On generalized games in H-spaces

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Abstract. We show that a recent existence result for the Nash equilibria of generalized games with strategy sets in H-spaces is false. We prove that it becomes true if we assume, in addition, that the feasible set of the game (the set of all feasible multistrategies) is closed.

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1. Introduction

Let E be a topological space, and let $\mathcal{F}_E$ be the family of all finite subsets of E. Let $\{\Gamma_A\}_{A\in\mathcal{F}_E}$ be a given family of nonempty contractible subsets of E, such that $\Gamma_A \subseteq \Gamma_B$ whenever $A \subseteq B$. The couple $(E, \{\Gamma_A\}_{A\in\mathcal{F}_E})$ is called an H-space ([3], [4], [5]). A set $K \subseteq E$ is said H-*convex* if for every $A \in \mathcal{F}_E$, with $A \subseteq K$, one has $\Gamma_A \subseteq K$; K is said H-*compact* if for each $A \in \mathcal{F}_E$ there is a compact H-convex set $D \subseteq K$ such that $K \cup A \subseteq D$. Any Hausdorff topological vector space T is an H-space with $\Gamma_A = \operatorname{conv}(A)$; moreover, any convex subset of T is H-convex and any nonempty compact convex subset of T is H-compact (see [3]). If $\{(E_i, \{\Gamma_{B_i}\}_{B_i\in\mathcal{F}_{E_i}})\}_{i\in I}$ $(I = \{1, \dots, N\})$ is a family of H-spaces, then the product $E := \prod_{i\in I} E_i$ can be regarded as an H-space in the following way: for each $A \in \mathcal{F}_E$, let $\Gamma_A = \prod_{i\in I} \Gamma_{A_i}$, where A_i denotes the image of A in the projection of E onto E_i. If for each $i \in I$ the set $D_i \subseteq E_i$ is H-convex [resp., H-compact], then the set $\prod_{i\in I} D_i$ is H-convex [resp., H-compact] (see [8]). We refer to [3], [4], [5] for more details on H-spaces.

Now, let $I = \{1, \dots, N\}$ be a set of N players. For each player $i \in I$, let X_i be a nonempty set (the strategy space), and let $X := \prod_{i\in I} X_i$, $X_{-i} := \prod_{j\in I, j\neq i} X_j$. The elements of X are called *multistrategies.* For each $i \in I$, let $F_i : X_{-i} \to 2^{X_i}$ a multifunction (the constraint correspondence), $u_i : X \to \mathbb{R}$ a single-valued function (the utility function). The family of triples $(X_i, F_i, u_i)_{i\in I}$ is called a *generalized game* (or an *abstract economy*). If $x \in X$ is a given multistrategy, we denote by $x_i \in X_i$ the i-th component of x, and by $x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \in X_{-i}$ the multistrategy x without its i-th component x_i.

In this model, the multi-valued mappings F_i represent the constraints imposed on each player from the other players' action. That is, for each fixed $i \in I$, when the other players choose their strategy $x_{-i} \in X_{-i}$, the player i can choose his strategy only in the set $F_i(x_{-i})$, and not in the whole set X_i. A multistrategy $x \in X$ is said to be *feasible* if $x_{-i} \in F_i(x_{-i})$ for each $i \in I$. A multistrategy $\hat{x} = (\hat{x}_1, \dots, \hat{x}_N) \in X$ is called a *Nash equilibrium* for the generalized game $(X_i, F_i, u_i)_{i\in I}$ if for each $i \in I$ one has

$$\hat{x}_i \in F_i(\hat{x}_{-i}) \quad \text{and} \quad u_i(\hat{x}) \geq u_i(\hat{x}_{-i}, v_i) \quad \text{for all} \quad v_i \in F_i(\hat{x}_{-i}),$$

where $(\hat{x}_{-i}, v_i)$ denotes the multistrategy obtained from $\hat{x}$ by replacing the i-th component $\hat{x}_i$ by v_i. Therefore, the multistrategy $\hat{x}$ is a Nash equilibrium for the game if it is feasible and for each $i \in I$ the strategy $\hat{x}_i$ maximizes the function $u_i(\hat{x}_{-i}, \cdot\,)$ over the set $F_i(\hat{x}_{-i})$. In the sequel, we shall denote by Σ the set of all feasible multistrategies (the *feasible set*). That is, we put

$$\Sigma = \big\{x \in X : x_i \in F_i(x_{-i}) \;\text{ for all }\; i \in I\big\}. \tag{1}$$

The reader is referred, for instance, to [1], [7], [10] and to the references therein for more details on generalized games. We also refer to [2], [9] for the basic facts about multifunctions.

Recently, in [8], the following fixed point theorem for multi-valued mappings in H-spaces has been established.

Theorem 1.1 ([8, Corollary 1]). *Let $(E, \{\Gamma_A\}_{A\in\mathcal{F}_E})$ be an H-space, $X \subseteq E$ a nonempty closed H-convex set, $S : X \to 2^X$ a multifunction, $D \subseteq X$ an H-compact set. Assume that:*

(i) *S has open lower sections; that is, for each $y \in X$ the set $S^-(y) = \{x \in X : y \in S(x)\}$ is open in the relative topology of X;*
(ii) *for each $x \in X$, $S(x)$ is a nonempty H-convex set;*
(iii) *$S(x) \cap D \neq \emptyset$ for each $x \in X \setminus D$.*

Then there exists $x^* \in X$ such that $x^* \in S(x^*)$.

As an application of Theorem 1.1, an existence result for the Nash equilibria of generalized games in H-spaces has been derived in [8, Theorem 3]. In that result, a special concavity condition on the utilities u_i was assumed ([8, Definition 3]), which is stronger than other H-concavity conditions used in the theory of H-spaces (see Definition 1 and Proposition 2 of [4]). As a matter of fact, in the proof of Theorem 3 of [8] such condition was used only to ensure that for each $x \in X$ the level set

$$\Big\{y = (y_1, \dots, y_N) \in \prod_{i\in I} F_i(x_{-i}) : \sum_{i\in I}(u_i(x) - u_i(x_{-i}, y_i)) < 0\Big\} \tag{2}$$

is H-convex. Therefore, we can paraphrase that result as follows.

Assertion 1.2 ([8, Theorem 3]). *Let (X_i, F_i, u_i) be a generalized game. For each $i \in I$, assume that:*

(i) *X_i is a closed H-convex subset of the H-space $(E_i, \{\Gamma_{B_i}\}_{B_i\in\mathcal{F}_{E_i}})$;*
(ii) *F_i has open lower sections and nonempty H-convex values;*
(iii) *u_i is continuous on X.*

Moreover, assume that:

(iv) *for each $x \in X$ the level set* (2) *is H-convex;*

(v) *there exist H-compact sets $D_i \subseteq X_i$ such that: for each $x = (x_1, \dots, x_N) \in X \setminus D$ ($D := \prod_{i \in I} D_i$), there exists $y = (y_1, \dots, y_N) \in D$ such that for each $i \in I$ one has*

$$y_i \in F_i(x_{-i}) \quad \text{and} \quad u_i(x) < u_i(x_{-i}, y_i).$$

Then there exists a Nash equilibrium for the generalized game.

Unfortunately, the proof of the latter result given in [8] is not correct. In this note we first show, by means of a simple counter-example, that Assertion 1.2 is not true in general. Then we show that it becomes true if we assume, in addition, that the feasible set Σ is closed. We also discuss in detail the mistake in the original proof of Assertion 1.2.

2. The counter-example and the result

The following counter-example shows that Theorem 1.2 above is false.

Example 2.1. Let $N = 2$, $X_1 = X_2 = [0, 1]$, $u_1(x_1, x_2) = u_2(x_1, x_2) = x_1 + x_2$, and let

$$F_1(x_2) =]0, 1[, \quad F_2(x_1) =]0, 1[\text{ for all } (x_1, x_2) \in [0, 1] \times [0, 1].$$

It is immediate to check that all the assumptions of Assertion 1.2 are satisfied (in particular, one can take $D_1 = D_2 = [0, 1]$). However, the conclusion of Assertion 1.2 does not hold. In fact, choose any $x = (x_1, x_2) \in \Sigma =]0, 1[\times]0, 1[$. If we pick any $v_1 \in]x_1, 1[\subseteq F_1(x_2)$ we get $u_1(v_1, x_2) > u_1(x_1, x_2)$. Therefore, the point (x_1, x_2) is not an equilibrium.

Consequently, Assertion 1.2 is not correct. Before analyzing the mistake in the original proof of Assertion 1.2, let us observe the following fact. First, define a multifunction $F : X \to 2^X$ by putting

$$F(x) := \prod_{i \in I} F_i(x_{-i}). \tag{3}$$

It is immediate to observe that a multistrategy $x \in X$ is feasible if and only if x is a fixed point for F. That is, $x \in \Sigma$ if and only if one has $x \in F(x)$. It is a well-known fact in game theory that a multistrategy $x \in X$ is a Nash equilibrium for the generalized game $(X_i, F_i, u_i)_{i \in I}$ if and only if one has

$$x \in F(x) \text{ and } \sum_{i \in I} u_i(x) \geq \sum_{i \in I} u_i(x_{-i}, y_i) \text{ for all } y \in F(x)$$

(a direct proof of this fact is also easy). Now we come to Assertion 1.2. The gap in the original proof (see [8, proof of Theorem 3]) resides in the fact that the technique used is not sufficient, in general, to ensure the existence of an equilibrium. In fact, the original proof of Theorem 1.2 consists of the two following steps.

First step. It is shown that F has a fixed point $x_0 \in F(x_0)$.

Second step. The conclusion is proved by assuming that for each $x \in X$ the set $h(x) = \{y \in F(x) : \sum_{i \in I} u_i(x) < \sum_{i \in I} u_i(x_{-i}, y_i)\}$ is nonempty and getting a contradiction. But the gap is exactly here, since the fact $h(x) = \emptyset$ does not imply, in general, that x is an equilibrium. In fact, we do not necessarily have $x \in F(x)$. In Example 2.1, for instance, we have $h((1,1)) = \emptyset$, but $(1,1)$ is not an equilibrium since $(1,1) \notin \Sigma$.

We point out that in Example 2.1 the feasible set Σ was not closed. As announced in the Introduction, we now prove that Assertion 1.2 becomes true if we assume, in addition, that the feasible set Σ is closed. The following is our result.

Theorem 2.2. *Let (X_i, F_i, u_i) be a generalized game. For each $i \in I$, assume that:*

(i) *X_i is a closed H-convex subset of the H-space $(E_i, \{\Gamma_{B_i}\})$;*
(ii) *F_i has open lower sections and nonempty H-convex values;*
(iii) *u_i is continuous on X.*

Moreover, assume that:

(iv) *for each $x \in X$ the level set (2) is H-convex;*
(v) *the feasible set Σ is closed;*
(vi) *there exists a nonempty H-compact set $D \subseteq X$ such that:*
  (a) *for each $x \in X \setminus (D \cup \Sigma)$ one has $F(x) \cap D \neq \emptyset$;*
  (b) *for each $x \in (X \setminus D) \cap \Sigma$, there exists $y \in F(x) \cap D$ such that*

$$\sum_{i \in I} (u_i(x) - u_i(x_{-i}, y_i)) < 0.$$

Then there exists a Nash equilibrium for the game.

Remark. It is easy to check that assumption (vi) of Theorem 2.2 is more general than the corresponding assumption (v) of Assertion 1.2. We point out that it is automatically satisfied if each set X_i is H-compact, by taking $D = X$.

Proof of Theorem 2.2: First, we observe that the multifunction $F : X \to 2^X$, defined as in (3), has open lower sections. To see this, let $y \in X$ be fixed, and let us show that the set $F^-(y) = \{x \in X : y \in F(x)\}$ is open in the relative topology of X. By assumption (ii), for each $i \in I$ the set $F_i^-(y_i) = \{x_{-i} \in X_{-i} : y_i \in F_i(x_i)\}$ is open in the relative topology of X_{-i}. For each $i \in I$, let $h_i : X \to X_{-i}$ be defined by $h_i(x) = x_{-i}$. Therefore, we have

$$F^-(y) = \bigcap_{i \in I} h_i^{-1}(F_i^-(y_i)).$$

Since each h_i is continuous, the set $F^-(y)$ is open in X, as claimed. Therefore, F has open lower sections. Moreover, by (ii), each set $F(x)$ is nonempty and H-convex. Now, consider the multifunctions $S : X \to 2^X$ and $\Psi : X \to 2^X$ defined by putting, for each $x \in X$,

$$S(x) = \Big\{y \in X : \sum_{i\in I}(u_i(x) - u_i(x_{-i}, y_i)) < 0\Big\},$$
$$\Psi(x) = S(x) \cap F(x).$$

By assumption (iv), each set $S(x)$ is H-convex, hence each set $\Psi(x)$ is H-convex. Moreover we claim that Ψ has open lower sections. To see this, let $z \in X$ be fixed. Since by (iii) the function

$$g_z : x \in X \to \sum_{i\in I}(u_i(x) - u_i(x_{-i}, z_i)) \in \mathbb{R}$$

is continuous, the set $S^-(z) = g_z^{-1}(\,]-\infty, 0[\,)$ is open in the relative topology of X. Since F has open lower sections (hence $F^-(z)$ is open in X) and $\Psi^-(z) = S^-(z) \cap F^-(z)$, we have that $\Psi^-(z)$ is open in X. Consequently, Ψ has open lower sections, as claimed. Now, let $\Phi : X \to 2^X$ be the multifunction defined by

$$\Phi(x) = \begin{cases} F(x) & \text{if } \ x \in X \setminus \Sigma \\ \Psi(x) & \text{if } \ x \in \Sigma. \end{cases}$$

Since F and Ψ have open lower sections, by assumption (v) and Proposition 4.1 of [6] it follows that Φ has open lower sections (a direct proof of this fact is not difficult). We claim that there exists $\hat{x} \in X$ such that $\Phi(\hat{x}) = \emptyset$. On the contrary, assume that $\Phi(x) \neq \emptyset$ for all $x \in X$. Therefore, Φ has nonempty H-convex values. Moreover, by assumption (vi), it is not difficult to check that one has $\Phi(x) \cap D \neq \emptyset$ for all $x \in X \setminus D$. Consequently, by Theorem 1.1, there exists $x^* \in X$ such that $x^* \in \Phi(x^*)$. Since $\Phi(x^*) \subseteq F(x^*)$, we get $x^* \in \Sigma$, hence $x^* \in \Psi(x^*)$, which is a contradiction by the definition of Ψ. Such an absurd implies that there exists $\hat{x} \in X$ such that $\Phi(\hat{x}) = \emptyset$. Since $F(x) \neq \emptyset$ for all $x \in X$, we get $\hat{x} \in \Sigma$ and $\Psi(\hat{x}) = \emptyset$, hence we have

$$\hat{x} \in F(\hat{x}) \ \text{ and } \ \sum_{i\in I} u_i(\hat{x}) \geq \sum_{i\in I} u_i(\hat{x}_{-i}, y_i) \ \text{ for all } \ y \in F(x).$$

Consequently, $\hat{x}$ is a generalized Nash equilibrium for the game. The proof is complete. □

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