, we say that *x is in the G_δ-closure* of A (and write $x \in [A]^\omega$), if every G_δ-subset of X containing x meets A. We say that A is *G_δ-dense* in X, if $x \in [A]^\omega$, for each $x \in X$.

We also need the next elementary and well known **observation (H)** ([7]): *Let Y be a dense subspace of a space X, and a a point of X such that every continuous function f on Y can be extended to a continuous function on $Y \cup \{a\}$. Then a belongs to the G_δ-closure of Y in X.*

An accumulation point of an (indexed) family η of subsets of a space X is a point $a \in X$ such that η is not locally finite at a. The set of all accumulation points of a family η will be denoted H_η.

Lemma 1.1. *Let $\{V_n : n \in \omega\}$ be a sequence of open neighbourhoods of the neutral element e of a topological group G, and $\{a_n : n \in \omega\}$ a sequence of points in G such that the following conditions are satisfied, for each $n \in \omega$:*

(1) $a_{n+1} \in V_n$;
(2) $V_{n+1}^2 \subset V_n$;
(3) $V_n = V_n^{-1}$.

Let $P = \cap\{V_n : n \in \omega\}$, and assume that the set H_η of accumulation points of the family $\eta = \{a_n V_{n+1} : n \in \omega\}$ in G is not empty. Then $H_\eta \subset P \subset \overline{\cup\{a_n V_n : n \in \omega\}}$.

Proof: Clearly,

$$H_\eta P \subset \overline{\cup\{a_n V_{n+1} : n \in \omega\}} P \subset \overline{\cup\{a_n V_{n+1} P : n \in \omega\}} \subset \overline{\cup\{a_n V_n : n \in \omega\}}.$$

Thus,

$$H_\eta P \subset \overline{\cup\{a_n V_n : n \in \omega\}}.$$

Obviously, P is a subgroup of G. Conditions (2) and (3) imply that $\bar{V}_{n+1} \subset V_n$ and $\cap\{\bar{V}_n : n \in \omega\} = \cap\{V_n : n \in \omega\} = P$. Therefore, *$P$ is a closed subgroup of G.*

From $a_{n+1} \in V_n$ we have: $a_n V_{n+1} \subset V_{n-1} V_{n+1} \subset V_{n-1}^2 \subset V_{n-2}$, for $n \geq 2$. Hence,

$$H_\eta \subset \overline{\cup\{a_n V_{n+1} : n \in \omega, n \geq k+1\}} \subset V_{k-1},$$

for each positive $k \in \omega$. It follows that $H_\eta \subset P$.

Now, since P is a subgroup of G, and H_η is not empty, it follows that $H_\eta P = P$. Therefore, $P \subset \overline{\cup\{a_n V_n : n \in \omega\}}$. Our key lemma is proved. □

The next result is a very particular case of Theorem 3.1, and its proof is only slightly simpler than that of Theorem 3.1. However, to formulate it, we do not have to introduce new notions; since the reader may be willing to restrict himself to this special case, which covers the classical one, we provide a separate proof of it.

Proposition 1.2. *Let G be a pseudocompact topological group, and Y a dense subspace of G such that e belongs to the G_δ-closure of Y. Then every continuous function f on Y can be extended to a continuous function f^* on $Y \cup \{e\}$.*

Proof: Since e is in the G_δ-closure of Y, there exists $k \in \omega$ such that e belongs to the closure of the set $\{y \in Y : |f(y)| \leq k\}$. Assume now that f cannot be extended to $Y \cup \{e\}$. Then, without loss of generality, we may assume that e belongs to the closure of each of the following subsets of Y: $A = \{y \in Y : f(y) > 1\}$ and $B = \{y \in Y : f(y) < 0\}$. Now, let us define a sequence $\{V_n : n \in \omega\}$ of open neighbourhoods of e, and sequences $\{a_n : n \in \omega\}$ and $\{b_n : n \in \omega\}$ of points in A and B, respectively, in the following way.

Choose a_0 to be any point of A, b_0 to be any point of B, and let V_0 be a symmetric open neighbourhood of e such that $f(y) > 1$, for each $y \in (a_0V_0) \cap Y$, and $f(y) < 0$, for each $y \in (b_0V_0) \cap Y$.

Assume now that an open neighbourhood V_k of e is already defined, for some $k \in \omega$. Then let a_{k+1} be any point of $A \cap V_k$, b_{k+1} any point of $B \cap V_k$, and V_{k+1} any symmetric open neighbourhood of e such that the following three conditions are satisfied: $V_{k+1}^2 \subset V_k$, $f(y) > 1$, for each $y \in (a_{k+1}V_{k+1}) \cap Y$, and $f(y) < 0$, for each $y \in (b_{k+1}V_{k+1}) \cap Y$. Obviously, this is possible. The construction of the sequences is complete.

Since G is pseudocompact, the indexed family $\eta = \{a_nV_{n+1} \cap Y : n \in \omega\}$ is not locally finite in G, that is, η accumulates at some point of G. It follows that all conditions of Lemma 1.1 are satisfied. Therefore, for the set $P = \cap\{V_n : n \in \omega\}$ we have: $P \subset \overline{\cup\{a_nV_n : n \in \omega\}}$. For the same reasons, $P \subset \overline{\cup\{b_nV_n : n \in \omega\}}$.

Since $e \in P$, and P is a G_δ-set in G, we have: $P \cap Y \neq \emptyset$. Take any $a \in P \cap Y$. Then $a \in \overline{\cup\{a_nV_n : n \in \omega\}}$, and $a \in \overline{\cup\{b_nV_n : n \in \omega\}}$.

Since Y is dense in G, $(a_nV_n) \cap Y$ is dense in a_nV_n and $(b_nV_n) \cap Y$ is dense in b_nV_n. Therefore, $a \in \overline{\cup\{(a_nV_n) \cap Y : n \in \omega\}}$ and $a \in \overline{\cup\{(b_nV_n) \cap Y : n \in \omega\}}$.

Notice, that $f(y) > 1$, for each $y \in (a_nV_n) \cap Y$, and $f(y) < 0$, for each $y \in (b_nV_n) \cap Y$, by the construction.

Since a is in Y, the function f is defined and continuous at a. It follows that $f(a) \leq 0$ and $f(a) \geq 1$, a contradiction. Proposition 1.2 is proved. □

From Proposition 1.2 we immediately get the next result:

Theorem 1.3. *Let G be a pseudocompact topological group, and Y a dense subspace of G. Then Y is C-embedded in the G_δ-closure of Y in G.*

It is well known, and easy to see, that if G is any compact space (not necessarily a topological group), and Y a dense subspace of G, then the G_δ-closure Z of Y in

G is Hewitt complete (that is, realcompact (see [7])). Therefore, from Theorem 1.3 we get:

Theorem 1.4 ([15]). *Let G be a compact topological group, and Y a dense subspace of G. Then the G_δ-closure of Y in G is the Hewitt completion of the space Y.*

Corollary 1.5. *Let G be a topological group, and Y a dense subspace of G. Then the next three conditions are equivalent:*

(a) *Y is G_δ-dense in G, and G is pseudocompact;*
(b) *Y is C-embedded in G, and G is pseudocompact;*
(c) *Y is pseudocompact.*

Proof: By Proposition 1.2, (a) implies (b). Obviously, (b) implies (c). Finally, if Y is pseudocompact, then G is pseudocompact, since Y is dense in G, and Y is G_δ-dense in G (this is easy to see and well known, [7]). Thus, (c) implies (a). □

Remark. Corollary 1.5, slightly generalizing one of the main results of Comfort and Ross in [5], should be qualified as essentially known, since it easily follows from Corollary 11 in [9].

Recall that a topological group is *totally bounded*, if it is a subgroup of a compact group. The next question was posed by V.G. Pestov and M.G. Tkachenko (see [10] for related results): *let G be a topological group. Is it always possible to extend the operations in G to continuous operations on the Hewitt completion νG (or on the Diedonne completion μG) of the space G so that νG (respectively, μG) becomes a topological group?* The next result in this direction is known and easily follows from Theorem 1.4. For more general results see [10].

Corollary 1.6 ([15]). *Let G be a totally bounded topological group. Then the Hewitt completion νG of G has a natural structure of a topological group, with continuous multiplication and inverse extending multiplication and inverse operations in G.*

Proof: Take a compact topological group bG, containing G as a subgroup. We may assume that G is dense in bG. By Theorem 1.4, the G_δ-closure Z of G in bG is the Hewitt completion of G, that is, $Z = \nu G$. It is also clear, that Z is a topological group. □

Corollary 1.7 ([5]). *Every continuous function f on a pseudocompact topological group G is uniformly continuous.*

Proof: Indeed, by Corollary 1.5, f can be extended to a continuous function on the compact group $b(G)$, containing G as a dense subgroup. And every continuous function on a compact group is obviously uniformly continuous ([13]). □

Corollary 1.8 ([5]). *Every pseudocompact group G is R-factorizable in the sense of M. Tkachenko* [15], *that is, for every continuous function f on G there exists* *a continuous homomorphism ϕ of G onto a compact metrizable group H, and a continuous function h on H such that $f = h \circ \phi$.*

Proof: Again by Corollary 1.5, the last assertion reduces to the case when G is compact. In this case the proof is straightforward. □

Let us call a space Y *a groupy space*, if there exists a topological group G such that Y is homeomorphic to a dense subspace of G.

Corollary 1.9. *The topological product of any family of pseudocompact groupy spaces Y_α is pseudocompact.*

Proof: Every pseudocompact groupy space is homeomorphic to a dense subspace of a compact topological group. It remains to refer to the fact that the product of any family of compact topological groups is a compact topological group, and to apply Corollary 1.5 (the G_δ-characterization of pseudocompactness for dense subspaces of topological groups, which is obviously productive). □

Corollary 1.9 also follows from the next result of A.C. Chigogidze [4]: *the product of every family of pseudocompact κ-metrizable spaces is pseudocompact.* But under his approach to Corollary 1.9 we need to know that all compact groups are κ-metrizable, as well as some other facts about κ-metrizability. Our approach is both more elementary and shorter.

§2. Pointwise pseudocompact topological groups and spaces

A point a of a space X will be called *a pseudocompactness point* of X, if there exists a sequence $\{U_n : n \in \omega\}$ of open neighbourhoods of a, satisfying the condition: for every sequence $\{V_n : n \in \omega\}$ of non-empty open sets such that $V_n \subset U_n$ for each $n \in \omega$, there exists a point of accumulation in X.

A space X is said to be *pointwise pseudocompact*, if each point of X is a pseudocompactness point. Recall that a space X is said to be of *point-countable type*, if for each $x \in X$ there exists a compact subset $F \subset X$ with a countable neighbourhood base in X ([3], [7]). All Čech-complete spaces, and, more generally, all p-spaces ([3]) are of point-countable type ([3], [7]). Note that a topological group G is a p-space if and only if the space G is of point-countable type (see [14]).

According to E. Michael (see [12]), a point x of a space X is said to be *a q-point*, if there exists a sequence $\{U_n : n \in \omega\}$ of open neighbourhoods of x satisfying the following condition:

(q) *for every sequence $\{x_n : n \in \omega\}$ of points in X such that $x_n \in U_n$ for each $n \in \omega$, there exists a point of accumulation in X.*

A space is called *a q-space* if all its points are q-points. Obviously, we have:

Proposition 2.1. *Every q-space is pointwise pseudocompact.*

Since each space X of point-countable type is a q-space ([12]), *all spaces of point-countable type are pointwise pseudocompact.*

Now we can formulate a much more general version of Proposition 1.2, with almost the same proof. We omit the proof, since this result is essentially covered by Theorem 3.3.

Proposition 2.2. *Let G be a topological group, Y a dense subspace of G, and p a pseudocompactness point of G. Then the next two conditions are equivalent:*

(1) *p belongs to the G_δ-closure of Y;*
(2) *Y is C-embedded in $Y \cup \{p\}$.*

The next result is a straightforward corollary of Proposition 2.2.

Theorem 2.3. *Let G be a pointwise pseudocompact topological group, and Y a dense subspace of G. Then Y is C-embedded in the G_δ-closure of Y in G, and therefore the next two conditions are equivalent:*

(a) *Y is G_δ-dense in G;*
(b) *Y is C-embedded in G.*

Corollary 2.4. *Let G be a topological group which is a p-space* (*in particular, this is so if G is Čech complete*), *and Y a dense subspace of G. Then Y is C-embedded in the G_δ-closure of Y in G. Therefore, under the above assumptions, Y is C-embedded in G if and only if Y is G_δ-dense in G.*

Proof: In view of Theorem 2.3, it suffices to notice that every p-space is of point-countable type, and every space of point-countable type is pointwise pseudocompact.

□

Corollary 2.5. *Let G_i be a topological group, which is a p-space, and Y_i a C-embedded dense subspace of G_i, for each $i \in \omega$. Then the product space $Y = \Pi\{Y_i : i \in \omega\}$ is C-embedded in the product space $G = \Pi\{G_i : i \in \omega\}$.*

Proof: Indeed, by Corollary 2.4, Y_i is G_δ-dense in G_i, for $i \in \omega$. It follows that Y is G_δ-dense in G. Obviously, G is a topological group which is a p-space, since the product of any countable family of p-spaces is a p-space ([3]). It remains to apply Corollary 2.4. □

Corollary 2.5 is closely related to the classical results of I. Glicksberg in [8] (and, of course, to the corresponding result on pseudocompact groups in [5]).

Some further applications of our results can be based on the next two obvious assertions:

Proposition 2.6. *If Y is a dense subspace of a space X, and y is a pseudocompactness point of Y, then y is also a pseudocompactness point of X.*

Proposition 2.7. *If a topologically homogeneous space X contains a dense pointwise pseudocompact subspace Y, then X is also pointwise pseudocompact.*

Corollary 2.8. *If a topological group G contains a dense pointwise pseudocompact subspace Y, then G is pointwise pseudocompact.*

Every topological group can be represented as a dense subgroup of its Rajkov completion G^*, which is a topological group complete with respect to the two-sided uniformity ([14]). By Corollary 2.8, we have:

Corollary 2.9. *If G is a pointwise pseudocompact topological group, then its Rajkov completion G^* is also a pointwise pseudocompact topological group.*

Theorem 2.10. *For each pointwise pseudocompact topological group G the operations on G can be continuously extended to the Diedonne completion μG of G, turning it into a topological group.*

Proof: The Rajkov completion G^* of G is a pointwise pseudocompact topological group, by Corollary 2.9. Let Z be the G_δ-closure of G in G^*. By Theorem 2.3, G is C-embedded in Z. Now the argument runs in a standard way (see [15], [10], [18]). We present it for the sake of completeness.

Clearly, Z is a subgroup of G^*. The space G^* is Rajkov complete; therefore, G^* is Diedonne complete. Since every point of $G^* \setminus Z$ can be separated from Z by a G_δ-set, it follows that the space Z is Diedonne complete. Let M be the smallest Diedonne complete subspace of Z such that $G \subset M$ (such subspace M exists since the intersection of any family of Diedonne complete subspaces of Z is a Diedonne complete space). Since G is C-embedded in M, it follows that the space M is the Diedonne completion of the space G.

It remains to show that M is a subgroup of the group Z. First, $M \subset M^{-1}$, since $G \subset M^{-1} \subset Z$ and M^{-1} is homeomorphic to M and, therefore, Diedonne complete. It follows that $M = M^{-1}$.

For every $a \in G$ we have: $G \subset aG \subset aM \subset aZ = Z$ which implies that $M \subset aM$, since aM is homeomorphic to M and, hence, Diedonne complete. Therefore, $M \subset a^{-1}M$ and $aM \subset aa^{-1}M = M$. Now take any $b \in M$. Then $G \subset Mb$. Indeed, take any $a \in G$. Then, as we just proved, $ab^{-1} \in M$, that is, $ab^{-1} = c$, for some $c \in M$. It follows that $a = cb \in Mb$. Obviously, $Mb \subset Z$ and Mb is homeomorphic to M. Therefore, Mb is Diedonne complete, which implies that $M \subset Mb$. Therefore, $M \subset Mb^{-1}$ and $Mb \subset Mb^{-1}b = M$, for each $b \in M$. Now it is clear that M is closed under multiplication. Hence, M is a subgroup of Z. □

Recall that a subset K of a space X is said to be *bounded in X* (or just *bounded*), if every continuous function on X is bounded on K. A space X is said to be *locally bounded*, if it can be covered by open bounded subsets.

The next two corollaries of Theorem 2.3 slightly generalize some results in [6]. Observe, that obviously *every locally bounded space is pointwise pseudocompact.*

Corollary 2.11. *Let G be a topological group, and Y a G_δ-dense subspace of G. Then the next three conditions are equivalent:*

(a) *G is locally bounded;*

(b) *Y is locally bounded;*
(c) *there exists a non-empty subset W of Y which is open in Y and bounded in G.*

Moreover, if at least one of the conditions (a)–(c) *is satisfied, then Y is C-embedded in G.*

Proof: Clearly, (b) implies (c). Since Y is dense in G, (a) implies (c). Let us show that (c) implies (a). Take any non-empty open subset V of Y such that V is bounded in G. Then $\bar{V}$ is bounded in G (where the closure is taken in G). Since Y is dense in G, $\bar{V}$ contains a non-empty open subset U of G. Obviously, U is bounded in G. Since G is a topologically homogeneous, it follows that the space G is locally bounded. Notice that we have not used G_δ-denseness of Y in G so far. We are going to use it now to show that (a) implies (b).

Indeed, every locally bounded space is pointwise pseudocompact. Therefore, given (a), Y is C-embedded in G by Theorem 2.3. Now take any $y \in Y$. Since G is locally bounded, there exists an open neighbourhood U of y in G such that U is bounded in G. Then $V = U \cap Y$ is a non-empty open subset of Y bounded in G. Since Y is C-embedded in G, it follows that V is bounded in Y.

The last assertion in Corollary 2.11 follows from Theorem 2.3, as we just saw it in the last portion of the argument above. □

Corollary 2.11 is closely related to the next result of Comfort and Trigos-Arrieta ([6]):

Proposition 2.12. *Let G be a topological group and Y a dense subgroup of G. Then the next assertions are equivalent:*

(a) *Y is locally bounded;*
(b) *G is locally bounded, and Y is G_δ-dense in G.*

Proof: By Corollary 2.11, (b) implies (a). Assume that Y is locally bounded. Then, as it was shown in the proof of Corollary 2.11, G is locally bounded. Now take any non-empty open subset V of the space Y such that V is bounded in Y. Let Z be the G_δ-closure of Y in G. Since Y is a subgroup of G, it follows that Z is a subgroup of G. Since V is bounded in Y, the closure $\bar{V}$ of V in G is contained in Z, by the standard reasoning. But $\bar{V}$ contains a non-empty open subset of G, since G is regular and Y is dense in G. Therefore, Z contains a non-empty open subset of G. It follows (see [13], [14]) that Z is an open subgroup of G. This implies that Z is closed in G ([13], [14]). Since Z contains Y, Z is dense in G. Hence $Z = G$. □

Every Rajkov complete locally bounded topological group is locally compact, since the closure of any bounded subset in a Diedonne complete space is compact ([7]). This observation, combined with the assertions 2.11 and 2.12, brings us to the following conclusion:

Corollary 2.13 ([6]). *For every locally bounded topological group G the Rajkov completion of G is a locally compact group in which G is C-embedded, and therefore, G_δ-dense.*

§3. Pointwise canonical weak pseudocompactness, Moscow spaces, and C-embeddings

Now we are going to introduce certain very weak forms of pointwise pseudocompactness, which will allow us to extend considerably the results obtained so far. Lemma 1.1 remains one of the main tools in our argument. However, some other technical results and notions play an important role as well.

Recall that a subset U of a space X is said to be *a canonical open subset* of X if U is the interior of its closure.

Let us call a point a of a space X *a point of canonical weak pseudocompactness*, if the following condition is satisfied:

(cwp) *for each canonical open subset U of X such that $a \in \bar{U}$ there exists a sequence $\{(A)_n : n \in \omega\}$ of subsets of U such that $a \in \overline{(A)_n}$, for each $n \in \omega$, and for each indexed family $\eta = \{O_n : n \in \omega\}$ of open subsets of X, satisfying the condition $O_n \cap A_n \neq \emptyset$, the family η has an accumulation point in X.*

If in the above condition (cwp) we drop the assumption that the open subset U is canonical, we obtain condition (wp); if a point of X satisfies this condition, we say that x is *a point of weak pseudocompactnes* of X. The condition (wp) is, formally, slightly stronger than the condition (cwp); therefore, every point of weak pseudocompactness is also a point of canonical weak pseudocompactness.

Clearly, every pseudocompactness point of a space X is also a point of weak pseudocompactnes of X. On the other hand, if X is (canonically) weakly Frechet-Urysohn at a point $a \in X$ (that is, if $a \in \bar{U}$, where U is a (canonical) open subset of G, implies that some sequence of points of U converges to a), then, obviously, a is also a point of (canonical) weak pseudocompactness of X.

Let us call a space X *pointwise (canonically) weakly pseudocompact*, if each point of X is a point of (canonical) weak pseudocompactness. All Frechet-Urysohn spaces, and all pointwise pseudocompact spaces, are pointwise weakly pseudocompact.

A space X is called *a Moscow space* ([2]), if for each open set U, the closure of U is the union of a family of G_δ-subsets of X (which can always be chosen to be closed). Of course, in this definition, U can be assumed to be a canonical open set.

The concept of a Moscow space is vital for C-embeddings; this was demonstrated in [17], [2], [18]; we will also see it shortly.

Theorem 3.1. *If a topological group G is pointwise canonically weakly pseudocompact, then G is a Moscow space.*

Proof: Let U be a canonical open subset of G. Clearly, it is enough to show that if $e \in \bar{U}$, then there exists a (closed) G_δ-subset P of G such that $e \in P \subset \bar{U}$. So let us assume that $e \in \bar{U}$.

Fix subsets $(A)_n$ of U such as in the condition (cpw) (where $a = e$).

Now, let us define a sequence $\{V_n : n \in \omega\}$ of open neighbourhoods of e, and a sequence $\{a_n : n \in \omega\}$ of points in U such that $a_n \in (A)_n$ (the construction is very similar to the one in the proof of Proposition 1.2). First, choose a_0 to be any point of $(A)_0$, and let V_0 be an open neighbourhood of e such that $a_0 V_0 \subset U$.

Assume now that an open neighbourhood V_k of e is already defined, for some $k \in \omega$. Then we let a_{k+1} to be any point of $(A)_{k+1} \cap V_k$. Now let V_{k+1} be any symmetric open neighbourhood of e such that $V_{k+1}^2 \subset V_k$ and $a_{k+1}V_{k+1} \subset U$. The recursive definition is complete.

By condition (cpw), the indexed family $\eta = \{a_n V_{n+1} : n \in \omega\}$ has a point of accumulation in G.

It follows that all conditions of Lemma 1.1 are satisfied.

Therefore, for the set $P = \cap\{V_n : n \in \omega\}$ we have:

$$e \in P \subset \overline{\cup\{a_n V_n : n \in \omega\}} \subset \bar{U}.$$

Since P is a (closed) G_δ-set, the proof is complete. $\square$

The next result essentially belongs to M.G. Tkachenko [17]. He formulated a weaker version of it, but all what one needs to generalize it to Moscow spaces is the notion of a Moscow space. V.V. Uspenskij formulated this generalization in [18].

Proposition 3.2. *If Y is a G_δ-dense subspace of a Moscow space X, then Y is C-embedded in X.*

Proof: Assume that Y is not C-embedded in X. Then, as it is easy to see, there are open subsets V_1 and V_2 of Y such that their closures in Y are disjoint, while the intersection of the closures of V_1 and V_2 in X is not empty. Fix a point x in $\overline{V_1} \cap \overline{V_2}$, and let U_i be the interior of the closure of V_i in X, $i = 1, 2$. Obviously, $V_i \subset U_i$; therefore, U_i is not empty.

Since X is a Moscow space, we can find G_δ-sets P_i in X such that $x \in P_i \subset \overline{U_i}$, $i = 1, 2$. Then $P = P_1 \cap P_2$ is a G_δ-subset of X and $x \in P$; therefore, $P \cap Y$ is not empty. Clearly, every point of $P \cap Y$ belongs to the intersection of closures of the sets V_1 and V_2 in Y, which is impossible, since this intersection is empty, by the choice of V_1 and V_2. $\square$

Theorem 3.3. *Let G be a pointwise canonically weakly pseudocompact topological group, and Y a dense subspace of G. Then Y is C-embedded in the G_δ-closure of Y in G, and therefore, such a subspace Y is C-embedded in G if and only if Y is G_δ-dense in G.*

Proof: Since every dense subspace of a Moscow space is, obviously, a Moscow space, it remains to apply Theorem 3.1 and Proposition 3.2. The last assertion in Theorem 3.3 follows from the first and observation (H). $\square$

A point a of X will be called *a* (*canonical*) *o-point of* X, if for each (canonical) open subset U of X such that $a \in \bar{U}$, there exists a set $K \subset X$, bounded in X, such that $a \in \overline{U \cap K}$. Notice that every subset of a bounded set is bounded, therefore, in the above definition, K may be assumed to be a subset of U. A space X will be called *a* (*canonical*) *o-space*, if each point of X is a (canonical) o-point.

It is well known, and easily verified, that *a set $A \subset X$ is bounded in X if and only if whenever $\eta = \{U_n : n \in \omega\}$ is a sequence of open sets in X such that $U_n \cap A$ is not empty, for each $n \in \omega$, there exists a point of accumulation of η in X*. Now the next assertion is obvious:

Proposition 3.4. *Each* (*canonical*) *o-point of a space X is a point of* (*canonical*) *weak pseudocompactness. Thus, every* (*canonical*) *o-space is pointwise* (*canonically*) *weakly pseudocompact.*

Therefore, the next result is an immediate corollary of Theorem 3.3 and Proposition 3.4:

Corollary 3.5. *If a topological group G is a canonical o-space, then every dense subspace Y of G is C-embedded in its G_δ-closure in G, and therefore, such Y is C-embedded in G if and only if it is G_δ-dense in G.*

A space X will be called *a* (*canonical*) *ok_1-space*, if for each (canonical) open subset U of X and each $a \in \bar{U}$, there exists a compact subset F of X such that $x \in \overline{U \cap F}$. Notice, that every (canonical) ok_1-space, and therefore every k_1-space, is a (canonical) o-space; hence, Corollary 3.5 is applicable to such spaces.

§4. Canonical uniform tightness and zip-spaces

If in the preceding sections the starting point was the approach of Comfort and Ross to C-embeddings in topological groups, in this last section of the paper we develop an approach of M.G. Tkachenko and V.V. Uspenskij to the same topic. In particular, we introduce a new notion of canonical uniform tightness of a topological group, which considerably simplifies proofs of some known results and allows to obtain new results.

Since the natural environment for the techniques we are going to introduce includes not only topological groups but some more general topologo-algebraic objects as well, we first recall some definitions.

A semitopological group G is a group with a topology on it such that the multiplication mapping $G \times G \to G$ is separately continuous.

A paratopological group G is a group with a topology on it such that the multiplication mapping $G \times G \to G$ is jointly continuous.

According to our general agreement, semitopological groups and paratopological groups considered in this paper are assumed to be Tychonoff.

Let G be a semitopological group, and U a subset of it. A subset F of U will be called *a deep subset* of U if there exists an open neighbourhood V of the neutral element e of G such that $FV \subset U$.

Let τ be a cardinal number. An element a of a semitopological group G is *uniformly τ-accessible from a subset U of G* if there exists a family γ of deep subsets of U such that $|\gamma| \leq \tau$ and $a \in \overline{\cup\gamma}$. Clearly, if U is open, then every point of U is uniformly τ-accessible from U for each positive cardinal τ.

Now we can formulate our second key lemma, even more transparent and easy to prove than the first one.

Lemma 4.1. *Let G be a semitopological group, U a subset of G, and a an element of G which is uniformly ω-accessible from U. Then there exists a* (*closed*) *G_δ-subset P of G such that $a \in P \subset \bar{U}$.*

Proof: Without loss of generality, we may assume that $a = e$. Fix a countable family γ of deep subsets of U such that $e \in \overline{\cup\gamma}$. For each $F \in \gamma$ choose an open neighbourhood V_F of e such that $FV_F \subset U$. Now let $P = \cap\{V_F : F \in \gamma\}$. Clearly, P is a G_δ-set, $e \in P$, and from $e \in \overline{\cup\gamma}$ we obtain:

$$P = eP \subset \overline{\cup\{FP : F \in \gamma\}} \subset \overline{\cup\{FV_F : F \in \gamma\}} \subset \bar{U}.$$

□

To properly identify the class of semitopological groups, to which Lemma 4.1 is applicable, we introduce the following definitions. Let τ be an infinite cardinal number, and G a semitopological group. Let us say that (*canonical*) *uniform tightness of G does not exceed τ* (notation: $ut(G) \leq \tau$ ($ut_c(G) \leq \tau$)) if for each (canonical) open subset U of G and each point a in the closure of U, the point a is uniformly τ-accessible from U. Of course, *the uniform tightness $ut(G)$* (*the canonical uniform tightness $ut_c(G)$*) of a topological group G is then defined as the smallest infinite cardinal number τ such that $ut(G) \leq \tau$ ($ut_c(G) \leq \tau$). Notice that these invariants are defined only for semitopological groups (though the definitions can be naturally extended to uniform spaces).

Theorem 4.2. *If the canonical uniform tightness of a topological group G is countable, then G is a Moscow space.*

Proof: This is an immediate corollary of Lemma 4.1. □

Theorem 4.3. *If the canonical uniform tightness of a semitopological group G is countable, and Y is a dense subspace of G, then Y is C-embedded in the G_δ-closure of Y in G. In particular, if Y is G_δ-dense in G, then Y is C-embedded in G.*

Proof: Since every dense subspace of a Moscow space is, obviously, a Moscow space, the assertion follows from Proposition 3.2 and Theorem 4.2. □

Let us now list a few classes of semitopological groups which are contained in the class of groups of countable uniform tightness.

The next result can be viewed as an improvement of Theorem 3.1.

Theorem 4.4. *Every pointwise (canonically) weakly pseudocompact topological group has countable (canonical) uniform tightness.*

Proof: We just repeat the proof of Theorem 3.1, making only the following change: choosing V_i, we now require that $a_i V_i^2 \subset U$. Then

$$e \in \overline{\cup\{a_n V_n : n \in \omega\}},$$

where each $a_n V_n$ is a deep subset of U. Therefore, $ut(G) \leq \omega$ ($ut_c(G) \leq \omega$). $\square$

A subset A of a space X is said to be *quasi-Lindelöf in X* if for every family η of open subsets of X such that $A \subset \cup\eta$ there exists a countable subfamily γ of η such that $A \subset \overline{\cup\gamma}$.

We will say that the (*canonical*) *zip-number* of a space X is countable (notation: $zp(X) \leq \omega$ ($zp_c(X) \leq \omega$)) if for each (canonical) open set U and each point x in the closure of U there exists a subspace A of U such that A is quasi-Lindelöf in X and $x \in \bar{A}$. Now it is clear how to define (*canonical*) *zip-number* of arbitrary space X. We will denote it $zp(X)$ (respectively, $zp_c(X)$). If $zp(X) \leq \omega$ ($zp_c(X) \leq \omega$), we also say that X is *a zip-space* (*a canonical zip-space*). Notice that unlike uniform tightness, zip-number is defined for all topological spaces. However, we have:

Theorem 4.5. *If G is a paratopological group such that the (canonical) zip-number of the space G is countable, then the (canonical) uniform tightness of G is countable.*

Proof: Let U be a (canonical) open subset of G and $b \in \bar{U}$. Since $zp(G) \leq \omega$ (since $zp_c(G) \leq \omega$), there exists $A \subset U$ such that A is quasi-Lindelöf in G and $b \in \bar{A}$. For each $a \in A$ we can fix an open neighbourhood V_a of the neutral element e such that $aV_a^2 \subset U$. Then the family $\eta = \{aV_a : a \in A\}$ covers A and consists of open sets. Therefore there exists a countable subfamily γ of η such that $A \subset \overline{\cup\gamma}$. Since $b \in \bar{A}$, it follows that $b \in \overline{\cup\gamma}$. It remains to notice that γ is countable and all elements of γ (and of η) are deep subsets of U. $\square$

Now we list a few straightforward corollaries of Theorems 4.4, 4.2, and 4.3.

Corollary 4.6. *If G is a paratopological group which is a canonical zip-space, then G is a Moscow space, and every dense subspace of G is C-embedded in its G_δ-closure in G.*

Corollary 4.7. *If the Souslin number of a paratopological group G is countable, then the uniform tightness of G is countable and G is a Moscow space. Therefore, every dense subspace of G is C-embedded in its G_δ-closure in G.*

Corollary 4.8. *If G is a paratopological group such that tightness of G is countable, then the uniform tightness of G is countable and G is a Moscow space. Therefore, every dense subspace of G is C-embedded in its G_δ-closure in G.*

In the case of topological groups the last parts of the assertions 4.7 and 4.8 are already known. Uspenskij [18] established (by a somewhat more involved argument) that if o-tightness of a topological group G is countable, then G is a Moscow space. Recall that, according to M.G. Tkachenko [17], o-tightness of a space X is countable (notation: $ot(X) \leq \omega$) if for every family γ of open sets and each point x in the closure of $\cup\gamma$ there exists a countable subfamily η of γ such that $x \in \overline{\cup\eta}$. In fact, we can easily obtain this result of Uspenskij from Theorems 4.4 and 4.2. Indeed, we have:

Theorem 4.9. *If o-tightness of a paratopological group G is countable, then the uniform tightness of G is also countable* (*general assertion: $ut(G) \leq ot(G)$, for every paratopological group G*).

PROOF: Let U be an open subset of G, and $a \in \bar{U}$. Consider the family η of all deep open subsets of U. Clearly, $U = \cup\eta$, and therefore $a \in \overline{\cup\eta}$. Since $ot(G) \leq \omega$, it follows that there exists a countable subfamily γ of η such that $x \in \overline{\cup\gamma}$. Since all elements of γ are deep subsets of U, the proof is complete. □

I do not know the answer to the next question:

Question 1. Is uniform tightness equal to o-tightness for every topological group G? For every paratopological group G?

I conjecture that the answer is "no". Here is a simple but interesting fact:

Theorem 4.10. *If G is an extremally disconnected semitopological group, then the canonical uniform tightness of G is countable.*

PROOF: Let U be a canonical open subset of G and $a \in \bar{U}$. Since in an extremally disconnected space the closure of any open set is open, all canonical open subsets of G are closed. Therefore $a \in U$. Then there exists an open neighbourhood V of e such that $aV \subset U$. Thus, the set $\{a\}$ is a deep subset of U, and $\gamma = \{\{a\}\}$ is the countable family of sets we are looking for. □

It is known that if there exists an Ulam-measurable cardinal, then there exists an extremally disconnected topological group G such that the uniform tightness of G (and, therefore, the o-tightness of G) is not countable.

Theorem 4.11. *Let G be a semitopological group such that all G_δ-subsets of G are open* (*the last condition means that G is a P-space*). *Then the canonical uniform tightness of G is countable if and only if G is extremally disconnected.*

PROOF: By Theorem 4.10, the condition is sufficient. Now let us assume that the canonical uniform tightness of G is countable. Take any open subset U of G. By Theorem 4.2, G is a Moscow space. Therefore, $\bar{U}$ is the union of a family η of G_δ-subsets of G. Since G is a P-space, it follows that the set $\bar{U}$ is open. Thus, the space G is extremally disconnected. □

Notice that if X is a P-space, then the uniform tightness of X is countable if and only if X is discrete.

Question 2. Does there exist in ZFC an extremally disconnected topological group G which is a non-discrete P-space?

Here are some more special corollaries of Theorem 4.3 and 4.5.

Corollary 4.12. *If G is a paratopological group such that $ut_c(G) \leq \omega$, (in particular, if G is a canonical zip-space), and Y is a dense Hewitt complete subspace of G, then the G_δ-closure of Y in G coincides with Y. Therefore, if in addition Y is G_δ-dense in G, then $Y = G$.*

Corollary 4.13. *If G is a topological group such that $ut_c(G) \leq \omega$ (in particular, if G is a canonical zip-space), and the space $G \setminus \{e\}$ is Hewitt complete, then G is submetrizable, that is, there exists a weaker metrizable topology on G; therefore, G is a space with G_δ-diagonal, every point in G is a G_δ, and all compact subspaces of G are metrizable.*

Proof: It is enough to show that $\{e\}$ is a G_δ-set in G, since then G, being a topological group, is submetrizable (see [1]), and all compacta in G are metrizable.

Now, if $\{e\}$ is not a G_δ-set in G, then $G \setminus \{e\}$ is G_δ-dense in G, and therefore, by Corollary 4.12, $G \setminus \{e\} = G$, a contradiction. □

Let us call a space Y *ω-normal*, if every two countable disjoint closed subsets of Y can be separated by a continuous function on Y. Clearly, each normal space is ω-normal.

Theorem 4.14. *If G is a Frechet-Urysohn paratopological group, and Y a dense ω-normal subspace of G, then the G_δ-closure of Y in G coincides with Y. Therefore, if in addition Y is G_δ-dense in G, then $Y = G$.*

Proof: Indeed, assume that $[Y]^\omega \setminus Y$ is not empty, and fix $x \in [Y]^\omega \setminus Y$. Since G is Frechet-Urysohn, there exists a sequence $\{y_n : n \in \omega\}$ of points in Y converging to x. Put $A = \{y_{2n} : n \in \omega\}$ and $B = \{y_{2n+1} : n \in \omega\}$. Clearly, we can assume that A and B are disjoint. Then, since A and B are closed in Y, there exists a continuous function f on Y such that $f(y) = 1$, for each $y \in A$, and $f(y) = 0$, for each $y \in B$. It is impossible to extend this function continuously to the point x. On the other hand, Y is C-embedded in G, by Corollary 4.6. This contradiction completes the proof. □

Corollary 4.15. *Let G be a sequential topological group such that the space $G \setminus \{e\}$ is normal. Then all points in G are G_δ's, and G is submetrizable.*

Proof: We argue almost in the same way as in the proof of Theorem 4.14. □

Corollary 4.15 implies that all hereditarily normal sequential topological groups are submetrizable.

The next corollary of the above results seems also to be of interest:

Theorem 4.16. *Let G be a topological group which is a Frechet-Urysohn Lindelöf Σ-space. Then the next conditions are equivalent:*

(1) *$G \setminus \{e\}$ is normal;*
(2) *$G \setminus \{e\}$ is Hewitt complete;*
(3) *$G \setminus \{e\}$ is Lindelöf;*
(4) *e is a G_δ-point in G;*
(5) *G is separable and metrizable.*

Proof: It is enough to refer to Theorem 4.14 and Corollary 4.15, and also to the well known theorem that every submetrizable Lindelöf Σ-space is separable and metrizable (see [3]). □

Theorem 4.17. *If G is a topological group such that the Rajkov completion G^* of G is Frechet-Urysohn, and the space G is normal, then the space G is Diedonne complete.*

Proof: Since G^* is a complete uniform space, it follows that G^* is a Diedonne complete topological space ([7]). On the other hand, by Theorem 4.14, G is G_δ-closed in G^*, that is, each $x \in G^* \setminus G$ is contained in a G_δ-set P, such that $P \cap G = \emptyset$. This implies (see [3], [7]) that the space G is also Diedonne complete. □

Theorem 4.18. *Let H be a dense subgroup of a topological group G. Then the canonical uniform tightness of H is countable if and only if the canonical uniform tightness of G is countable.*

Proof: Notice first, that U is a canonical open subset of the space H if and only if there exists a canonical open subset U^* of G such that $U = U^* \cap H$. This is, actually, well known to be true in all spaces, only denseness of H in G matters.

Now fix U and U^* such as above, and let F be a deep subset of U^*. Then there exists an open neighbourhood V of the neutral element e in G such that $FV^2 \subset U^*$. Then FV is an open deep subset of U^*, and $P = U \cap FV$ is, obviously, a deep (in H) subset of U such that the closure of P in G contains the closure of F in G. From this it follows that $ut_c(H) \leq ut_c(G)$.

Now, with the same U and U^* in mind, take any deep (in H) subset B of U. Since U^* is a canonical open set, $U^* \cap H = U$, and H is dense in G, it follows that U^* is the maximal open subset of G contained in the closure of U in G. It is easy to derive from this that B is a deep (in G) subset of U^* as well. Therefore, $ut_c(G) \leq ut_c(H)$. □

Corollary 4.19. *The canonical uniform tightness of a topological group G coincides with the canonical uniform tightness of its Rajkov completion.*

It follows from Theorem 4.10 and Corollary 4.19 that if G is an extremally disconnected topological group, then the canonical uniform tightness of the Rajkov completion of G is countable. But, actually, we can prove much more in this case.

Proposition 4.20. *If X is a topologically homogeneous space and Y a dense extremally disconnected subspace of X, then X is also extremally disconnected.*

Proof: If X is not extremally disconnected, then we can find two disjoint open sets U and V in X and a point $b \in X$ such that $b \in \bar{U} \cap \bar{V}$. Fix a point $c \in Y$ and take a homeomorphism h of X onto X such that $h(b) = c$, and put $U_Y = Y \cap h(U)$, $V_Y = Y \cap h(V)$. Then U_Y and V_Y are disjoint open subsets of Y such that c is in the intersection of the closures of U_Y and V_Y in Y. Therefore, the closure of U_Y in Y is not open, and the space Y is not extremally disconnected, a contradiction. □

From Proposition 4.20 we immediately obtain the next result:

Theorem 4.21. *If G is an extremally disconnected topological group, then its Rajkov completion G^* is also an extremally disconnected topological group.*

PROOF: Indeed, G is dense in G^*, and the space G^* is topologically homogeneous. □

Here is an interesting corollary of Theorem 4.21, which I could not find in the literature:

Theorem 4.22. *Every totally bounded extremally disconnected topological group G is discrete* (*and, therefore, finite*).

PROOF: Indeed, the Rajkov completion G^* of G is a compact topological group, since G is totally bounded. From Theorem 4.21 it follows that G^* is extremally disconnected. Now, it is well known that a compact extremally disconnected topological group is finite (for example, since every infinite compact topological group contains a non-trivial convergent sequence, while an extremally disconnected space cannot contain such sequences (see [1], [7])). □

Theorem 4.23. *Let G be a topological group of the countable canonical uniform tightness. Then the operations in G can be continuously extended to the Diedonne completion of the space G, making it into a topological group, containing G as a subgroup.*

PROOF: We argue as in the proof of Theorem 2.10, using as the starting point Theorem 4.3 and Corollary 4.19. □

Notice, that a similar result for topological groups of countable o-tightness was obtained by V.V. Uspenskij in [18]. Because of Theorem 4.9, his result follows from Theorem 4.23. From Corollary 4.19, Proposition 4.20 and Theorem 4.23 we obtain the next curious corollary:

Theorem 4.24. *If G is an extremally disconnected topological group, then the Diedonne completion of the space G is an extremally disconnected topologically homogeneous space* (*in fact, homeomorphic to a topological group*).

Let us now present some relevant examples.

Example 4.25. There exists a topological group G such that G is a Lindelöf P-space and the weight of G is exactly ω_1. Obviously, we can define by a transfinite recursion a transfinite sequence $\{U_\alpha : \alpha < \omega_1\}$ of disjoint non-empty open subsets of G converging to the neutral element e of G. Now take any two disjoint uncountable subsequences η and ξ of this sequence, and put $U = \cup\eta$, $V = \cup\xi$. Then U and V are disjoint open subset of G and $e \in \bar{U} \cap \bar{V}$. Therefore, the space G is not extremally disconnected. From Theorem 4.11 it follows that the canonical uniform tightness of G is uncountable. Thus, *not for every Lindelöf topological group G the canonical uniform tightness of G is countable.*

Example 4.26. Let X be the Alexandroff one-point compactification of an uncountable discrete space Z. Notice that the pair Z, X shows that a subspace may be G_δ-dense in a compact Frechet-Urysohn space without being C-embedded in it (which, of course, is well known). This demonstrates that the assumption, that G is a topological group, is crucial for our main results.

Now let $G = F(X)$ be the free topological group of Y, and Y the subspace of G, algebraically generated by Z. The set Z is G_δ-dense in X, which implies that Y is G_δ-dense in G (see [1]). The space G is sequential (see [1]), and the topological group G is Rajkov complete ([14]). It is known that G is not metrizable (see [1], [14]). Thus, G is a non-metrizable sequential Rajkov complete topological group.

Question 3. Is there in ZFC a non-metrizable, Frechet-Urysohn, Rajkov complete topological group? Is there in ZFC a countable, non-metrizable, Frechet-Urysohn, Rajkov complete topological group?

Example 4.27. Let D be the two-point discrete group, G the topological product of an uncountable family of copies of D, and Y the Σ-product subgroup of G. Then Y is G_δ-dense in G, Y is normal, Frechet-Urysohn, and G is compact, and, therefore, G is a k_1-space. Nevertheless, Y and G do not coincide. Thus, Theorem 4.14 cannot be extended to the case when G is a k_1-space.

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