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On a characterization of the unit interval in terms of clones

Artur Barkhudaryan

Abstract. This paper gives a partial solution to a problem of W. Taylor on characterization of the unit interval in the class of all topological spaces by means of the first order properties of their clones. A characterization within the class of compact spaces is obtained.

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Introduction

To any topological space X, one can relate an algebraic structure consisting of all continuous maps between finite powers of X, with their composition as operation — the clone of X. The clone of a topological space carries considerable amount of information about that space (see [10]). For example, in the class of completely regular spaces containing an arc even the monoid of self-maps determines the topology. On the other hand, the clone does not characterize the space in general; for related results see [9], [11] and the proposition at the end of this section.

More precisely, following Taylor [10], we define the *clone* of a topological space X to be the ω-sorted universal algebra

$$Cl(X) = \langle C_n; e_n^i; S_m^n \rangle$$

with underlying sets $C_n = \{f : X^n \to X;\ f \text{ continuous}\}$ for $n \in \omega$, constants e_n^i denoting the projection of X^n onto the i-th component for $i < n \in \omega$, and the "composition" operations $S_m^n : C_n \times (C_m)^n \to C_m$, sending any $(n+1)$-tuple $(F, G_1, \dots, G_n)$ with $F \in C_n, G_1, \dots, G_n \in C_m$ to the map $S_m^n(F, G_1, \dots, G_n) = H \in C_m$, defined by

$$H(x_1, \dots, x_m) = F(G_1(x_1, \dots, x_m), \dots, G_n(x_1, \dots, x_m)).$$

Thus C_0 is just the underlying set of X, and the set C_1 with the operation S_1^1 (which is actually a binary operation on C_1) is the monoid $M(X)$ of all continuous

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self-maps of X. Of course, clones can be defined more generally for objects of any category with finite products. Clones can also be treated as abstract categories — *algebraic theories* (see [6]). Treating them as universal algebras is more convenient for us, as we are interested in their *first order properties*, i.e. those properties which can be expressed by formulas of the first order language of clone theory. We briefly recall the formal construction of this language now.

The symbols of the language of clones consist of ω types of variables $f_i^{(n)}$, $i, n \in \omega$, each $f_i^{(n)}$ referring to the underlying set C_n, constant symbols e_n^i for $i < n \in \omega$ and operation symbols S_m^n for $m, n \in \omega$. The n-th sort terms are constructed by the following rules:

- each $f_i^{(n)}$ is an n-th sort term;
- each e_n^i is an n-th sort term;
- whenever $t_1, \dots, t_m$ are n-th sort terms, then also $S_n^m(f_i^{(m)}, t_1, \dots, t_m)$ is.

Atomic formulas have the form $t_1 = t_2$ for terms t_1 and t_2 of the same sort. Formulas are then constructed in the usual way:

- each atomic formula is a formula;
- whenever φ and ψ are formulas, so are also $\neg\varphi$, $\varphi \ \& \ \psi$, $\varphi \ \vee \ \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$, $(\forall\, f_i^{(n)})\varphi$, $(\exists\, f_i^{(n)})\varphi$.

It is clear what it means that a formula of clone theory holds (is true, etc.) in a clone. Since the formulas of the above described language are somewhat cumbersome, we shall use a better-looking convention throughout this paper: the variables of type 0 are denoted by lower-case latin letters (e.g. x, y, a, b, $\dots$) here, the type (arity) of higher-type variables is denoted by dashes in parentheses rather than by a superscript, so $f(-)$, $g(-)$ stand for $f_i^{(1)}$, $f_j^{(1)}$, $F(-,-)$ stands for $f_i^{(2)}$, etc. Composition operators are omitted and the composition written in an implicit manner. Thus the formula

$$(\forall\, f(-))(\exists\, x)(f(x) = x)$$

is easily recognized to express the well-known fixed-point property. Formally it should have been written as $(\forall\, f_0^{(1)})(\exists\, f_0^{(0)})(S_0^1(f_0^{(1)}, f_0^{(0)}) = f_0^{(0)})$.

In [10], W. Taylor showed that in the class of metrizable spaces containing an arc many topological properties are definable in terms of first order formulas. The assumption that the space contains an arc is very strong, however, so the following problem naturally arose:

Problem (No. 9.2 in [10]). *Is there any first order condition on $Cl(X)$ which is equivalent to the existence of an arc in X? Is there any first order condition on $Cl(X)$ which characterizes those spaces X homeomorphic to a closed bounded interval?*

There are some topological characterizations of the interval. For example, a topological space is homeomorphic to the unit interval iff it is compact, Hausdorff, separable and has exactly two non-cut points (see [5]). So if compactness, separability, Hausdorffness and having two non-cut points were first order properties, we would have a first order characterization of the unit interval. Unfortunately, neither compactness nor separability is characterizable by the clone of the space. Recall that a topological space is called *rigid* if each its continuous self-map is either constant or the identity. It is known that if X is a rigid space and $f : X^n \to X$ is continuous then f is either constant or a projection (see [3], [4] or [10] for the proof). So if X and Y are rigid spaces of the same cardinality then they have isomorphic clones. Now observe that there exist compact and non-compact rigid spaces of the same cardinality and, similarly, there exist separable and non-separable rigid spaces of the same cardinality (methods and results of [8] can serve as a powerful tool for constructing those and many other spaces). So we get the following

Proposition. *There exist compact and non-compact spaces with isomorphic clones; the same is true for separability.* □

Note that even the property of being Hausdorff is not determined by clones; the discrete and the indiscrete topologies on a set have the same clone.

We approach the problem in a manner that will allow us to show that, in particular, such a characterization is possible within the class of compact spaces.

Throughout the paper, I always denotes the closed unit interval of the real line.

Playing with clones

Let X be an arbitrary topological space. On the clone of X, we will impose certain first order conditions that hold for the clone of the unit interval I, and then derive topological properties of X from these conditions.

(1) I is non-degenerate; one can express this property by the first order clone-theoretic formula

$$NDeg \equiv (\exists\, x, y)(x \neq y).$$

So our first condition will be that $NDeg$ holds in $Cl(X)$, and we will abbreviate this fact as

$$Cl(X) \models NDeg.$$

This precisely means that X is non-degenerate.

(2) I is a connected space, which can be expressed, for example, by

$$Con \equiv (\forall\, f(-))[(\exists\, x, y)(f(x) \neq f(y)) \to (\forall\, a, b)(\exists\, x, y)(a \neq y \neq b \;\;\&\;\; f(x) = y)].$$

So the second condition,

$$Cl(X) \models Con,$$

precisely says that every non-constant self-map of X attains at least three values. Therefore X **is connected**. Selecting id_X for f in Con and using the non-degeneracy of X, one concludes that X **has at least three points**.

(3) I has two boundary points, which can be characterized, among others, by the formula

$$Bd(x,y) \equiv x \neq y \ \& \ (\forall f(-))[(\exists a,b)(f(a)=x \ \& \ f(b)=y) \to \\ \to (\forall z)(\exists u)(f(u)=z)]$$

($Bd(x,y)$ means: whenever x and y are in the range of f, then f is *onto*). In $Cl(I)$, $Bd(x,y)$ if and only if $\{x,y\}=\{0,1\}$, so that

$$Cl(I) \models (\exists x,y)Bd(x,y) \ \& \ (\forall a,b,c,d)(Bd(a,b) \ \& \ Bd(c,d) \to \\ \to (a=c \ \& \ b=d \ \vee \ a=d \ \& \ b=c).$$

Now, letting *Bound* denote the latter formula, we insist that

$$Cl(X) \models Bound,$$

i.e. X contains exactly two distinct points such that whenever they belong to the range of a function then the function is onto.

For the sake of convenience, we also introduce the abbreviation

$$Bp(x) \equiv (\exists y)Bd(x,y),$$

meaning x is a "boundary" point.

(4) I is "homogeneous", by which we mean the following:

$$\begin{aligned} Cl(I) \models & (\forall x,y,z,t)\{(x \neq y \ \& \ z \neq t) \to \\ & \to [(\neg Bp(x) \ \& \ \neg Bp(y) \ \& \ \neg Bp(z) \ \& \ \neg Bp(t) \ \vee \\ & \vee \ \neg Bp(x) \ \& \ Bp(y) \ \& \ \neg Bp(z) \ \& \ Bp(t) \ \vee \\ & \vee \ Bp(x) \ \& \ Bp(y) \ \& \ Bp(z) \ \& \ Bp(t)) \to \\ & \to (\exists f(-),g(-))(f(x)=z \ \& \ f(y)=t \ \& \\ & \& \ (\forall u)(f(g(u))=u \ \& \ g(f(u))=u)))]\} \end{aligned}$$

and

$$\begin{aligned} Cl(I) \models & (\forall x,y,z,t)(Bd(x,y) \ \& \ z \neq t \to \\ & \to (\exists f(-),g(-))[f(x)=z \ \& \ f(y)=t \ \& \ (\forall u)(g(f(u))=u) \ \& \\ & \& \ (\neg Bp(z) \ \& \ \neg Bp(t) \to (\forall u)(\neg Bp(f(u))))]). \end{aligned}$$

In informal language, the first formula says that there is a homeomorphism of the unit interval onto itself, taking x to z and y to t, whenever x,y,z,t are points such that each of the pairs $\{x,z\}$ and $\{y,t\}$ consists of either two boundary or two interior points. The second one expresses the fact that I is homeomorphic to any of its subintervals. Having denoted by *Homog* the conjunction of these two formulas, from now on we shall also require that

$$Cl(X) \models Homog.$$

Now we are able to prove some simple facts about X.

Lemma. *X has at least four points, or, equivalently,*

$$Cl(X) \models (\exists\, x, y)(x \neq y \ \ \& \ \ \neg Bp(x) \ \ \& \ \ \neg Bp(y)).$$

Proof: As we know, X has exactly two "boundary" points, say a and b. Furthermore, X has at least three points (see **(2)**), so there exists an $x \in X$ different from a and b. By homogeneity there exist $f, g : X \to X$ such that $f(a) = x$, $f(b) = b$ and $g \circ f = id_X$. Then obviously $f(x) \neq b$ and $f(x) \neq x$. If X were three-pointed, then $X = \{a, b, x\}$, $f(x) = a$ and $f = g$, so that f would be a homeomorphism and hence $Bp(x)$ would hold in $Cl(X)$, because $Bp(a)$ and $x = f(a)$. So X must have at least four points. □

Proposition. *X is a T_1-space.*

Proof: Let $a \neq b \in X$ be the "boundary" points of X, that is, let $Bd(a, b)$ hold. Denote $\overset{\circ}{X} = X \setminus \{a, b\}$ and suppose $x \in \overline{\{y\}}$, where $x, y \in \overset{\circ}{X}$ and $x \neq y$. By homogeneity we then get $u \in \overline{\{t\}}$ for any $u, t \in \overset{\circ}{X}$, hence $\overline{\{t\}} \supseteq \overset{\circ}{X}$ for each $t \in \overset{\circ}{X}$, so each closed subset of X that intersects $\overset{\circ}{X}$ must include $\overset{\circ}{X}$. By the previous lemma, $\overset{\circ}{X}$ has at least two points. Let u and v be different points of $\overset{\circ}{X}$. Define

$$f(x) = \begin{cases} u, & \text{if } \ x \in \{a, b\} \\ v, & \text{if } \ x \in \overset{\circ}{X}. \end{cases}$$

Then $f : X \to X$ is a continuous map (as the preimage of each closed set is either $\emptyset$ or X) with exactly two values, which contradicts Con. This proves that $\overset{\circ}{X}$ is a T_1-space.

Take now two different points $u, v \in \overset{\circ}{X}$ and, using homogeneity, find $f, g : X \to X$ such that $f(a) = u$, $f(b) = v$, $g \circ f = id_X$ and $f(X) \subseteq \overset{\circ}{X}$. Suppose that $x \in \overline{\{y\}}$. Then also $f(x) \in \overline{\{f(y)\}}$ and $f(x), f(y) \in \overset{\circ}{X}$, so $f(x) = f(y)$ and $x = g(f(x)) = g(f(y)) = y$. □

We now continue to impose further conditions on X.

(5) One can define the order of the unit interval in terms of the language of clones. We do it in the following way. Let $\min(-,-)$ and $\max(-,-)$ be two different binary variables (this notation may seem to be suggestive, but the author hopes that this suggestion is the right one). Define

$$\begin{aligned} lo(\min, \max) \equiv (\forall\, x, y, z)[&\min(x, y) = \min(y, x) \ \ \& \\ &\& \ \ \min(\min(x, y), z) = \min(x, \min(y, z)) \ \ \& \\ &\& \ \ (\min(x, y) = x \ \ \vee \ \ \min(x, y) = y) \ \ \& \\ &\& \ \ (\max(x, y) = x \ \ \vee \ \ \max(x, y) = y) \ \ \& \\ &\& \ \ (\min(x, y) = \max(x, y) \to x = y)]. \end{aligned}$$

The above formula tells us that the maps $\min(-,-)$ and $\max(-,-)$ are lattice operations defining a linear order. So we introduce the abbreviation $x \preccurlyeq y$ for $\min(x,y) = x$ (or, equivalently, $\max(x,y) = y$). $x \prec y$ will stand for $x \preccurlyeq y$ & $x \neq y$. The symbol $\preccurlyeq$ will also stand for the pair $(\min, \max)$, so we will use notions like $lo(\preccurlyeq)$, $(\exists \preccurlyeq) \ldots$ etc. Write

$$Order(\preccurlyeq) \equiv lo(\preccurlyeq) \ \& \ (\forall f(-))(\forall a,b,y)(f(a) \preccurlyeq y \preccurlyeq f(b) \rightarrow (\exists x)(f(x) = y)),$$

saying that "$\preccurlyeq$ is a linear order with Darboux property". Obviously, up to the duality, there is exactly one such order on the unit interval. Hence our fifth condition will be

$$Cl(X) \models (\exists \preccurlyeq) Order(\preccurlyeq).$$

Notice that **the "boundary" points guaranteed by** $Bound$ **are the least and the greatest point in any specific order given by this condition** and thus, in particular, **the least and the greatest points exist**. This is obvious when one realizes that, for $a \preccurlyeq b$, the map $x \mapsto \min(b, \max(a,x))$ is continuous and onto the subinterval $[a,b]$. Actually, we could also impose the last fact onto X directly by the requirement that

$$Cl(X) \models Order(\preccurlyeq) \ \& \ Bd(a,b) \rightarrow (\forall x)(a \preccurlyeq x \preccurlyeq b \ \vee \ b \preccurlyeq x \preccurlyeq a).$$

(6) The following formula expresses continuity:

$$\begin{aligned}
Cont \equiv & Order(\preccurlyeq) \rightarrow (\forall f(-))(\forall x,a,b) \\
& [(\neg Bp(x) \ \& \ a \prec f(x) \prec b \rightarrow (\exists c,d)(c \prec x \prec d \ \& \\
& \qquad\qquad \& \ (\forall z)(c \prec z \prec d \rightarrow a \prec f(z) \prec b))) \ \& \\
& (\neg Bp(x) \ \& \ Bp(f(x)) \ \& \ a \prec f(x) \rightarrow (\exists c,d)(c \prec x \prec d \ \& \\
& \qquad\qquad \& \ (\forall z)(c \prec z \prec d \rightarrow a \prec f(z)))) \ \& \\
& (\neg Bp(x) \ \& \ Bp(f(x)) \ \& \ f(x) \prec b \rightarrow (\exists c,d)(c \prec x \prec d \ \& \\
& \qquad\qquad \& \ (\forall z)(c \prec z \prec d \rightarrow f(z) \prec b))) \ \& \\
& (Bp(x) \ \& \ (\exists u)(x \prec u) \ \& \ a \prec f(x) \prec b \rightarrow (\exists d)(x \prec d \ \& \\
& \qquad\qquad \& \ (\forall z)(z \prec d \rightarrow a \prec f(z) \prec b))) \ \& \\
& (Bp(x) \ \& \ (\exists u)(x \prec u) \ \& \ Bp(f(x)) \ \& \ a \prec f(x) \rightarrow (\exists d)(x \prec d \ \& \\
& \qquad\qquad \& \ (\forall z)(z \prec d \rightarrow a \prec f(z)))) \ \& \\
& (Bp(x) \ \& \ (\exists u)(x \prec u) \ \& \ Bp(f(x)) \ \& \ f(x) \prec b \rightarrow (\exists d)(x \prec d \ \& \\
& \qquad\qquad \& \ (\forall z)(z \prec d \rightarrow f(z) \prec b))) \ \& \\
& (Bp(x) \ \& \ (\exists v)(v \prec x) \ \& \ a \prec f(x) \prec b \rightarrow (\exists c)(c \prec x \ \& \\
& \qquad\qquad \& \ (\forall z)(c \prec z \rightarrow a \prec f(z) \prec b))) \ \&
\end{aligned}$$

$$(Bp(x) \ \& \ (\exists\, v)(v \prec x) \ \& \, Bp(f(x)) \ \& \ a \prec f(x) \rightarrow (\exists\, c)(c \prec x \ \&$$
$$\& \ (\forall\, z)(c \prec z \rightarrow a \prec f(z)))) \ \&$$
$$(Bp(x) \ \& \ (\exists\, v)(v \prec x) \ \& \, Bp(f(x)) \ \& \ f(x) \prec b \rightarrow (\exists\, c)(c \prec x \ \&$$
$$\& \ (\forall\, z)(c \prec z \rightarrow f(z) \prec b)))].$$

Obviously $Cl(I) \models Cont$. So we require that

$$Cl(X) \models Cont,$$

meaning that each function continuous in the topology of X is continuous also in the topology induced by the specific order $\preccurlyeq$.

(7) In $Cl(I)$ the formula

$$OIO \equiv Order(\preccurlyeq) \rightarrow (\forall\, a, b)[(a \prec b) \rightarrow$$
$$\rightarrow (\exists\, f(-))(\exists\, x)(\forall\, z)(a \prec z \prec b \leftrightarrow f(z) \neq x) \ \&$$
$$\& \ (\exists\, g(-))(\exists\, x)(\forall\, z)(a \prec z \leftrightarrow g(z) \neq x) \ \&$$
$$\& \ (\exists\, h(-))(\exists\, x)(\forall\, z)(z \prec b \leftrightarrow h(z) \neq x)]$$

holds, i.e. for every open interval there exists a continuous map and a point such that the interval is the preimage of the complement to the point. So our seventh condition

$$Cl(X) \models OIO$$

implies, using the fact that X is a T_1-space, that **every open interval in the order $\preccurlyeq$ is open in the topology of X**. Using Con one now gets that **the order $\preccurlyeq$ is dense, complete and has a greatest and a least point**.

(8) In I, one can continuously choose a midpoint for each interval; in the language of clones

$$Cl(I) \models Order(\preccurlyeq) \rightarrow (\exists\, F(-,-))[\{F \text{ is continuous in the topology}$$
$$\text{induced by } \preccurlyeq\} \ \& \ (\forall\, x, y)(x \prec y \rightarrow x \prec F(x, y) \prec y)].$$

The phrase enclosed in braces is the analogy of $Cont$ for binary maps. Let us denote the latter formula by Cen, and impose

$$Cl(X) \models Cen.$$

Now we prove that X is quite close to the bounded interval. Fix an order $\preccurlyeq$ on X satisfying $Order(\preccurlyeq)$. Let $x_1^{(0)}$ be the least and $x_2^{(0)}$ the greatest point of X. Put $I_1^{(0)} = [x_1^{(0)}, x_2^{(0)}]$. We will halve inductively the intervals by means of the function F, whose existence is guaranteed by Cen. So, having at the k-th step the intervals $I_1^{(k)} = [x_1^{(k)}, x_2^{(k)}]$, $I_2^{(k)} = [x_2^{(k)}, x_3^{(k)}]$, $\dots$, $I_{2^k}^{(k)} = [x_{2^k}^{(k)}, x_{2^k+1}^{(k)}]$, at the $(k+1)$-st step we set $I_1^{(k+1)} = [x_1^{(k+1)} = x_1^{(k)}, x_2^{(k+1)} = F(x_1^{(k)}, x_2^{(k)})]$, $I_2^{(k+1)} = [F(x_1^{(k)}, x_2^{(k)}), x_2^{(k)}]$, ... , $I_{2^{k+1}-1}^{(k+1)} = [x_{2^k}^{(k)}, F(x_{2^k}^{(k)}, x_{2^k+1}^{(k)})]$, $I_{2^{k+1}}^{(k+1)} = [F(x_{2^k}^{(k)}, x_{2^k+1}^{(k)}), x_{2^k+1}^{(k)}]$. Thus we get a binary tree of closed intervals. Let M denote the set of their endpoints ($M = \{x_i^{(k)}; k = 0, 1, 2, \dots, i = 1, 2, \dots, 2^k+1\}$); obviously $|M| = \aleph_0$. We prove that M is dense in the topology given by $\preccurlyeq$. Let $x \in X \setminus M$. Then the set of those intervals $I_i^{(k)}$ including x forms an infinite branch of the binary tree; let $I_1 \supseteq I_2 \supseteq \dots$ be the sequence of those ordered decreasingly. Let $I_i = [a_i, b_i]$. We thus have

$$a_1 \preccurlyeq a_2 \preccurlyeq \dots \preccurlyeq x \preccurlyeq \dots \preccurlyeq b_2 \preccurlyeq b_1.$$

Because $\preccurlyeq$ is complete we can define $a = \sup a_i$, $b = \inf b_i$. Suppose that $a \neq b$, that is, $a \prec b$. One has $a \prec F(a,b) \prec b$ and, as F is continuous in the topology of $\preccurlyeq$, there exist $\preccurlyeq$-neighbourhoods $O_a \ni a$, $O_b \ni b$ such that $x \in O_a, y \in O_b \Rightarrow F(x,y) \in (a,b)$. But for some large enough i we have $a_i \in O_a$ and $b_i \in O_b$ and thus either $a_{i+1} \in (a,b)$ or $b_{i+1} \in (a,b)$, which is a contradiction. Hence $a = b = x$ and x lies in the $\preccurlyeq$-closure of M.

We proved that $\preccurlyeq$ is a complete dense order with a greatest and a least point and a countable dense subset, and hence it is isomorphic to the order of the unit interval. Thus we have the following

Theorem 1. *Let X be an arbitrary topological space. Suppose the clone $Cl(X)$ satisfies the first order formula*

$$\begin{aligned} &NDeg \ \& \ Con \ \& \ Bound \ \& \ Homog \ \& \\ &\& \ (\exists \preccurlyeq) Order(\preccurlyeq) \ \& \ Cont \ \& \ OIO \ \& \ Cen. \end{aligned}$$

Then there exists an order $\preccurlyeq$ on X such that

(i) *$(X, \preccurlyeq)$ is order isomorphic to $(I, \leqslant)$;*
(ii) *the topology of X is finer than that induced by the order $\preccurlyeq$;*
(iii) *each continuous self-map of X is continuous also in the topology induced by $\preccurlyeq$;*
(iv) *the maps $(x,y) \mapsto \max(x,y)$ and $(x,y) \mapsto \min(x,y)$ are continuous in the topology of X.*

In particular, X is Hausdorff. □

Remark. Adding an n-ary version of *Cont* to the assumptions, one obtains point (iii) of Theorem 1 for n-ary maps. So if we assume elementary equivalence of $Cl(X)$ and $Cl(I)$, then (iii) holds for maps of arbitrary arity.

As compact topologies are the coarsest Hausdorff ones we have a

Corollary. *Under the assumptions of Theorem 1, if X is compact, then it is homeomorphic to I.* □

The monoid

Now we translate the conditions for $Cl(X)$ into the language of monoids of continuous self-maps $M(X)$. The conditions $NDeg$, Con, $Bound$, $Homog$ are translated with no difficulty since they include only unary and nullary variables and the nullary ones can be replaced by unary constant ones, which are characterized in the language of monoids by $(\forall\, g, h)(f \circ g = f \circ h)$. Thus the respective formulas $NDeg_M$, Con_M, $Bound_M$, $Homog_M$ in the language of monoids are equivalent to the original ones.

Next we define the order in the language of monoids. Note that there exists a continuous function $o : I \to I \times I$ such that its image is the upper triangle $\{(x, y) \in I \times I; x \leqslant y\}$. Actually o is a pair of unary functions, $o = (o_1, o_2)$, that define the order of the interval. So now for us the "order" will be a pair of self-maps $o_1,\ o_2$. The symbol $\preccurlyeq$ will be an abbreviation for the pair (o_1, o_2) and for the relation defined by the following way:

$$x \preccurlyeq y \iff (\exists\, t)(x = o_1(t)\ \ \&\ \ y = o_2(t)).$$

The property of being a linear order is expressed by

$$\begin{aligned} lo_M(\preccurlyeq) \equiv (\forall\, x, y, z)(x \preccurlyeq x\ \ \&\ \ (x \preccurlyeq y\ \ \&\ \ y \preccurlyeq x \to x = y)\ \ \& \\ (x \preccurlyeq y\ \ \&\ \ y \preccurlyeq z \to x \preccurlyeq z)\ \ \&\ \ (x \preccurlyeq y\ \ \vee\ \ y \preccurlyeq x)). \end{aligned}$$

(Where x, y, z mean constant self-maps; more precisely it would look like $(\forall\, x, y, z)$ $[(\forall\, u, v)(x \circ u = x \circ v\ \ \&\ \ y \circ u = y \circ v\ \ \&\ \ z \circ u = z \circ v) \to \ldots]$. Let us agree that lower-case letters from the end of the latin alphabet will always be taken relativised to constant maps.) Now we can define the formula $Order_M(\preccurlyeq)$ just as $Order(\preccurlyeq)$, using lo_M instead of lo. Again it holds $M(I) \models (\exists \preccurlyeq) Order_M(\preccurlyeq)$. Of course there are many pairs (o_1, o_2) satisfying $Order_M$ but the order defined by them is unique up to duality thanks to the Darboux property which is wanted by $Order_M$.

Of course, $(\exists \preccurlyeq) Order_M(\preccurlyeq)$ is not quite equivalent to $(\exists \preccurlyeq) Order(\preccurlyeq)$ (or at least it is not quite obvious) as it does not need continuity of maps min and max.

Using the latter fact, in **(5)** we proved that the boundary points guaranteed by the formula $Bound$ are those of the order. But as pointed out already, we can require this directly by the formula

$$Ep_M \equiv Order_M(\preccurlyeq)\ \ \&\ \ Bd_M(x, y) \to (\forall\, z)(x \preccurlyeq z \preccurlyeq y\ \ \vee\ \ y \preccurlyeq z \preccurlyeq x).$$

(Actually we use it only for the fact that the boundary points of the order exist, which could be required directly as well.)

The formulas $Cont$ and OIO are now easily translated replacing $Order$ by $Order_M$. Denote the resulting formulas by $Cont_M$ and OIO_M, respectively.

Now it suffices to translate the formula Cen, saying that one can choose a midpoint for each subinterval, continuously in the topology of X and $\preccurlyeq$. There exists a continuous map $f = (f_1, f_2)$ of the unit interval I onto the square $I \times I$. The triple $(f_1, f_2, \dfrac{f_1+f_2}{2})$ now shows that in $M(I)$ it holds

$$\begin{aligned} Cen_M \equiv & Order_M(\preccurlyeq) \to (\exists\, f_1, f_2, g)[(\forall\, x, y)(\exists\, t)(x = f_1(t) \ \& \ y = f_2(t)) \ \& \\ & \& \ (\forall\, t)(\forall\, u)(f_1(t) = f_1(u) \ \& \ f_2(t) = f_2(u) \to g(t) = g(u)) \ \& \\ & \& \ (\forall\, t)(f_1(t) \prec f_2(t) \to f_1(t) \prec g(t) \prec f_2(t)) \ \& \\ & \& \ (\forall\, x, y, t, a, b)\{\neg Bp(x) \ \& \ \neg Bp(y) \ \& \ f_1(t) = x \ \& \ f_2(t) = y \ \& \\ & \& \ a \prec g(t) \prec b \to (\exists\, c_1, d_1, c_2, d_2)(c_1 \prec x \prec d_1 \ \& \ c_2 \prec y \prec d_2 \ \& \\ & (\forall\, z)(c_1 \prec f_1(z) \prec d_1 \ \& \ c_2 \prec f_2(z) \prec d_2 \to a \prec g(z) \prec b)) \ \& \ \ldots\}] \end{aligned}$$

(the first row announces the existence of a self-map $g : I \to I$ and a surjection $(f_1, f_2) : I \twoheadrightarrow I \times I$, the second one says that the map $F(-,-) = g \circ (f_1, f_2)^{-1}$ is correctly defined, the third one says that $F(x, y)$ lies between x and y, and the rest expresses the continuity of F in the topology induced by $\preccurlyeq$).

Of course this formula does not say that the map $F(-,-) = g \circ (f_1, f_2)^{-1}$ is continuous also in the topology of X, but in the above considerations we used only continuity in the topology of $\preccurlyeq$. So we have the following

Theorem 2. *Let X be a topological space such that*

$$\begin{aligned} M(X) \models & NDeg_M \ \& \ Con_M \ \& \ Bound_M \ \& \ Homog_M \ \& \\ & \& \ (\exists \preccurlyeq) Order_M(\preccurlyeq) \ \& \ Ep_M \ \& \ Cont_M \ \& \ OIO_M \ \& \ Cen_M. \end{aligned}$$

Then there exists a linear order $\preccurlyeq$ on X satisfying the points (i), (ii), (iii) of Theorem 1. □

Corollary. *If X is also compact then it is homeomorphic to I.* □

Some additions

Now we will strengthen our conditions radically, wanting an **isomorphism** of $M(X)$ and $M(I)$ and, moreover, we will assume the existence of a convergent sequence in X.

Theorem. *Suppose that X contains a countable sequence $\{x_i\}_{i\in\mathbb{N}}$ of pairwise distinct points converging to a point $x \in X$, and let there exist an isomorphism $\Phi : M(X) \to M(I)$. Then X and I are homeomorphic.*

Proof: Any topological space Y can be identified with the set of constant maps of $M(Y)$, which are well-defined within the monoidal structure of $M(Y)$, as we saw above. Hence the isomorphism $\Phi : M(X) \to M(I)$ defines a bijection $F : X \to I$. On the other hand, the bijection F determines Φ uniquely. Thus we can suppose that X and I are identical as sets and $M(X) = M(I)$. We want to prove that X and I are identical also topologically.

Since isomorphism is stronger than elementary equivalence, $M(X)$ satisfies, in particular, $Ndeg_M$, Con_M, $Bound_M$ and $Homog_M$, and hence X is a T_1-space, as we proved in **(4)**. This implies that each subset closed in the topology of I is closed in the topology of X, as the preimage of a point under a continuous map.

Let K be an X-closed set and let y be in the I-closure of K. This means that there exists a sequence $\{y_i\}_{i\in\mathbb{N}}$ of points of K converging to y in I. We know that $x_i \xrightarrow{X} x$ and, as I has a coarser topology than X, also $x_i \xrightarrow{I} x$. Without loss of generality $x \neq x_i$ for any $i \in \mathbb{N}$. So the map defined by $x_i \mapsto y_i$, $x \mapsto y$ is I-continuous on an I-closed subset $\{x_i\}_{i\in\mathbb{N}} \cup \{x\}$ and can be continuously extended to a function $f : I \to I$. But then f is continuous also in the topology of X and hence $y_i = f(x_i) \xrightarrow{X} f(x) = y$. So y lies in the X-closure of K, which is K itself. Hence every X-closed set is I-closed, and the proof is complete. □

Now we discuss some consequences of our theorems. Notice that the first order theory of the clone (the monoid) of the unit interval determines the cardinality of it. Let $Th_{Cl}(I)$ ($Th_M(I)$) be that theory, i.e. the set of all first order clone- (monoid-) theoretic formulas that hold in $Cl(I)$ ($M(I)$). Using, say, the Lövenheim-Skolem theorem, one has models of those theories of any infinite cardinality. Let Cl_0 (M_0) be a countable model of $Th_{Cl}(I)$ ($Th_M(I)$). If Cl_0 (M_0) were isomorphic to the clone (monoid) of a topological space X, then X would satisfy the assumption of Theorem 1 (Theorem 2) and, consequently, X would have cardinality $2^{\aleph_0}$. The last fact means Cl_0 (M_0) has $2^{\aleph_0}$ constants, which is not the case. So Cl_0 (M_0) cannot be the clone (monoid) of a topological space. This answers a simplified version of the problem 9.3 of [10], which asks whether each clone with constants is equivalent to a clone of a topological space in the lattice of interpretability (two clones are equivalent in the lattice of interpretability if there exist homomorphisms from each of them into the other one; see [1], [7]).

Note that on the other hand, according to [8], the category of topological spaces is almost universal, which means, particularly, that for each monoid M there exists a topological space X such that the non-constant continuous self-maps of X form a monoid isomorphic to M. Of course it is very easy to find a monoid that is not a monoid of a topological space; it suffices to take a monoid with no constants, e.g. a non-trivial group. Note also that a monoid (clone) elementarily equivalent to $M(I)$ ($Cl(I)$) is a monoid of self-maps (clone of finitary operations) of a set, containing all constant ones.

References

[1] García O.C., Taylor W., *The lattice of interpretability types of varieties*, Mem. Amer. Math. Soc. **50** (1984), no. 305.

[2] Grätzer G., *Universal algebra*, 2-nd ed., Springer-Verlag, Berlin and New York, 1979.

[3] Herrlich H., *On the concept of reflection in general topology*, Conf. on Contributions to Extension Theory of Topological Structures, Berlin, 1967.

[4] Herrlich H., *Topologische Reflexionen und Coreflexionen*, Lecture Notes in Math., vol. 78, Springer-Verlag, Berlin; Heidelberg, New York, 1968.

[5] Kuratowski K., *Topologie I, II*, Monogr. Mat., Warsaw, 1950.

[6] Lawvere F.W., *Functorial semantics of algebraic theories*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 869–872.
[7] Neumann W.N., *On Malcev conditions*, J. Austral. Math. Soc. **17** (1974), 376–384.
[8] Pultr A., Trnková V., *Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories*, North-Holland, Amsterdam, 1980.
[9] Sichler J., Trnková V., *On elementary equivalence and isomorphism of clone segments*, Period. Math. Hungar. **32** (1996), 113–128.
[10] Taylor W., *The Clone of a Topological Space*, Res. Exp. Math., vol. 13, Heldermann Verlag, 1986.
[11] Trnková V., *Semirigid spaces*, Trans. Amer. Math. Soc. **343** (1994), no. 1, 305–325.

Mathematical Institute, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

21 Yervand Kochar str., apt. 74, Yerevan 375070, Republic of Armenia
E-mail: ABar5696@menza.mff.cuni.cz

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