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On the fusion problem for degenerate elliptic equations II

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Abstract. Let F be a relatively closed subset of a Euclidean domain Ω. We investigate when solutions u to certain elliptic equations on $\Omega \setminus F$ are restrictions of solutions on all of Ω. Specifically, we show that if ∂F is not too large, and u has a suitable decay rate near F, then u can be so extended.

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In this paper, we study removability of a set F for solutions to certain degenerate elliptic partial differential equations which are defined on $\Omega \setminus F$ and decay in the vicinity of F. Here and throughout this paper, Ω is an open set in $\mathbb{R}^n$, $n \geq 2$, F is a relatively closed proper subset of Ω, and $1 < p \leq n$.

The results in this paper are closely related to those in [4]. Roughly speaking, both papers show that if the dimension of ∂F is less than a critical index dependent on the rate of decay, then F is removable. The innovation in this paper is that we measure dimension by means of Hausdorff measure rather than lower Minkowski density. Since it is easy to give examples of sets whose Hausdorff dimension is strictly less than their lower Minkowski dimension, this improves the results in the earlier paper.

We shall be concerned with partial differential equations of the form

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0 \tag{1}$$

where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping that satisfies the following assumptions for some constants $0 < \alpha \leq \beta < \infty$:

(a) the mapping $x \mapsto \mathcal{A}(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^n$, and the mapping $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous for a.e. $x \in \mathbb{R}^n$;
(b) $\mathcal{A}(x, \xi) \cdot \xi \geq \alpha |\xi|^p$;
(c) $|\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1}$;
(d) $(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) \geq 0$ whenever $\xi_1 \neq \xi_2$;
(e) $\mathcal{A}(x, \lambda\xi) = |\lambda|^{p-2} \lambda \mathcal{A}(x, \xi)$ for $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

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In particular, taking $p = 2$, and $\mathcal{A}(x,\xi) = A(x)\xi$ for some bounded measurable matrix-valued function A satisfying a uniform ellipticity condition, we see that the above class contains the class of self-adjoint linear elliptic equations with measurable coefficients. Another example (for any $p > 1$) is the p-Laplace equation

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

Throughout this paper, $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ refer to functions satisfying conditions (a)–(e) above.

By a solution of (1) in Ω, we shall mean a function u in the local Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\phi\,dx = 0 \tag{2}$$

for all test functions $\phi \in C_0^\infty(\Omega)$. An excellent source for the potential theory of such solutions (which arise naturally in the theory of quasiregular mappings) is the monograph of Heinonen, Kilpeläinen, and Martio [2].

By an $\mathcal{A}$-harmonic function, we mean a continuous solution of (1) (in the linear case $\mathcal{A}(x,\xi) = A(x)\xi$, where A is bounded, measurable, and uniformly elliptic, we say that u is A-harmonic). We now record some basic properties possessed by $\mathcal{A}$-harmonic functions u — proofs can be found in Chapters 3 and 6 of [2]. We note first that any solution of (1) can be regarded as an $\mathcal{A}$-harmonic function, since it differs from a continuous function only on a set of measure zero. Next, we note that (2) is actually true for all test functions ϕ in the Sobolev space $W_0^{1,p}(\Omega)$. Finally, u is Hölder continuous with some exponent $0 < \alpha \le 1$ depending only on n, p, and β/α.

For any non-decreasing gauge function $h : [0,\infty) \to [0,\infty)$ satisfying $h(0) = 0$, we can define a Hausdorff measure $\mathcal{H}^h$ (in fact, we only need h to be defined near 0); see, for example, [1]. This refines the more well-known notion of Hausdorff measure $\mathcal{H}^s$, where s is a positive number, since $\mathcal{H}^s = \mathcal{H}^h$ if $h(t) \equiv t^s$.

The main result of this paper, Theorem 6, says roughly that if a solution u in $\Omega \setminus F$ has some rate of decay near F, and ∂F is a null set for a related Hausdorff-type measure, then u is a solution in all of Ω. For simplicity, we first state and prove our main result in the case where the gauge function has the form $h(t) = t^s$. From here on, $\delta_A(x)$ denotes the distance from the point x to the closed set A.

Theorem 1. *Suppose that u is $\mathcal{A}$-harmonic (with parameter $p > 1$) in $\Omega \setminus F$. Suppose also that $\mathcal{H}^{n-p+(p-1)\alpha}(\partial F) = 0$ and $|u(x)| \le C\delta_F^\alpha(x)$ for some $0 < \alpha \le p/(p-1)$ and all $0 < \delta_F(x) < \min\{1, \delta_{\partial\Omega}(x)\}/2$. If we extend u to be zero on F, then u is $\mathcal{A}$-harmonic in Ω.*

In the linear case, Theorem 1 immediately yields the following corollary, which we believe is new.

Corollary 2. *Suppose that u is A-harmonic on $\Omega \setminus F$, i.e. it is a continuous solution in $\Omega \setminus F$ of the linear equation* $\operatorname{div}(A(x)\nabla u(x)) = 0$, *where A is a bounded measurable matrix-valued function satisfying a uniform ellipticity condition. Suppose also that $\mathcal{H}^{n-2+\alpha}(\partial F) = 0$ and $|u(x)| \le C\delta_F^\alpha(x)$ for some $0 < \alpha \le 2$ and all $0 < \delta_F(x) < \min\{1, \delta_{\partial\Omega}(x)\}/2$. If we extend u to be zero on F, then u is A-harmonic in Ω.*

Related theorems have been considered elsewhere. For example, Král [6] showed that for the Laplace equation (i.e. $\mathcal{A}(x,\xi) = \xi$), a $C^1(\Omega)$ function which is harmonic on $\{x \in \Omega : u(x) \neq 0\}$ is harmonic on all of Ω; Kilpeläinen [3] proves a similar result for the p-Laplace equation in the plane. In our result, the decay of u near F takes the place of the smoothness assumption (note that $\mathcal{A}$-harmonic functions are not necessarily C^1, or even locally Lipschitz). Results even more closely related to Theorem 1 are to be found in [5] and [4]. In particular, Theorem 1.7 in the latter paper is a weaker version of Theorem 1 in which the Hausdorff measure condition on the size of ∂F is replaced by a condition on the lower Minkowski density of F. Example 5.1 in [4] shows that Theorem 1 is essentially sharp and that Corollary 2 is sharp for $1 \le \alpha < 2$.

In the linear case, Corollary 2 also allows us to say something about the fusion problem, which asks when two solutions can be spliced together to give a single solution. More precisely, the *fusion problem* is as follows:

Suppose that u_1 is $\mathcal{A}_1$-harmonic in Ω and that u_2 is $\mathcal{A}_2$-harmonic in $\Omega \setminus F$. Define

$$u = \begin{cases} u_1 & \text{in } F \\ u_2 & \text{in } \Omega \setminus F \end{cases}$$

and

$$\mathcal{A}(x,\xi) = \begin{cases} \mathcal{A}_1(x,\xi), & \text{if } x \in F \\ \mathcal{A}_2(x,\xi), & \text{otherwise.} \end{cases}$$

Is u $\mathcal{A}$-harmonic in Ω?

We now state a result which addresses the fusion problem in the special case where $\mathcal{A}_1 = \mathcal{A}_2$ and the equation is linear; this corollary follows immediately by applying Corollary 2 to $u \equiv u_2 - u_1$.

Corollary 3. *Suppose that u_1 is A-harmonic in Ω and u_2 is A-harmonic in $\Omega \setminus F$, i.e. u_1, u_2 are continuous solutions in the indicated open sets of the linear equation* $\operatorname{div}(A(x)\nabla u) = 0$, *where A is a bounded measurable matrix-valued function satisfying a uniform ellipticity condition. Suppose also that $\mathcal{H}^{n-2+\alpha}(\partial F) = 0$ and $|u_1(x) - u_2(x)| \le C\delta_F^\alpha(x)$ for some $0 < \alpha \le 2$ and all $0 < \delta_F(x) < \min\{1, \delta_{\partial\Omega}(x)\}/2$. Then the function*

$$u = \begin{cases} u_1 & \textit{in } F \\ u_2 & \textit{in } \Omega \setminus F \end{cases}$$

is A-harmonic in Ω.

Note that for the equations under consideration in the above corollary, there is no unique continuation property. In fact, Miller [7] showed that certain equations of the form $\mathrm{div}(A\nabla u) = 0$ have non-trivial smooth weak solutions that vanish on an open set.

Before proving Theorem 1, we first state a couple of useful lemmas, the first of which is Lemma 2.2 of [4].

Lemma 4. *Suppose that F is a relatively closed subset of $\Omega \subset \mathbb{R}^n$ and that $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is continuous. Let h be $\mathcal{A}$-harmonic in $\Omega \setminus F$ such that*

$$\lim_{x \to y} h(x) = v(y)$$

for every $y \in \partial F \cap \Omega$. Then the function

$$w = \begin{cases} h & \text{in } \Omega \setminus F \\ v & \text{in } F \end{cases}$$

belongs to $C(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$.

Lemma 5. *Suppose that F is a relatively closed subset of Ω. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is continuous in Ω, $\mathcal{A}$-harmonic on $\Omega \setminus F$, and zero on F, then*

$$\int_{B(x,r)} |\nabla u|^p \le C r^{-p} \int_{B(x,2r)} |u - u(x)|^p$$

whenever the ball $B(x, 3r) \subset \Omega$. Here, C depends only on n, p, and β/α.

This last lemma is a type of Caccioppoli Lemma. It is proved in the usual fashion, but there is one obstacle to be overcome: we need to choose u as the test function in (2), and so we would like to know that u lies in $W^{1,p}_0(\Omega \setminus F)$ and not just in $W^{1,p}_{\mathrm{loc}}(\Omega)$. By multiplying by a suitable bump function, we first kill off u outside a suitably large ball, for instance $B(x, 11r/4)$, without changing it on $B(x, 5r/2)$. Thus we may assume that $u \in W^{1,p}_0(\Omega)$; of course, u is now only $\mathcal{A}$-harmonic on $B(x, 5r/2) \setminus F$, but this is good enough for the proof. Because u is continuous on Ω, and zero on F, it is not hard to see that we actually have $u \in W^{1,p}_0(\Omega \setminus F)$ (hint: write u as the limit of the compactly supported functions $u_\epsilon = \max\{0, u - \epsilon\}$, $\epsilon > 0$). With this one obstacle removed, the rest of the proof is standard, so we omit the details.

Proof of Theorem 1: Let $\epsilon > 0$ be given and let $\phi \in C^\infty_0(\Omega)$ be a test function with support K. We cover ∂F by balls $\{B_i\}$, where $B_i = B(x_i, r_i)$ and $\sum_i r_i^{n-p+(p-1)\alpha} < \epsilon$. We may additionally assume that $8r_i < \min\{1, \mathrm{dist}(K, \partial\Omega)\}$. Letting $G = \bigcup 2B_i$, we note that $|G| < C\epsilon$ since $n - p + (p-1)\alpha \le n$. We next choose $\psi_i \in C^\infty_0$ such that $\psi_i \equiv 1$ on B_i, $\psi_i \equiv 0$ on $(2B_i)^c$, and $\nabla \psi_i \lesssim r_i^{-1}$. Let $\psi = \min\{1, \sum_{i=1}^\infty \psi_i\}$. Since $\phi(1-\psi)$ is Lipschitz and is compactly supported in

$\Omega \setminus F$, we have $\phi(1-\psi) \in W_0^{1,p}(\Omega \setminus F)$, and so $\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla(\phi(1-\psi))\, dx = 0$. Next

$$\begin{aligned}\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla(\phi\psi)\,dx &= \int_\Omega \mathcal{A}(x,\nabla u)\cdot\psi\nabla\phi\,dx + \int_\Omega \mathcal{A}(x,\nabla u)\cdot\phi\nabla\psi\,dx \\ &= I + II.\end{aligned}$$

Now ψ is supported on G, and both $\nabla\phi$ and ψ are bounded. Therefore

$$\begin{aligned}|I| &\lesssim |G|^{1/p}\left(\int_{G\cap K} |\mathcal{A}(x,\nabla u)|^{p/(p-1)}\,dx\right)^{(p-1)/p} \\ &\lesssim |G|^{1/p}\left(\int_{G\cap K} |\nabla u|^p\right)^{(p-1)/p}.\end{aligned}$$

Lemma 4 implies that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, and so $|I| \lesssim \epsilon^{1/p}$.

As for II, we first note that

$$|II| \lesssim {\sum_i}' \int_{2B_i} |\nabla u|^{p-1}|\nabla\psi_i|,$$

where ${\sum_i}'$ indicates that we sum over only those values of i for which $\delta_K(x_i) \le 2r_i$ (other terms give no contribution). Since also $8r_i < \operatorname{dist}(K, \partial\Omega)$, it follows that $6B_i \subset \Omega$, and so we may use Lemma 5. We now use the bound on $\nabla\psi_i$, Hölder's inequality, Lemma 5, and the decay estimate for u (in that order), to get

$$\begin{aligned}|II| &\lesssim {\sum_i}' r_i^{-1+n} ⨍_{2B_i} |\nabla u|^{p-1} \\ &\lesssim {\sum_i}' r_i^{-1+n}\left(⨍_{2B_i} |\nabla u|^p\right)^{(p-1)/p} \\ &\lesssim {\sum_i}' r_i^{n-p}\left(⨍_{4B_i} |u-u(x_i)|^p\right)^{(p-1)/p} \\ &\lesssim {\sum_i}' r_i^{n-p+(p-1)\alpha} < \epsilon,\end{aligned}$$

as required. □

We now consider more general decay rates for u near F. We omit the proof of this more general result, as it requires only straightforward modifications to the proof of Theorem 1. Corollary 3 can be generalized in an analogous fashion.

Theorem 6. *Let $h : [0,1) \to [0,\infty)$ be a non-decreasing function satisfying $h(0) = 0$, the doubling condition $h(t) \le Ch(t/2)$, and the growth condition $t^n \le Ch(t)$ (both for some constant C and all $0 < t < 1$). Let $g(t) \equiv [t^{p-n}h(t)]^{1/(p-1)}$, and suppose that $\lim_{t\to 0^+} g(t) = 0$. Suppose also that u is $\mathcal{A}$-harmonic (with* *parameter $p > 1$) in $\Omega \setminus F$, that $\mathcal{H}^h(\partial F) = 0$, and that $|u(x)| \le g(\delta_F(x))$ for all $0 < \delta_F(x) < \min\{1, \delta_{\partial\Omega}(x)\}/2$. If we extend u to be zero on F, then u is $\mathcal{A}$-harmonic in Ω.*

Finally note that, if $|u(x)|/\delta_F^\alpha(x)$ tends to zero as $\delta_F(x)$ tends to zero, then the assumption $\mathcal{H}^{n-p+(p-1)\alpha}(\partial F) = 0$ in Theorem 1 can be replaced by the weaker assumption that this quantity is merely finite, as is clear from the proof; similar comments applies to the other results above.

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