Relations between weighted Orlicz and BMO_ϕ spaces through fractional integrals

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Abstract. We characterize the class of weights, invariant under dilations, for which a modified fractional integral operator I_α maps weak weighted Orlicz$-\phi$ spaces into appropriate weighted versions of the spaces BMO_ψ, where $\psi(t) = t^{\alpha/n}\phi^{-1}(1/t)$. This generalizes known results about boundedness of I_α from weak L^p into Lipschitz spaces for $p > n/\alpha$ and from weak $L^{n/\alpha}$ into BMO. It turns out that the class of weights corresponding to I_α acting on weak$-L_\phi$ for ϕ of lower type equal or greater than n/α, is the same as the one solving the problem for weak$-L^p$ with p the lower index of Orlicz-Maligranda of ϕ, namely $\omega^{p'}$ belongs to the A_1 class of Muckenhoupt.

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1. Introduction and statement of results

In this work we are going to deal with non-negative functions ϕ defined and increasing on $[0,\infty)$ such that $\lim_{t\to 0^+}\phi(t) = 0$ and $\lim_{t\to\infty}\phi(t) = \infty$. In addition, we shall also assume that the following conditions are satisfied.

(1.1) *ϕ is of lower type p, $p > 1$, that is there exists a constant C such that*

$$\phi(st) \le C s^p \phi(t)$$

holds for every $s \in [0,1]$ and every $t \ge 0$.

(1.2) *ϕ is of upper type q, that is there exists a constant C such that*

$$\phi(st) \le C s^q \phi(t)$$

holds for every $s \ge 1$ and every $t \ge 0$.

In connection with the above conditions, we introduce the notion of lower and upper indices.

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1.3 Definition. *Let ϕ be a function as above. Set*

$$h(s) = \sup_{t>0} \frac{\phi(st)}{\phi(t)}\,,$$

for $s > 0$, and define the lower index of ϕ by

$$i(\phi) = \lim{}_{s\to 0^+} \frac{\log h(s)}{\log s} = \sup_{0<s<1} \frac{\log h(s)}{\log s}$$

and the upper index of ϕ by

$$I(\phi) = \lim{}_{s\to\infty} \frac{\log h(s)}{\log s} = \inf_{1<s<\infty} \frac{\log h(s)}{\log s}\,.$$

The existence of the above limits follows from the theory of submultiplicative functions and the details can be found for instance in [B] or in [GP]. Clearly, for any function ϕ we have $i(\phi) \le I(\phi)$. Also, under our assumptions on ϕ, both indices are finite and bigger than one.

It is easy to see that ϕ is of lower type $i(\phi) - \epsilon$, and of upper type $I(\phi) + \epsilon$ for every $\epsilon > 0$, where the constant appearing in (1.1) and (1.2) may depend on ϵ. We also mention that $i(\phi)$ and $I(\phi)$ may be viewed as the supremum of the lower types of ϕ and the infimum of upper types, respectively. For these reasons the assumption that ϕ is of lower type greater than one is equivalent to say that $i(\phi) > 1$. A similar statement is true for the upper index.

Given ϕ, the complementary function (with respect to ϕ) is defined by

$$\tilde{\phi}(s) = \sup_{t>0}(st - \phi(t))$$

for $s \ge 0$.

It is known (see for example [KK]) that $\tilde{\phi}$ satisfies similar properties to ϕ. In particular,

$$i(\tilde{\phi}) = (I(\phi))' \quad\text{and}\quad I(\tilde{\phi}) = (i(\phi))', \tag{1.4}$$

where r' means $r/(r-1)$. Moreover, it can be proved that there exist two constants C_1 and C_2 such that

$$C_1 s \le \phi^{-1}(s)\,\tilde{\phi}^{-1}(s) \le C_2 s \tag{1.5}$$

for every $s > 0$.

Let ϕ be a function with $1 < i(\phi) \le I(\phi) < \infty$. We remind that under even more general conditions on ϕ (see for example [RR]) the Orlicz space L_ϕ is defined as the class of measurable functions $f : \mathbf{R}^n \to \mathbf{R}$ such that

$$\int_{\mathbf{R}^n} \phi(|f(x)|)\,dx < \infty.$$

In this class we introduce the following analogue to the Luxemburg norm

$$\|f\|_\phi = \inf\left\{\lambda > 0 : \int_{\mathbf{R}^n} \phi(|f(x)|/\lambda) \le 1\right\}.$$

Let us note that $\|\cdot\|_\phi$ is not a norm but in view of the properties of ϕ, it can be shown that it is equivalent to a norm. Moreover, the Hölder type inequality

$$\int f(x)\, g(x)\, dx \le C\, \|f\|_\phi\, \|g\|_{\tilde{\phi}}$$

holds for every $f \in L_\phi$ and every $g \in L_{\tilde{\phi}}$.

We also introduce a version of weak-Orlicz spaces L^*_ϕ as the class of measurable functions f satisfying

$$\sup_{t>0} \phi(t)\, |\{x \in \mathbf{R}^n : |f(x)| > t\}| < \infty.$$

For these functions, we set

$$[f]_\phi = \inf\left\{\lambda > 0 : \sup_{t>0} \phi(t) \left|\left\{x \in \mathbf{R}^n : \left|\frac{f(x)}{\lambda}\right| > t\right\}\right| \le 1\right\}.$$

As in the strong case, $[\cdot]_\phi$ is equivalent to a norm.

We also consider families of spaces $\{L_{\phi_t}\}_{t>0}$ and $\{L^*_{\phi_t}\}$, where

$$\phi_t(s) = \frac{\phi(s)}{t} \quad \text{for every} \quad s > 0. \tag{1.6}$$

It is not difficult to prove that

$$i(\phi_t) = i(\phi) \qquad \text{and} \qquad I(\phi_t) = I(\phi)$$

for every $t > 0$. Moreover, it is clear that lower and upper types of ϕ_t are those of ϕ with constants independent of t. On the other hand, using again the types of ϕ, it is easy to check that $\|\cdot\|_{\phi_t}$ and $[\cdot]_{\phi_t}$ are equivalent to $\|\cdot\|_\phi$ and $[\cdot]_\phi$ respectively, but this time the constants would depend upon t. To deal with the dual families $\{L_{\widetilde{\phi_t}}\}_{t>0}$ and $\{L^*_{\widetilde{\phi_t}}\}_{t>0}$, we note that there exist constants C_1 and C_2 such that

$$C_1 \frac{\tilde{\phi}(ts)}{t} \le \widetilde{\phi_t}(s) \le C_2 \frac{\tilde{\phi}(ts)}{t} \tag{1.7}$$

for every $s > 0$ and $t > 0$. This relationship follows easily from (1.5) and (1.6).

In what follows a non-negative function ω defined in $\mathbf{R}^n$ will be called a weight if it is locally integrable. We will denote by $|E|$ the Lebesgue measure of E and by $\omega(E) = \int_E \omega(x)\,dx$. Given a ball $B = B(x_B, R)$, and $\theta > 0$, θB and B^θ will mean the balls $B(x_B, \theta R)$ and $B(x_B, R^\theta)$, respectively. Also, for a locally integrable function f and a ball B in $\mathbf{R}^n$, $m_B f$ stands for the usual average

$$\frac{1}{|B|}\int_B f(x)\,dx.$$

For a given weight ω we define the weighted Orlicz space $L_{\phi,\omega}$ as the class of functions f such that $\|f\|_{\phi,\omega} \equiv \|f/\omega\|_\phi$ is finite. Similarly, we shall say that f belongs to $L^*_{\phi,\omega}$ if $[f]_{\phi,\omega} \equiv [f/\omega]_\phi$ is finite. Denoting by $\delta_t f(x) = f(tx)$, $t > 0$, it is not too hard to see that

$$\|f\|_{\phi,\delta_\epsilon\omega} = \left\|\delta_{1/\epsilon} f\right\|_{\phi_{\epsilon^n},\omega}$$

and that

$$[f]_{\phi,\delta_\epsilon\omega} = \left[\delta_{1/\epsilon} f\right]_{\phi_{\epsilon^n},\omega}.$$

We now introduce the classes of weights $C(\phi)$, which will be used throughout this work.

1.8 Definition. *Given a function ϕ, we say that $\omega \in C(\phi)$ if there exists a constant C such that*

$$\phi^{-1}(1/|B|)\,\|\chi_B \delta_t\omega\|_\phi \le C \inf{}_B \delta_t\omega$$

for every ball $B \subset \mathbf{R}^n$ and every $t > 0$.

Notice that these classes have been defined to make them invariant under dilations. That means that if $\omega \in C(\phi)$ then $\delta_t\omega \in C(\phi)$ for all $t > 0$ with a constant independent of t. Furthermore, in Section 2, we shall study the connection between $C(\phi)$ and the A_1 class of Muckenhoupt, that is those weights satisfying

$$\frac{\omega(B)}{|B|} \le C \inf{}_B\, \omega$$

for every ball $B \in \mathbf{R}^n$.

For $0 < \alpha < n$, the fractional integral operator of order α is defined by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} f(y)\,|x-y|^{\alpha-n}\,dy \tag{1.9}$$

whenever this integral is finite almost everywhere. Since the functions f we are interested in may not have the necessary decay at infinity to make the above integral convergent, we will use a modified version of this operator which will be also denoted by I_α. We set

$$I_\alpha f(x) = \int_{\mathbf{R}^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1-\chi_{B_{(0,1)}}(y)}{|y|^{n-\alpha}} \right) f(y)\, dy,$$

where $\chi_{B_{(0,1)}}(y)$ is the characteristic function of the unit ball. We point out that for functions good enough to make the integral in (1.9) convergent, the modified version is also finite and both agree upon a constant, but this means equality as functions in Lipschitz type spaces. This new operator is well defined for functions belonging to weighted Orlicz spaces $L^*_{\phi,\omega}$ as long as the upper index q satisfies $q < n/(\alpha-1)^+$. With this notation we mean $q < \infty$ if $\alpha \leq 1$ and $q < n/(\alpha-1)$ otherwise. This result is contained in the following theorem.

1.11 Theorem. *Let $0 < \alpha < n$ and let ϕ be a non-decreasing function with lower index $p > 1$ and upper index $q < n/(\alpha-1)^+$. Then the following conditions are equivalent.*

(1.12) *The operator I_α is well defined on $L^*_{\phi,\omega}$ and there exists a constant C independent of t such that*

$$\sup_B \frac{\left\|\chi_B \omega^{-1}\right\|_\infty}{|B|^{1+\alpha/n}\phi_t^{-1}(1/|B|)} \int_B |I_\alpha f(x) - m_B(I_\alpha f)|\, dx \leq C\, [f/\omega]_{\phi_t}$$

for every $t > 0$, where the sup *is taken over all the balls $B \subset \mathbf{R}^n$.*

(1.13) *The operator I_α is well defined on $L^*_{\phi,\omega}$ and there exists a constant C independent of t such that*

$$\sup_B \frac{\left\|\chi_B \delta_t \omega^{-1}\right\|_\infty}{\psi(|B|)\, |B|} \int_B |I_\alpha f(x) - m_B(I_\alpha f)|\, dx \leq C\, [f/(\delta_t\omega)]_{\phi}$$

for every $t > 0$, where $\psi(s) = s^{\alpha/n}\phi^{-1}(1/s)$ and the sup *is taken over all the balls B in $\mathbf{R}^n$.*

(1.14) *The weight ω belongs to $C(\tilde{\phi})$.*

(1.15) *The weight $\omega^{p'}$ belongs to A_1.*

1.16 Remark. Let us note that if in (1.12) and (1.13) the weak norms $[\cdot]_{\phi_t}$ and $[\cdot]_\phi$ are replaced by the strong norms $\|\cdot\|_{\phi_t}$ and $\|\cdot\|_\phi$ respectively, then the corresponding statements are equivalent for each $t > 0$.

1.17 Remark. We would like to point out that for $\omega \equiv 1$ and $\phi(t) = t^p$, Theorem 1.11 gives the classical results:

$$I_\alpha : \quad \text{weak} \ \ L^{n/\alpha} \to BMO$$

and

$$I_\alpha : \quad \text{weak} \ \ L^p \to \Lambda(\beta),$$

$\beta = \alpha - n/p$, $n/\alpha < p < n/(\alpha-1)^+$, where $\Lambda(\beta)$ means the space of Lipschitz functions of order β.

For $\omega \equiv 1$ and general ϕ, the theorem recovers the results about the boundedness of I_α on weak Orlicz spaces proved by the authors in [HSV1], and for $\phi(t) = t^{n/\alpha}$ and general ω, the ones obtained by B. Muckenhoupt an R. Wheeden in [MW] (see Theorems 7 and 8). Finally for general ω and $\phi(t) = t^p$ it gives Theorem 2.5 of [HSV2] but for a slightly different class of weights.

1.18 Remark. We observe that when $\alpha \le 1$ the operator I_α acts on any weighted Orlicz space with $1 < p \le q < \infty$. On the other hand, if $\alpha > 1$ we restrict the function ϕ to have $q < \frac{n}{\alpha-1}$. This range could be extended modifying the definition of I_α and the left hand side of (1.13) as to involve higher order differences.

In the next section we give some properties of the class $C(\phi)$ and the proof of Theorem 1.1.

2. Classes $C(\phi)$ and proof of Theorem 1.11

2.1 Proposition. *The following statements are equivalent.*

(2.2) *ω belongs to $C(\phi)$.*

(2.3) *There exists a constant C such that*

$$\phi_t^{-1}(1/|B|) \ \|\chi_B \, \omega\|_{\phi_t} \le C \inf_{x\in B} \omega(x)$$

for every ball $B \subset \mathbf{R}^n$ and every $t > 0$.

(2.4) *$\phi(t\omega)$ belongs to A_1 for every $t > 0$, with constant independent of t, that is there exists a constant C such that*

$$\frac{1}{|B|} \int_B \phi(t\omega(x)) \, dx \le C \inf_{x\in B} \phi(t\omega(x))$$

for every ball $B \subset \mathbf{R}^n$ and every $t > 0$.

Proof: Let us assume (2.2) holds. Writing down this inequality for $t^{1/n}$ instead of t and $t^{-1/n}B$ instead of B we get

$$\int_B \phi\left(\frac{\omega(x)\, \phi^{-1}(t/|B|)}{C \ \inf_B \ \omega}\right) \frac{dx}{t} \le 1,$$

which proves (2.3). Arguing in a similar way, we obtain that (2.3) implies (2.2). Now we prove that (2.2) follows from (2.4). In fact, taking $t = \phi^{-1}(\epsilon^n/|B|)/\inf_{x\in B}\omega(x)$ with $\epsilon > 0$ in (2.4), we get

$$\frac{1}{|B|}\int_B \phi\left(\frac{\omega(x)}{\inf_B \omega}\phi^{-1}\left(\frac{\epsilon^n}{|B|}\right)\right)\,dx \le \frac{C\,\epsilon^n}{|B|}\,.$$

Then, an obvious change of variable yields to (2.2). Proceeding similarly, (2.4) can be obtained from (2.2). □

2.5 Corollary. *Let ω be a weight in $C(\phi)$. Then there exists $\epsilon > 0$ such that $\omega \in C(\phi^\beta)$ for every β in $(0, 1+\epsilon)$.*

Proof: From Proposition 2.1 and the fact that A_1-weight satisfies a reverse-Hölder inequality there exists $\epsilon > 0$ such that, for any β, $1 \le \beta < 1+\epsilon$, we have

$$\frac{1}{|B|}\int_B \phi^\beta(t\omega(x))\,dx \le C\inf_{x\in B}\phi^\beta(t\omega(x))$$

for every $t > 0$. Then, by using Proposition 2.1 again, we get that $\omega \in C(\phi^\beta)$ for every β in $[1, 1+\epsilon)$. On the other hand, if β belongs to $(0,1)$, by Hölder inequality, we obtain that

$$\int_B \phi^\beta\left(\frac{\delta_t\,\omega(x)}{C\ \inf_B(\delta_t\omega)}(\phi^\beta)^{-1}(1/|B|)\right)dx$$
$$\le |B|^{1-\beta}\left(\int_B \phi\left(\frac{\delta_t\omega(x)}{\inf_B(\delta_t\omega)}\phi^{-1}\left(\frac{1}{|B|^{1/\beta}}\right)\right)dx\right)^\beta.$$

Replacing t by $\epsilon|B|^{(1-\beta)/n\beta}$ and changing variables, we get that the last expression is bounded by

$$\left(\int_{B^{1/\beta}} \phi\left(\frac{\delta_\epsilon\,\omega(x)}{C\ \inf_{B^{1/\beta}}(\delta_\epsilon\omega)}\phi^{-1}\left(\frac{1}{|B|^{1/\beta}}\right)\right)dx\right)^\beta.$$

Then, since $\omega \in C(\phi)$, we have that $\omega \in C(\phi^\beta)$ for every β in $(0,1)$. □

Now we prove two technical lemmas that will be used in the proof of our main result.

2.6 Lemma. *Let ω be a weight in $C(\tilde{\phi})$, where ϕ is a function with lower type $p > 1$ and upper type q. Let f be a function in $L^*_{\phi,\omega}$. Then, there exists a constant C, independent of f, such that*

$$\int_B |f(x)|\,dx \le C\,|B|\,\phi_\epsilon^{-1}(1/|B|)\,\inf_B \omega\,[f]_{\phi_\epsilon,\omega}$$

holds for every ball B in $\mathbf{R}^n$ and for every $\epsilon > 0$.

Proof: Let B be a ball in $\mathbf{R}^n$. From Corollary 2.5 there exists $r > 1$ such that $\omega \in C(\widetilde{\phi}^{\,r})$. Then, by Hölder inequality, we have

$$\begin{aligned}\int_B |f(x)|\,dx &\le \|\chi_B\,\omega\|_{\widetilde{\phi_t}^{\,r}}\;\|f/\omega\|_{\widetilde{\widetilde{\phi_t}^{\,r}}}\\ &\le C\frac{\inf_B \omega}{(\widetilde{\phi_t}^{\,r})^{-1}(1/|B|)}\;\|f/\omega\|_{\widetilde{\widetilde{\phi_t}^{\,r}}}\end{aligned}\tag{2.7}$$

for every $t > 0$. Now, we estimate the norm on the right hand side of (2.7). Denoting $\psi = \widetilde{\widetilde{\phi_t}^{\,r}}$ and $g = f/\omega$, for some $\lambda > 0$ to be determined, we get

$$\begin{aligned}\int_B \psi\left(\frac{g(x)}{\lambda}\right)dx &= \int_0^\infty \left|\{x \in B : |g(x)| > \psi^{-1}(s)\lambda\}\right|\,ds\\ &= \left(\int_0^{(2|B|)^{-1}} + \int_{(2|B|)^{-1}}^\infty\right)\left|\{x \in B : |g(x)| > \psi^{-1}(s)\lambda\}\right|\,ds\\ &\le \frac{1}{2} + I,\end{aligned}\tag{2.8}$$

where I is the integral over $[(2|B|)^{-1}, \infty)$. Since $g \in L^*_{\phi_\epsilon,\omega}$ for any $\epsilon > 0$, we have

$$\begin{aligned}I &\le \int_{(2|B|)^{-1}}^\infty \frac{ds}{\phi_\epsilon(\psi^{-1}(s)\lambda/[g]_{\phi_\epsilon,\omega})}\\ &= \frac{1}{2|B|}\int_1^\infty \frac{ds}{\phi_\epsilon(\psi^{-1}(s(2\,|B|)^{-1})\lambda/[g]_{\phi_\epsilon,\omega})}.\end{aligned}$$

Notice that $\psi^{-1}(s) \approx s^{1/r'}\phi_t^{-1}(s^{1/r})$ with $r' = r/(r-1)$. Therefore, choosing $t = \epsilon/\,|B|^{1/r'}$ and using the upper type of ϕ, we obtain

$$I \le \frac{1}{2\,|B|}\int_1^\infty \frac{ds}{\phi_\epsilon(C\phi_\epsilon^{-1}(s^{1/r}/|B|)\;s^{1/r'}\lambda/(|B|^{1/r'}[g]_{\phi_\epsilon,\omega}))}\,.$$

Then, taking $\lambda = H|B|^{1/r'}[g]_{\phi_\epsilon,\omega}$ for some constant H to be determined and using that ϕ is of lower type $p > 1$, we get that

$$
\begin{aligned}
I &\leq \frac{1}{2\,|B|} \int_1^\infty \frac{ds}{\phi_\epsilon(c\,H\phi_\epsilon^{-1}(s^{1/r}/|B|)\,s^{1/r'})} \\
&\leq \frac{1}{2\,|B|\,C^p\,H^p} \int_1^\infty \frac{ds}{s^{p/r'}\,\phi_\epsilon(\phi_\epsilon^{-1}(s^{1/r}/|B|))} \\
&= \frac{C}{H^p} \int_1^\infty \frac{ds}{s^{p/r'+1/r}} \\
&= \frac{C}{H^p}\,.
\end{aligned}
$$

Choosing H sufficiently large, we have that $I \leq 1/2$. Therefore, from (2.8), we have

$$
\|g\|_\psi \leq H\,|B|^{1/r'}\,[g]_{\phi_\epsilon,\omega}. \tag{2.9}
$$

Finally, since $(\widetilde{\phi_t^{\,r}})^{-1}(s) \approx s^{1/r}/\phi^{-1}(ts^{1/r})$, from (2.7), (2.9) and our choice of t, we obtain the desired conclusion. $\square$

2.10 Lemma. *Let α belong to $(0,n)$. Let ω be a weight in $C(\tilde{\phi})$, where ϕ is a function with lower type $p > 1$ and upper type $q < n/(\alpha-1)^+$. Then, there exists a constant C such that*

$$
\int_{\mathbf{R}^n\setminus B} \frac{|f(y)|}{|x_B - y|^{n-\alpha+1}}\,dy \leq C\,|B|^{\alpha/n-1/n}\,\phi_\epsilon^{-1}(1/|B|)\,\inf_B \omega\,[f]_{\phi_\epsilon,\omega}
$$

holds for any ball $B = B(x_B, R)$ in $\mathbf{R}^n$, every $\epsilon > 0$ and $f \in L^*_{\phi,\omega}$.

Proof: Denoting $g = f/\omega$, we can write

$$
g = g^a + g_a,
$$

where $g^a = g\,\chi_{\{x:|g(x)|>a\}}$ with a a constant to be determined. Therefore,

$$
\begin{aligned}
\int_{\mathbf{R}^n\setminus B} \frac{|f(y)|}{|x_B - y|^{n-\alpha+1}}\,dy &= \int_{\mathbf{R}^n\setminus B} \frac{|g^a(y)|\,\omega(y)}{|x_B - y|^{n-\alpha+1}}\,dy \\
&\quad + \int_{\mathbf{R}^n\setminus B} \frac{|g_a(y)|\,\omega(y)}{|x_B - y|^{n-\alpha+1}}\,dy \\
&= I_1 + I_2.
\end{aligned}
$$

Let us estimate I_1. From Corollary 2.5, there exists $r > 1$ such that $\omega \in C(\widetilde{\phi}^{\,r})$. Then, by Hölder inequality, it follows that

$$I_1 \leq \left\| \frac{\chi_{\mathbf{R}^n\setminus B} \quad \omega}{|x_B - .|^{n-\alpha+1}} \right\|_{\widetilde{\phi_t}^{\,r}} \|g^a\|_{\widetilde{\widetilde{\phi_t}^{\,r}}} \tag{2.11}$$

for any $t > 0$. Now, for a positive constant λ to be determined later, denoting by $B_j = B(x_B, 2^j R)$, we have

$$\begin{aligned}
&\int\limits_{\mathbf{R}^n\setminus B} \widetilde{\phi_t}^{\,r} \left(\frac{\omega(y)}{|x_B - y|^{n-\alpha+1}\,\lambda} \right) dy \\
&= \sum_{j=1}^{\infty} \int\limits_{2^j R \leq |x_B - y| < 2^{j+1} R} \widetilde{\phi_t}^{\,r} \left(\frac{\omega(y)}{|x_B - y|^{n-\alpha+1}\,\lambda} \right) dy \\
&\leq \sum_{j=0}^{\infty} \int_{B_j} \widetilde{\phi_t}^{\,r} \left(\frac{\omega(y)(\widetilde{\phi_t}^{\,r})^{-1}\,(|B_j|^{-1})}{\lambda(|B|^{1/n} 2^j)^{n-\alpha+1}(\widetilde{\phi_t}^{\,r})^{-1}(|B_j|^{-1})} \right) dy.
\end{aligned} \tag{2.12}$$

Therefore, using the relation $(\widetilde{\phi_t}^{\,r})^{-1}(s) \approx s^{1/r}/\phi_t^{-1}(s^{1/r})$ and having in mind that ϕ_t^{-1} is of lower type $1/q$ and $\widetilde{\phi_t}^{\,r}$ is of upper type rq', the above series is bounded by

$$\begin{aligned}
&\sum_{j=0}^{\infty} \int\limits_{B_j} \widetilde{\phi_t}^{\,r} \left(\frac{C\omega(y)(\widetilde{\phi_t}^{\,r})^{-1}\,(|B_j|^{-1})}{\lambda} \frac{\phi_t^{-1}(|B|^{-1/r} 2^{-jn/r})}{|B|^{1-\alpha/n+1/n-1/r}\, 2^{j(n-\alpha+1-n/r)}} \right) dy \\
&\quad \leq C \sum_{j=0}^{\infty} \frac{1}{2^{j(n-\alpha+1-n/rq')rq'}} \\
&\qquad \times \int\limits_{B_j} \widetilde{\phi_t}^{\,r} \left(\frac{C\omega(y)(\widetilde{\phi_t}^{\,r})^{-1}(|B_j|^{-1})}{\lambda \inf_{B_j} \omega} \frac{\phi_t^{-1}(|B|^{-1/r})\ \inf_B \omega}{|B|^{1-\alpha/n+1/n-1/r}} \right) dy.
\end{aligned}$$

Now, we choose $t = \epsilon |B|^{-1/r'}$, $\epsilon > 0$, and

$$\lambda = H\, C\, \phi^{-1}(\epsilon\, |B|^{-1})\, |B|^{\alpha/n - 1/n - 1/r'} \inf_B \omega$$

with $H > 1$ to be fixed later. Since ϕ is of upper type q and $\omega \in C(\widetilde{\phi}^{\,r})$, we have that the above expression is bounded by

$$\frac{C}{H^{q'r}} \sum_{j=0}^{\infty} \frac{1}{2^{j(n-\alpha+1-n/rq')rq'}}\,.$$

Then, since $q < n/(\alpha-1)^+$, the last series converges and we can take H large enough to have (2.12) bounded by one. So, we obtain

$$\left\| \frac{\chi_{\mathbf{R}^n\setminus B} \quad \omega}{|x_B - .|^{n-\alpha+1}} \right\|_{\widetilde{\phi_t}^r} \leq C \frac{\widetilde{\phi_\epsilon}^{-1}(1/|B|) \inf_B \omega}{|B|^{1/n+1/r'-\alpha/n}} \tag{2.13}$$

for $t = \epsilon|B|^{-1/r'}$, $\epsilon > 0$. On the other hand, in order to estimate the second factor in (2.11), we denote by $\psi = \widetilde{\widetilde{\phi_t}^r}$, with $t = \epsilon\,|B|^{-1/r'}$ as before. Then, since $g \in L^*_\phi$, we get

$$\begin{aligned}
\int_{\mathbf{R}^n} \psi\left(g^a(x)/\lambda\right) dx &= \int_0^\infty \left|\{x : |g^a(x)| > \psi^{-1}(s)\lambda\}\right| ds \\
&= \left(\int_0^{\psi(a/\lambda)} + \int_{\psi(a/\lambda)}^\infty \right) \left|\{x : |g^a(x)| > \psi^{-1}(s)\lambda\}\right| ds \\
&\leq \psi(a/\lambda)\,|\{x : |g^a(x)| > a\}| \\
&\quad + \int_{\psi(a/\lambda)}^\infty \left|\{x : |g^a(x)| > \psi^{-1}(s)\lambda\}\right| ds \\
&\leq \frac{\psi(a/\lambda)}{\phi_\epsilon(a/[g]_{\phi_\epsilon})} + \int_{\psi(a/\lambda)}^\infty \frac{ds}{\phi_\epsilon(\psi^{-1}(s)\lambda/[g]_{\phi_\epsilon})}.
\end{aligned} \tag{2.14}$$

From (1.5) it follows easily that

$$\psi^{-1}(s) \approx s^{1/r'} \phi_\epsilon^{-1}\left(s^{1/r}/\,|B|^{1/r\prime}\right).$$

Therefore, taking $a = H[g]_{\phi_\epsilon}\phi_\epsilon^{-1}(|B|^{-1})$ with H to be determined and λ such that $\psi(a/\lambda) \approx |B|^{-1}$, that is $\lambda = H[g]_{\phi_\epsilon}|B|^{1/r'}$ and using that ϕ is of lower type p, inequality (2.14) allows us to obtain

$$\begin{aligned}
\int_{\mathbf{R}} \psi\left(g^a(x)/\lambda\right) dx &\leq \frac{1}{|B|\,\phi_\epsilon\left(H\,\phi_\epsilon^{-1}(|B|^{-1})\right)} \\
&\quad + \frac{1}{|B|} \int_1^\infty \frac{ds}{\phi_\epsilon\left(\psi^{-1}(s/\,|B|)\,H\,|B|^{1/r'}\right)} \\
&\leq \frac{C}{H^p} + \frac{1}{|B|} \int_1^\infty \frac{ds}{\phi_\epsilon\left(cH\,s^{1/r'}\,\phi_\epsilon^{-1}(s^{1/r}/\,|B|)\right)}
\end{aligned}$$

$$\leq \frac{C}{H^p}\left(1+\int_1^\infty \frac{ds}{s^{p/r'+1/r}}\right)$$
$$\leq \frac{C}{H^p}.$$

Consequently, for H large enough, we have

$$\|g^a\|_\psi \leq C\,|B|^{1/r'}\,[g]_{\phi_\epsilon}.$$

Therefore, from (2.11) and (2.13), it follows

$$I_1 \leq \frac{C\ \phi_\epsilon^{-1}(1/|B|)\ \inf_B \omega}{|B|^{1/n-\alpha/n}}[g]_{\phi_\epsilon}. \tag{2.15}$$

Now we estimate I_2. Since $q < n/(\alpha-1)^+$, there exists $\delta < 1$ such that $(q'\delta)' < n/(\alpha-1)^+$. Applying Hölder inequality, we get

$$I_2 \leq \left\| \frac{\chi_{\mathbf{R}^n\setminus B}\quad \omega}{|x_B - .|^{n-\alpha+1}} \right\|_{\tilde{\phi}_t{}^\delta} \|g_a\|_{\widetilde{\tilde{\phi}_t{}^\delta}} \tag{2.16}$$

for every $t > 0$. Proceeding as in (2.12), denoting $B_j = B(x_B, 2^jR)$ and using that the lower type of $\tilde{\phi}$ is q', we obtain

$$\int_{\mathbf{R}^n\setminus B} \tilde{\phi}_t{}^\delta\left(\frac{\omega(y)}{|x_B-y|^{n-\alpha+1}\,\lambda}\right)dy \leq C\sum_{j=0}^\infty \frac{1}{2^{j(n-\alpha+1-n/(\delta q'))\delta q'}}$$
$$\times \int_{B_j} \tilde{\phi}_t{}^\delta\left(\frac{C\omega(y)(\tilde{\phi}_t{}^\delta)^{-1}(|B_j|^{-1})}{\lambda \inf_{B_j}\omega}\,\frac{(\phi_t{}^\delta)^{-1}(|B|^{-1})\ \inf_B\omega}{|B|^{1-\alpha/n+1/n-1/\delta}}\right)dy.$$

Let us take $t = \epsilon\,|B|^{1/\delta-1}$, $\epsilon > 0$ and
$\lambda = CH\ \phi^{-1}(\epsilon|B|^{-1})|B|^{\alpha/n-1/n+1/\delta-1}\inf_B\omega$. From Corollary 2.5 and the fact that $(q'\delta)' < n/(\alpha-1)^+$, we have that the above series is bounded by

$$\frac{C}{H^{q'\delta}}\sum_{j=0}^\infty 2^{-j(n-\alpha+1-n/(\delta q'))\delta q'} \leq 1$$

for H large enough. Hence, we get

$$\left\| \frac{\chi_{\mathbf{R}^n\setminus B}\quad \omega}{|x_B - .|^{n-\alpha+1}} \right\|_{\tilde{\phi}_t{}^\delta} \leq \frac{C\tilde{\phi}_\epsilon^{-1}(1/|B|)\inf_B\omega}{|B|^{1-1/\delta+1/n-\alpha/n}}, \tag{2.17}$$

where $t = \epsilon|B|^{1/\epsilon - 1}$, $\epsilon > 0$. With this choice of t we are going to estimate the second norm in (2.16). Setting $\psi = \widetilde{\widetilde{\phi_t}}^{\delta}$, we have

$$\begin{aligned}\int \psi\left(\frac{g_a(x)}{\lambda}\right)dx &= \int\limits_0^{\psi(a/\lambda)} \left|\{x : |g(x)| > \lambda\psi^{-1}(s)\}\right| ds \\ &\leq \int\limits_0^{\psi(a/\lambda)} \frac{ds}{\phi_\epsilon(\psi^{-1}(s)\lambda/[g]_{\phi_\epsilon})}\,,\end{aligned} \tag{2.18}$$

where $a = H[g]_{\phi_\epsilon}\phi_\epsilon^{-1}(|B|^{-1})$ as before. From (1.5) we get that

$$\psi^{-1}(s) \approx s^{1-1/\delta}\phi_\epsilon^{-1}(s^{1/\delta}/|B|^{1-1/\delta}).$$

So, choosing $\lambda = H[g]_{\phi_\epsilon}|B|^{1-1/\delta}$, it is easy to see that $\psi(a/\lambda) \approx |B|^{-1}$. Moreover, since ϕ is of lower type $p > 1$, from (2.18), we obtain that

$$\begin{aligned}\int \psi\left(\frac{g_a(x)}{\lambda}\right)dx &\leq \psi(a/\lambda)\int\limits_0^1 \frac{du}{\phi_\epsilon(\psi^{-1}(u\psi(a/\lambda))H\,|B|^{1-1/\delta})} \\ &\leq \frac{C}{|B|}\int\limits_0^1 \frac{du}{\phi_\epsilon(cHu^{1-1/\delta}\phi_\epsilon^{-1}(u^{1/\delta}\,|B|^{-1}))} \\ &\leq \frac{C}{H^p}\int\limits_0^1 u^{(1/\delta-1)p-1/\delta}\,du \\ &\leq 1\end{aligned}$$

for H large enough. Therefore

$$\|g_a\|_\psi \leq c[g]_{\phi_\epsilon}\,|B|^{1-1/\delta}\,. \tag{2.19}$$

From (2.16), (2.17) and (2.19), we conclude that

$$I_2 \leq \frac{C\phi_\epsilon^{-1}(1/\,|B|)\inf_B \omega}{|B|^{1/n-\alpha/n}}\,,$$

which together with the estimate for I_1 yield the conclusion of the lemma. □

We are in position to prove our main result.

Proof of Theorem 1.11: Assuming (1.12) let us prove (1.13). It is easy to check that

$$[f/\omega]_{\phi_{t^n}} = [\delta_t f/\delta_t \omega]_\phi \tag{2.20}$$

holds for every $t > 0$. On the other hand, since

$$I_\alpha f(x) = t^\alpha\, \delta_{1/t}(I_\alpha(\delta_t f))(x),$$

we have

$$\begin{aligned}&\frac{\left\|\chi_B \omega^{-1}\right\|_\infty}{|B|^{1+\alpha/n}\ \phi_{t^n}^{-1}(1/|B|)} \int_B |I_\alpha f(x) - m_B(I_\alpha f)|\, dx\\ &= \frac{\left\|\chi_{t^{-1}B}(\delta_t\omega)^{-1}\right\|_\infty}{|t^{-1}B|^{\alpha/n}\ \phi^{-1}(1/|t^{-1}B|)} \frac{1}{|t^{-1}B|} \int_{t^{-1}B} |I_\alpha(\delta_t f)(x) - m_{t^{-1}B}(I_\alpha \delta_t f)|\, dx.\end{aligned}$$

So, from (2.20), (1.13) is clear. A similar reasoning allows us to prove that (1.13) implies (1.12).

Next, we are going to show that (1.14) can be obtained from (1.12). First note that, since $[f/\omega]_{\phi_t} \le C\, \|f/\omega\|_{\phi_t}$, we also get (1.12) with $[f/\omega]_{\phi_t}$ replaced by the strong norm. Then taking $B = B(x_B, R)$ and $\tilde{B} = 12B$, we have

$$\frac{\left\|\chi_{\tilde{B}} \omega^{-1}\right\|_\infty}{|\tilde{B}|^{1+\alpha/n}\, \phi_t^{-1}(1/|\tilde{B}|)} \int_B \left|I_\alpha f(x) - m_{\tilde{B}}(I_\alpha f)\right|\, dx \le C\, \|f/\omega\|_{\phi_t} \tag{2.21}$$

for every $f \in L_{\phi,\omega}$. We denote by B_1 and B_2 the translates of B defined by $B + e_1$, $B + e_2$ with $|e_1| = 4R$ and $|e_2| = 10R$. A straightforward calculation shows that

$$\begin{aligned}&|B_1| = |B_2| = |B|,\\ &B \cup B_1 \cup B_2 \subset \tilde{B} \qquad \text{with}\\ &|\tilde{B}| = 12^n\, |B|.\end{aligned}$$

Moreover, for every $y \in B_1$, $z \in B_2$ and $x \in B$, we get that

$$|y - x| \le 6R \quad \text{and} \quad |z - x| \ge 8R.$$

For a non-negative f supported in B, the integral on the left side of (2.21) can be bounded from below by

$$\begin{aligned}&\frac{1}{2|\tilde{B}|} \int_{\tilde{B}} \int_{\tilde{B}} \left| \int_B \left(\frac{1}{|y-x|^{n-\alpha}} - \frac{1}{|z-x|^{n-\alpha}} \right) f(x)\, dx \right| dy\, dz\\ &\qquad \ge \frac{1}{2|\tilde{B}|} \int_{B_2} \int_{B_1} \int_B \left(\frac{1}{|y-x|^{n-\alpha}} - \frac{1}{|z-x|^{n-\alpha}} \right) f(x)\, dx\, dy\, dz\\ &\qquad \ge C\, |B|^{\alpha/n} \int_B f(x)\, dx.\end{aligned}$$

Hence, recalling that ϕ has upper index q and combining the above inequality with (2.21) for ωf instead of f, we have that

$$\left|\int_B f(x)\omega(x)\,dx\right| \le C\frac{|B|\,\phi_t^{-1}(1/|B|)}{\|\chi_B\omega^{-1}\|_\infty}\,\|f\chi_B\|_{\phi_t}$$

for every $f \in L_{\phi_t}(B)$ with $t > 0$. Therefore, ω must belong to the dual of $L_{\phi_t}(B)$, that is $L_{\tilde{\phi}_t}(B)$, concluding that $\omega \in C(\tilde{\phi})$ by using (1.5).

In order to prove the reciprocal, let us assume (1.12). For $B = B(x_B, R)$ we can write

$$\begin{aligned}\int_B |I_\alpha f(x) - m_B(I_\alpha f)|\,dx &\le \int_B |I_\alpha(\chi_{2B}f)(x) - m_B I_\alpha(\chi_{2B}f)|\,dx\\ &\quad+\int_B \left|I_\alpha(\chi_{\mathbf{R}^n\setminus 2B}f)(x) - m_B I_\alpha(\chi_{\mathbf{R}^n\setminus 2B}f)\right|\,dx \qquad (2.22)\\ &= I_1 + I_2.\end{aligned}$$

We first estimate I_1. Applying Lemma 2.6 and Fubini's theorem we get that

$$\begin{aligned}|I_1| &\le 2\int_B \left(\int_{2B}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy\right)dx\\ &\le C\,|B|^{\alpha/n}\int_{2B}|f(y)|\,dy\\ &\le C\,|B|^{1+\alpha/n}\ \phi_\epsilon^{-1}(1/|B|)\,\inf_B \omega\,[f]_{\phi_\epsilon,\omega}.\end{aligned}$$

for any $\epsilon > 0$.

On the other hand, by Lemma 2.10, we have that

$$\begin{aligned}|I_2| &\le C\,|B|^{1+1/n}\int_{\mathbf{R}^n\setminus 2B}\frac{|f(y)|}{|x_B-y|^{n-\alpha+1}}\,dy\\ &\le C\,|B|^{1+\alpha/n}\ \phi_t^{-1}(1/|B|)\,\inf_B \omega\,[f]_{\phi_\epsilon,\omega}.\end{aligned}$$

Therefore, from (2.2) and the estimates for I_1 and I_2, (1.12) follows immediately.

Let us prove the equivalence between (1.14) and (1.15). Suppose that $\omega^{p'} \in A_1$. Then, there exists $\epsilon > 0$ such that $\omega^{p'+\epsilon} \in A_1$. Since ϕ has lower index p and upper index q, it follows that $\tilde{\phi}$ is, in particular, of lower type $p' + \epsilon$ and it is of upper type $q'-\delta$ for some $\delta>0$. Hence, for λ large enough, we get that

$$\begin{aligned}\int_B \tilde{\phi}\left(\frac{\omega(x)\,\tilde{\phi}^{-1}(t/|B|)}{\lambda\,\inf_B\omega}\right)\frac{dx}{t} &\le \frac{C}{\lambda^{q'-\delta}}\frac{1}{|B|}\int_B\left(\frac{\omega(x)}{\inf_B\omega}\right)^{p'+\epsilon}dx\\ &\le \frac{C}{\lambda^{q'-\delta}}\\ &\le 1\end{aligned}$$

for every ball $B\subset\mathbf{R}^n$, which proves that $\omega\in C(\tilde{\phi})$. For the converse note that in view of Corollary 2.5, we only need to prove that $\omega\in C(\tilde{\phi})$ implies $\omega^{p'-\epsilon}\in A_1$ for every $\epsilon>0$. From Definition 1.3 it is clear that for each $r>1$ there exists $s=s(r)>0$ satisfying

$$r^{p'}\tilde{\phi}(s)\le 2\tilde{\phi}(rs). \tag{2.23}$$

On the other hand, since $\omega\in C(\tilde{\phi})$, taking a ball B in $\mathbf{R}^n$ and defining

$$E_k=\{x\in B: 2^k\le\omega(x)/\inf_B\omega<2^{k+1}\},\quad k\ge 0,$$

we have

$$\begin{aligned}1&\ge\int_B\tilde{\phi}\left(\frac{\omega(x)}{C\,\inf_B\omega}\tilde{\phi}^{-1}(t/|B|)\right)\frac{dx}{t}\\ &\ge\tilde{\phi}\left(\frac{2^k}{C}\tilde{\phi}^{-1}(t/|B|)\right)\frac{|E_k|}{t}\end{aligned}\tag{2.24}$$

for any $t>0$ and $k\ge0$. Applying (2.23) for $r=2^k$, $k\ge0$, we get a sequence s_k such that

$$2^{kp'}\tilde{\phi}(s_k)\le 2\tilde{\phi}(2^k s_k).$$

Now, for each $k>0$, we use (2.24) with $t=\tilde{\phi}(c\,s_k)|B|$. Therefore, having in mind that $\tilde{\phi}$ is of upper type p', we get

$$\begin{aligned}1&\ge\tilde{\phi}(2^k s_k)\frac{|E_k|}{\tilde{\phi}(C\,s_k)\,|B|}\\ &\ge\frac{2^{kp'}\tilde{\phi}(s_k)}{2\tilde{\phi}(C\,s_k)}\frac{|E_k|}{|B|}\\ &\ge C2^{kp'}\frac{|E_k|}{|B|}\\ &\ge C\,2^{k\epsilon}\frac{1}{|B|}\int_{E_k}\left(\frac{\omega(x)}{\inf_B\omega}\right)^{p'-\epsilon}dx\end{aligned}$$

for $0 < \epsilon < p'$. So, we have that

$$\frac{1}{|B|} \int\limits_{E_k} \left(\frac{\omega(x)}{\inf_B \omega} \right)^{p'-\epsilon} dx \leq c\, 2^{-k\epsilon}$$

holds for every $k \geq 0$. Summing up over k these estimates we obtain the desired conclusion. □

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