Irresolvable countable spaces of weight less than $\mathfrak{C}$

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Abstract. We construct in Bell-Kunen's model: (a) a group maximal topology on a countable infinite Boolean group of weight $\aleph_1 < \mathfrak{C}$ and (b) a countable irresolvable dense subspace of 2^{ω_1}. In this model $\mathfrak{C} = \aleph_{\omega_1}$.

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0. Introduction

In 1943, E. Hewitt [He] called a space resolvable if it has two disjoint dense subsets, and irresolvable otherwise. Recall that a space X is called k-resolvable, where k is a cardinal, if X contains k disjoint dense subsets. When we talk about resolvable or irresolvable spaces we assume that they have no isolated points. All spaces are assumed to be infinite.

A singularity of irresolvable topologies is that they are constructed only by transfinite induction; as a rule, by enlargement of topologies. Therefore it is difficult to obtain irresolvable topologies with some given properties.

In this note we consider mainly topologies on countable sets and the main result is the following.

The existence of countable regular irresolvable space of weight less than $\mathfrak{C}$ is consistent with ZFC.

Such spaces exist in a model constructed by M. Bell and K. Kunen [BK].

1. Resolvability of topologies on countable set

It is not difficult to construct an irresolvable T_1-topology. Let ξ be a free ultrafilter on ω. Let $\tau = \{\emptyset\} \cup \xi$. Then τ is an irresolvable T_1-topology and its weight is equal to the character of ξ. But the following proposition shows the difference between T_1- and Hausdorff topologies.

1.1 Proposition. *If a Hausdorff topology on a countable set contains a base of a free ultrafilter which is a P-point then this topology has an isolated point.*

Proof: In a Hausdorff space X, an ultrafilter ξ can have at most one limit point. Let x be such a point. Now, each point y of the rest $Y = X \setminus \{x\}$ has an open neighbourhood V_y which does not belong to ξ. As ξ is a P-point, there is an element $W \in \xi$ such that $W \cap V_y$ is finite for each $y \in Y$. There is a $Q \subset W$,

$Q \in \xi$ which is open. Necessarily $Q \cap V_z$ is not empty for some $z \in Y$. It implies that there is an isolated point. $\square$

Remark that according to A.G. El'kin [E] each irresolvable topology contains a base of some set-theoretic ultrafilter. Proposition 1.1 implies that this ultrafilter cannot be a P-point if our topology is a Hausdorff topology on a countable set. By this reason we cannot use some known ultrafilters for constructing Hausdorff irresolvable topologies (for example, we cannot use Kunen's P-point of weight $\aleph_1 < \mathfrak{C}$ [K].)

Let us recall a weak version of Martin's Axiom

$MA_{countable}$. *Let P be a countable partially ordered (p. o.) set and $\mathcal{D}$ be a family of dense subsets, $|\mathcal{D}| < \mathfrak{C}$. Then there exists a $\mathcal{D}$-generic subset $G \subset P$.*

It is known that $MA_{countable}$ is consistent with any admissible cardinal arithmetic.

Now let us recall that a π-net is a family of subsets such that every nonempty open subset contains a member of this family.

1.2 Theorem ($MA_{countable}$)**.** *Let a topology on a countable set have a π-net of cardinality less than $\mathfrak{C}$ consisting of infinite subsets. Then this topology is $\aleph_0$-resolvable.*

The proof is very standard and will be omitted.

2. A sketch of Bell-Kunen's model [BK]

Let M_0 be a countable standard transitive model for ZFC in which GCH is true. Bell and Kunen construct in M_0 an increasing transfinite family of p. o. sets $P_\alpha : \alpha \leq \omega_1$ such that

(i) the Souslin number of each P_α is countable,

(ii) if α is limit then $P_\alpha = \bigcup\{P_\beta : \beta < \alpha\}$,

(iii) if α is not limit then P_α is such that $M_0^{P_\alpha} \to [MA + \mathfrak{C} = \aleph_\alpha"]$.

Let $G = G_{\omega_1}$ be a M_0-generic subset of a p. o. set P_{ω_1} and $G_\alpha = G \cap P_\alpha$ for every $\alpha \leq \omega_1$. Then in M_{ω_1} there is a transfinite increasing family of models $\{M_\alpha = M[G_\alpha] : \alpha \leq \omega_1\}$ and if $\alpha > 0$ is a non-limit countable ordinal then the assertion "$MA + \mathfrak{C} = \aleph_\alpha$" is true in M_α.

Let us note too, that in M_{ω_1} the power set $\mathcal{P}(\omega)$ of all subsets of ω is the union of the increasing chain $\{\mathcal{P}(\omega) \cap M_{\alpha+1} : \alpha < \omega_1\}$.

Let us find in the Bell-Kunen's model M_{ω_1} at least two interesting irresolvable countable spaces with weight less than $\mathfrak{C}$.

3. A maximal group of weight less than $\mathfrak{C}$ in Bell-Kunen's model

The first example of a Hausdorff maximal group was constructed under Martin's Axiom in [Ma] by the author of this note in 1975. Algebraically it is an infinite countable Boolean group. Every Boolean group is Abelian and contains only elements of order 2, i.e. $x + x = 0$ for every x. An infinite countable Boolean group can be identified, for example, with the set Ω of all finite subsets of ω with symmetric difference as the group operation. The neutral element is the empty set, which we denote by 0. We call a topology maximal if it admits no isolated points but if there is an isolated point in any its proper refinement.

The maximal topology is irresolvable in the strongest sense: no point can be limit for two disjoint subsets; this implies, of course, irresolvability of the whole space and its extremal disconnectedness.

We constructed the desired group topology in Bell-Kunen's model using the following example from [Ma].

3.1 Example ([BL]). *On an infinite countable Boolean group there exists a non-discrete Hausdorff group maximal topology.*

BL denotes Booth's Lemma or Combinatorial Principle $P(\mathfrak{C})$:

If ξ is a centered family of infinite subsets of ω (i.e. the intersection of any finite subfamily is infinite) and $|\xi| < \mathfrak{C}$ then there exists an infinite $B \subseteq \omega$ such that $|B \setminus A| < \aleph_0$ for each $A \in \xi$.

BL is known as one of the important consequences of Martin's Axiom.

Sketch of the construction of Example 3.1 (the details can be found in [Ma]). In the construction of Example 3.1 we deal in fact only with filters of neighbourhoods of 0 in the group Ω. Such a filter is called *linear* if it has a base composed of subgroups. Such filters, their bases and corresponding group topologies will be denoted F, P, bF, $\tau(F)$.

3.2 Proposition ([BL]). *Suppose that a linear filter F has a base of size less than $\mathfrak{C}$. Then there exists a linear filter P with a countable base, bigger than F.*

3.3 Proposition ([BL]). *Let F be a linear filter with countable base and let $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 \cap \Omega_2 = \emptyset$. Then there exists a linear filter P, bigger than F, with a countable base such that at least one of the subsets $\{0\} \cup \Omega_1$, $\{0\} \cup \Omega_2$, contains an element of P (i.e. a neighbourhood of 0 with respect to the topology $\tau(P)$).*

The final construction of Example 3.1. The set of all decompositions of the group Ω into two disjoint subsets Ω_1, Ω_2 will be numerated by ordinals smaller than $2^{\aleph_0}$. Now an increasing system of greater and greater filters F_α is constructed, using Propositions 3.2–3.3. The filter-join of all filters F_α will be as desired. It generates a group maximal topology. $\square$

Now we work in Bell-Kunen's model using the construction of this example.

Let τ be any group topology without isolated points on Ω in M_0. Let us consider it in the model M_2. In this model τ has weight less than $\mathfrak{C}$, so we apply Proposition 3.2 and obtain some larger than τ topology ν_2 with countable base $\mathcal{B}_2$. Then we may use the construction of the example from [Ma] and obtain a larger group topology τ_2 which is maximal in M_2. And so on.

Let in M_{ω_1} the topology μ be generated by the union $\bigcup\{\tau_{\alpha+1} : \alpha < \omega_1\}$. But this union has a family $\bigcup\{\nu_{\alpha+1} : \alpha < \omega_1\}$ as its base and this base has cardinality $\aleph_1$. We can prove maximality of μ with the aid of ideas of the construction of Example 3.1.

4. A countable irresolvable dense subset in 2^{ω_1} in Bell-Kunen's model

The following proposition is a "topological" version of Booth's Lemma. Its proof is also very familiar.

4.1 Proposition ([BL]). *Let (X, τ) be a countable space without isolated points with π-weight less than $\mathfrak{C}$. Let $\mathcal{S}$ be a family of dense subsets, $|\mathcal{S}| < \mathfrak{C}$ and the intersection of every finite subfamily of $\mathcal{S}$ is also dense. Then there exists a dense subset A such that $|A \setminus S| < \aleph_0$ for every $S \in \mathcal{S}$.*

Proof: Let $\mathcal{E}$ be a π-base of cardinality less than $\mathfrak{C}$. Let us consider the following p. o. set P. An element $p \in P$ is a pair (a, δ), where a is a finite subset of X and δ is a finite subfamily of $\mathcal{S}$. Let $p \leq q$ iff $a_p \supset a_q, \delta_p \supset \delta_q$ and $a_p \setminus a_q \subset \cap\delta_q$. Let us define a system $\mathcal{D}$ of dense subsets.

For every $S \in \mathcal{S}$ let $D_S = \{(a, \delta) : S \in \delta\}$.

For every $E \in \mathcal{E}$ and $m \in \omega$ let $D_{Em} = \{(a, \delta) : (a \cap E) \setminus m \neq \emptyset\}$.

Let $\mathcal{D} = \{D_S : S \in \mathcal{S}\} \cup \{D_{Em} : E \in \mathcal{E}, m \in \omega\}$. It is clear that $|\mathcal{D}| < \mathfrak{C}$. It is clear also that the p. o. set P is σ-centered, so we may apply BL and obtain a $\mathcal{D}$-generic subset $G \subset P$. It remains to check that $A = \bigcup\{a : (a, \delta) \in G$ for some $\delta\}$ is the desired subset.

Now we work in Bell-Kunen's model using this proposition. Let X be an infinite countable set and $X_0 = (X, \tau_0)$ be a dense subspace of 2^ω. Describe a general α-step of transfinite induction, $0 < \alpha < \omega_1$.. Let $X_\alpha = (X, \tau_\alpha)$ be a dense countable subspace of $2^{\omega\times\alpha}$. Let us consider this space in the model $M_{\alpha+1}$. Let ξ_α be a maximal centered family of τ_α-dense subsets of X belonging to M_α. In the model $M_{\alpha+1}$ $|\xi_\alpha| < \mathfrak{C}$, so we apply Proposition 4.1 and obtain some dense subset A_α (see Proposition 4.1). Later we can find by induction pairs of disjoint subsets (K_i^0, K_i^1) of A_α such that $K_i^0 \cup K_i^1 = A_\alpha$ and each K_{i+1}^j is a dense subset concerning the topology generated by a family $\tau_\alpha \cup \{K_l^n : l \leq i, n = 0, 1\}$. After this we may enlarge points of X on coordinates from the set $B_\alpha = \{\omega \times \alpha + n : n \in \omega\}$ by the rule: if $x \notin A_\alpha$ then $x(k) = 0$ for every $k \in B_\alpha$ and if $x \in K_i^j$ then $x(\omega\times\alpha+j) = i$. After this we have dense subset $X_{\alpha+1} = (X, \tau_{\alpha+1}) \subset 2^{\omega\times(\alpha+1)}$ and A_α is open in this space. Let $\mathcal{A}_\alpha = \{A_\alpha \setminus \delta : \delta$ is a finite subset of $A_\alpha\}$. Then every dense subset of X_α belonging to M_α contains some element from $\mathcal{A}_\alpha$.

After finishing this transfinite induction we obtain in M_{ω_1} a dense subset $X_{\omega_1} = (X, \tau_{\omega_1}) \subset 2^{\omega\times\omega_1}$. Let us prove that it is irresolvable. Let C be a dense subset of (X, τ_{ω_1}). Then $C \in M_\alpha$ for some countable α. But then $C \in \xi_\alpha$ and hence C contains an element of $\mathcal{A}_\alpha$. Hence C contains some $\tau_{\alpha+1}$-open subset. Therefore $X \setminus C$ is not τ_{ω_1}-dense. □

4.5 Remark. *In ZFC there exists a countable dense irresolvable subset in* $2^{\mathfrak{C}}$.

Indeed, let S be any countable dense subset of $2^{\mathfrak{C}}$. If it is irresolvable we are done. If not, we may enlarge points of S on next coordinates and at the end we will obtain some countable dense irresolvable subspace of $2^{\mathfrak{C}+m}$. But $m \leq \mathfrak{C}$ so we obtain a countable dense irresolvable subset of $2^{\mathfrak{C}}$.

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