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The local solution of a parabolic-elliptic equation with a nonlinear Neumann boundary condition

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Abstract. We investigate a parabolic-elliptic problem, where the time derivative is multiplied by a coefficient which may vanish on time-dependent spatial subdomains. The linear equation is supplemented by a nonlinear Neumann boundary condition $-\partial u/\partial\nu_A = g(\cdot,\cdot,u)$ with a locally defined, L_r-bounded function $g(t,\cdot,\xi)$. We prove the existence of a local weak solution to the problem by means of the Rothe method. A uniform a priori estimate for the Rothe approximations in L_∞, which is required by the *local* assumptions on g, is derived by a technique due to J. Moser.

Keywords: parabolic-elliptic problem, nonlinear Neumann boundary condition, Rothe method

Classification: 35K65, 65N40, 35M10

Introduction

In this paper we prove the weak solvability of a time-dependent partial differential equation with a nonlinear Neumann boundary condition. The evolution problem which shall be investigated shows the following special features.

(i) The time derivative is multiplied by a coefficient $\psi(t,x)$, $(t,x) \in [0,T]\times G$ which may vanish in certain time-dependent subdomains $\mathcal{E}(t)$ of G (cf. Assumption 1.3). Hence, the differential equation we consider is parabolic-elliptic.

(ii) Though we show the weak solvability (up to a certain point of time) in a Sobolev space, any growth restrictions of the nonlinearity, arising in the boundary condition $Bu = g(\cdot,\cdot,u)$, are omitted. Instead, the function $g(\cdot,\cdot,\xi)$ is assumed to be defined and bounded only on a set $\{\ \xi \in \mathbb{R} : |\xi| \le R\ \}$ (cf. Assumption 1.6).

We derive our existence result by means of the Rothe method (cf. e.g. [6], [13]) which is based on a semidiscretization with respect to the time variable, whereby the given evolution problem is approximated by a sequence of linear elliptic problems.

In view of (ii), the approximations obtained by solving these "discretized" problems have to be estimated in L_∞. For that purpose, we fall back on a technique introduced by J. Moser (cf. [8]), where appropriate L_p-estimates uniformly approach the desired boundedness statement as $p \longrightarrow \infty$. In various papers which treat parabolic Dirichlet problems, a method has been developed to derive such L_p-bounds, which are uniform with respect to both p and the stepsize h of the discretization (cf. e.g. [10]–[12]). Its principle consists in showing that L_p-norms of the approximations, $p \geq 2$, may be traced back recursively to L_2, where appropriate estimates easily can be derived by means of a well-known technique.

Using the Rothe method, nonlinear Neumann problems have also been investigated, for instance, by such authors as J. Kačur, J. Filo, and M. Slodička (cf. e.g. [2], [7], [15]). However, the nonlinearities arising in these problems were assumed to satisfy global growth conditions.

For the treatment of the degenerate differential equation, the outlined L_∞-technique is combined with the use of weighted Lebesgue norms. In contrast to [12] or [17, Section 3.1], where the coefficient of the time derivative may vanish only at a set of zero measure, these norms do not supply us with information on the behaviour of the approximations on the "elliptic" subdomains $\mathcal{E}(t)$. This fact complicates our proofs and entails the simple form of the differential operator.

Nonlinear degenerations in sets, depending upon the function searched for, have been investigated in fixed L_p or Orlicz spaces, for instance, by J. Kačur (cf. e.g. [7]). However, we consider the case of degeneration domains $\mathcal{E}(t)$ which are not influenced by the solution sought and estimate the Rothe approximations in L_∞.

The present paper generalizes results of [17, Section 3.2], where the Rothe method was applied to parabolic-elliptic equations in which the coefficient of the time derivative may vanish in an invariable subdomain $\mathcal{E}(t) \equiv \mathcal{E}$.

1. The problem and the assumptions

Let $G \subset \mathbb{R}^N$, $2 \leq N \leq 5$, be a simply connected, bounded domain of the C^∞-class, and I_T the time interval $[0, T]$. Moreover, we use the abbreviations $Q_T := I_T \times G$, $\Gamma_T := I_T \times \partial G$.

In the course of this paper, $\|\cdot\|_{p,\Omega}$ denotes the norm of $L_p(\Omega)$, $1 \leq p \leq \infty$, and $(\cdot,\cdot)_\Omega$ the duality between $L_p(\Omega)$ and $L_{p'}(\Omega)$, where p' is the conjugate exponent of p, i.e., $1/p + 1/p' = 1$. In particular, if $\Omega = G$, we write $\|\cdot\|_p := \|\cdot\|_{p,G}$, $(\cdot,\cdot) := (\cdot,\cdot)_G$. The norm of the Sobolev-Slobodeckiĭ space $W_p^\mu(G)$, $1 \leq p \leq \infty$, $\mu \geq 0$, shall be denoted by $\|\cdot\|_{\mu,p}$. Moreover, we introduce the functional $\|\cdot\|_{\nabla,2}$, defined as

$$\|u\|_{\nabla,2} := \left\{ \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_2^2 \right\}^{\frac{1}{2}}$$

on $W_2^1(G)$. Let X be a normed linear space. Then $L_p(I_T, X)$, $C(I_T, X)$, and $C^{0,1}(I_T, X)$ denote the sets of the L_p-integrable, continuous, or Lipschitz continuous mappings $\varphi : I_T \longrightarrow X$, respectively. Moreover, $B_R[X]$ is the closed ball $\{x \in X : \|x\|_X \leq R\}$.

In the course of this paper, the letter c is often used to denote a constant, which may differ from occurrence to occurrence. If it depends upon additional parameters, say t, we sometimes indicate this by $c(t)$. Finally, $\mathbb{R}^+$ is the set of nonnegative real numbers.

Note that all presented results remain valid for $N = 1$. Here we discuss $N \geq 2$ to avoid an extensive distinction of cases (e.g. for p_* in Lemma 1.9) which is necessary if $N = 1$.

Moreover, we shall not search for the weakest possible regularity assumption on the boundary ∂G. In general, the weak solvability theory for nondegenerate parabolic problems in $L_2(I_T, W_q^1(G))$ requires only $\partial G \in C^1$. In our proofs, however, we refer to known results on elliptic equations (cf. proof of Theorem 1.17) as well as to trace and interpolation theorems (cf. Lemma 1.8 and Lemma 1.9) which are formulated for $\partial G \in C^\infty$. For this reason this assumption is adopted. An analogous situation regards the regularity assumption on the coefficients of the differential operator (cf. Assumption 1.2).

Problem 1.1. We consider the initial boundary value problem

$$\psi(t,x)\frac{\partial u}{\partial t} + Au = 0 \quad \text{on} \quad Q_T, \quad -\frac{\partial u}{\partial \nu_A} = g(t,x,u) \quad \text{on } \Gamma_T, \quad u(0,x) = U_0(x),$$

where A denotes the differential operator

$$Au := -\sum_{i,k=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right),$$

and $\partial/\partial\nu_A$ the corresponding conormal derivative

$$\frac{\partial u}{\partial \nu_A} := \sum_{i,k=1}^{N} a_{ik}(x)\frac{\partial u}{\partial x_k}\ \cos\left(x_i, \vec{n}\right), \ \vec{n}\ldots \quad \text{exterior normal on} \quad \partial G.$$

Assumption 1.2. The operator A, which contains only second partial derivatives, is assumed to be symmetric and uniformly elliptic. Its coefficients a_{ik} belong to $C^\infty(\bar{G})$.

As a consequence of Assumption 1.2, the positive definite and symmetric bilinear form $(\cdot,\cdot)_A$, given by

$$(u,v)_A = (u,v)_{A,G} := \sum_{i,k=1}^{N} \int_G a_{ik}(x)\frac{\partial u}{\partial x_k}(x)\frac{\partial v}{\partial x_i}(x)\,\mathrm{d}x,$$
$$\forall\,(u,v) \in W_q^1(G) \times W_{q'}^1(G), \ q \geq 1,$$

satisfies the inequality

$$\text{(1)} \qquad \left(u, |u|^{p-2}u\right)_A \geq c_* \left\| |u|^{\frac{p-2}{2}} u \right\|_{\nabla,2}^2, \ c_* = \mathcal{O}\left(p^{-1}\right),$$
$$\forall\, p \geq 2, \ \forall\, u \in W_2^1(G) \cap L_\infty(G)$$

(cf. e.g. [9, Lemma 3]).

In order to formulate our assumptions on the function $\psi(t,x)$, we introduce the families of open sets $\mathcal{E}(t) := G \setminus \operatorname{supp}[\psi(t,\cdot)]$ and $\mathcal{P}(t) := G \setminus \overline{\mathcal{E}(t)}$, $t \in I_T$. Moreover, $\{(t,x) \in Q_T : x \in \mathcal{E}(t)\}$ and $\{(t,x) \in Q_T : x \in \mathcal{P}(t)\}$ will be denoted by $\mathcal{E}_T$ or $\mathcal{P}_T$ respectively.

Assumption 1.3. Let $\psi : I_T \times G \longrightarrow \mathbb{R}^+$ be an element of $C^{0,1}(I_T, L_\kappa(G))$, where $\kappa \in \mathbb{R}$ fulfills $\kappa > \max\{2, N/2\}$.

The above defined subsets $\mathcal{P}(t) \subseteq G$ are supposed to be nonempty C^∞-domains with $\partial\mathcal{P}(t) \supseteq \partial G$, $\forall\, t \in I_T$. Then, we assume that $1/\psi(t,\cdot)$, $t \in I_T$, belongs to $L_\beta(\mathcal{P}(t))$, $\beta > \kappa/(\kappa-2)$, and satisfies

$$\left\|\psi^{-1}(t,\cdot)\right\|_{\beta,\mathcal{P}(t)} \le c,\ \forall\, t \in I_T. \tag{2}$$

Due to our assumptions on ψ, the functional

$$\|u\|_{p,[\psi(t,\cdot)]} = \|u\|_{p,[\psi(t,\cdot)],\mathcal{P}(t)} := \left\{\int_G \psi(t,x)|u(x)|^p \,\mathrm{d}x\right\}^{\frac{1}{p}}$$

defines a norm on $L_{p\kappa'}(\mathcal{P}(t))$, $t \in I_T$, but in general, only a semi-norm on $L_{p\kappa'}(G)$. Since ψ is assumed to be an element of $C(I_T, L_\kappa(G))$ and satisfies the estimate (2), we obtain

$$\begin{aligned}\|u\|_{\frac{\beta}{1+\beta}p,\mathcal{P}(t)} &\le \left\|\psi^{-1}(t,\cdot)\right\|^{1/p}_{\beta,\mathcal{P}(t)} \|u\|_{p,[\psi(t,\cdot)]} \le c^{1/p}\|u\|_{p,[\psi(t,\cdot)]}\\ &\le c^{1/p}\|\psi(t,\cdot)\|_\kappa^{1/p}\|u\|_{p\kappa',\mathcal{P}(t)} \le c^{1/p}\|u\|_{p\kappa',\mathcal{P}(t)},\\ &\qquad \forall\, u \in L_{p\kappa'}(\mathcal{P}(t)),\ \forall\, p \ge \frac{1+\beta}{\beta},\ \forall\, t \in I_T.\end{aligned} \tag{3}$$

Remark 1.4. As a consequence of our assumptions on ψ, the following property of the domains $\mathcal{P}(t)$ can be derived: Let t' and t'' be arbitrary points of the time interval I_T. Then, using Hölder's inequality, we obtain the estimate

$$\begin{aligned}\operatorname{meas}[\mathcal{P}(t'') \setminus \mathcal{P}(t')] &= \int_{\mathcal{P}(t'')\setminus\mathcal{P}(t')} \psi^{-\frac{\beta\kappa}{\beta+\kappa}}(t'',x)\psi^{\frac{\beta\kappa}{\beta+\kappa}}(t'',x)\,\mathrm{d}x\\ &\le \left\|\psi^{-1}(t'',\cdot)\right\|^{\frac{\beta\kappa}{\beta+\kappa}}_{\beta,\mathcal{P}(t'')\setminus\mathcal{P}(t')} \|\psi(t'',\cdot)\|^{\frac{\beta\kappa}{\beta+\kappa}}_{\kappa,\mathcal{P}(t'')\setminus\mathcal{P}(t')}\\ &\le c\|\psi(t'',\cdot) - \psi(t',\cdot)\|_\kappa^{\frac{\beta\kappa}{\beta+\kappa}}.\end{aligned}$$

Thus, the measure of $\mathcal{P}(t'') \setminus \mathcal{P}(t')$ satisfies the Hölder condition

$$\operatorname{meas}[\mathcal{P}(t'') \setminus \mathcal{P}(t')] \le c|t'' - t'|^{\frac{\beta\kappa}{\beta+\kappa}}.$$

According to the assumptions on ψ (and A), Problem 1.1 is parabolic on $\mathcal{P}_T$ and elliptic on $\mathcal{E}_T$. Therefore, out of $\mathcal{P}(0)$ a definition of an initial function U_0 makes no sense. On the other hand, an extension of U_0 to G is required to carry out the Rothe method. So our assumption on the initial value U_0 reads as follows.

Assumption 1.5. Assume U_0 is the restriction of a function $U_0^* \in W_2^1(G) \cap L_\infty(G)$ to the subdomain $\mathcal{P}(0) \subseteq G$.

Without loss of generality we may assume that $\|U_0\|_{\infty,\mathcal{P}(0)} = \|U_0^*\|_\infty$. If not, the function $U_0^{**} \in W_2^1(G) \cap L_\infty(G)$, defined by

$$U_0^{**}(x) := \begin{cases} U_0^*(x), & \text{if } |U_0^*(x)| \leq \|U_0\|_{\infty,\mathcal{P}(0)} \\ \operatorname{sign}[U_0^*(x)]\|U_0\|_{\infty,\mathcal{P}(0)}, & \text{if } |U_0^*(x)| > \|U_0\|_{\infty,\mathcal{P}(0)} \end{cases},$$

might be chosen instead of U_0^*.

Assumption 1.6. Let the function $g : I_T \times \partial G \times [-R, R] \longrightarrow \mathbb{R}$, $R > \|U_0\|_{\infty,\mathcal{P}(0)}$, satisfy the following conditions.

(C_1) (Carathéodory Condition)
- (a) For all $(t, \xi) \in I_T \times [-R, R]$ the mapping $x \longmapsto g(t, x, \xi)$ is measurable on ∂G.
- (b) For almost all $x \in \partial G$ the mapping $(t, \xi) \longmapsto g(t, x, \xi)$ is continuous on $I_T \times [-R, R]$.

(C_2) There is a function $\tilde{g} \in L_r(\partial G)$, $r > N - 1$, such that the inequality $|g(t, x, \xi)| \leq \tilde{g}(x)$ holds for all $(t, x, \xi) \in I_T \times \partial G \times [-R, R]$.

Thus, $\mathcal{G}(t, v)[x] := g(t, x, v(x))$ defines a continuous mapping $\mathcal{G} : I_T \times B_R[L_\infty(\partial G)] \longrightarrow L_r(\partial G)$. Moreover, we obtain the local boundedness property

$$\|g(t, \cdot, v)\|_{r,\partial G} \leq c, \ \forall\, (t, v) \in I_T \times B_R[L_\infty(\partial G)].$$

According to our assumptions formulated above, the classical solvability of the initial boundary value Problem 1.1 may not be expected. Hence we introduce the following concept of a weak solution.

Definition 1.7. A function $u \in L_2(I_T, W_2^1(G)) \cap B_R[L_\infty(Q_T)]$ is called *a weak solution* to the parabolic-elliptic Problem 1.1 if the following conditions are satisfied.

(C_1) For almost all $t \in I_T$, $u(t, \cdot)$ belongs to $B_R[L_\infty(\partial G)]$.

(C_2) Let $V(Q_T)$ be the set of all $v \in L_2(I_T, W_2^1(G))$ which have a time derivative $v_t \in L_1(I_T, L_{\kappa'}(G))$ and fulfil $v(T, x) \equiv 0$. Then the integral relation

$$-\left(u, \frac{\partial}{\partial t}(\psi v)\right)_{\mathcal{P}_T} - (\psi(0, \cdot)U_0, v(0, \cdot)) + \int_{I_T} (u(t, \cdot), v(t, \cdot))_A \ \mathrm{d}t = -(g(\cdot, \cdot, u), v)_{\Gamma_T} \tag{4}$$

is satisfied for all $v \in V(Q_T)$.

Note that our assumptions on κ and r imply $L_2(I_T, W_2^1(G)) \subset L_2(I_T, L_{\kappa'}(G) \cap L_{r'}(\partial G))$. Moreover, Assumption 1.3 guarantees the existence of the weak derivative $\psi_t \in L_\infty(I_T, L_\kappa(G))$, and therefore, $(\psi v)_t \in L_1(Q_T)$ for $v \in V(Q_T)$. Consequently, the integral relation (4) is well-defined.

In the following discussion, we provide some statements which are required within the scope of the Rothe method. Using the assumption $G \in C^\infty$, our first lemma was proved in [16] (cf. 4.7.1 Theorem). It reads as follows:

Lemma 1.8. *The real numbers p and δ are assumed to satisfy the conditions $1 < p < \infty$, $\delta > 0$. Then there exists a linear continuous trace operator $T : W_p^{1/p+\delta}(G) \longrightarrow L_p(\partial G)$.*

The following interpolation inequality can be found in [17, Section 1.2.2], and is based on [16, 1.3.3 Theorem, 4.3.1 Theorem, and 2.4.2 Remark 2].

Lemma 1.9 (Nirenberg-Gagliardo Interpolation). *Let p_* be an arbitrary, but fixed real number with $p_* < \frac{2N}{N-2(1-\mu)}$, where $\mu \in \mathbb{R}$ satisfies $0 \le \mu < 1$. Then there exists some $\theta \in (0,1)$, such that the inequality $\|u\|_{\mu,p} \le c\|u\|_{1,2}^{\theta}\|u\|_{\gamma}^{1-\theta}$, $\gamma > 1$, holds for all $p \in [1, p_*]$ and $u \in W_2^1(G) \cap L_\infty(G)$.*

In the course of this paper both Lemma 1.8 and 1.9 shall be applied at the domains $\mathcal{P}(t)$. Thus, the constants arising in the resulting inequalities

(5)
$$\|u\|_{p,\partial G} \le \|u\|_{p,\partial\mathcal{P}(t)} \le c(t)\|u\|_{W_p^{1/p+\delta}(\mathcal{P}(t))},\ 1 < p < \infty,\ \delta > 0,\ t \in I_T,$$

(6)
$$\|u\|_{W_p^{\mu}(\mathcal{P}(t))} \le c(t)\|u\|_{1,2}^{\theta}\|u\|_{\gamma,\mathcal{P}(t)}^{1-\theta},\ \gamma > 1,\ 0 \le \mu < 1,$$
$$\forall\, p \le p_* < \frac{2N}{N-2(1-\mu)},\ t \in I_T,$$

depend on the time variable. On the other hand, the outlined technique used to estimate the Rothe approximations (uniformly with respect to the stepsize of the discretization) requires the boundedness of $\{c(t)\}_{t\in I_T}$. For this reason, we assume the following:

Assumption 1.10. The “parabolic” domains $\mathcal{P}(t)$ are assumed to behave in a manner such that the families of constants $\{c(t)\}_{t\in I_T}$, occurring in (5) and (6), are bounded.

Example 1.11. Obviously, Assumption 1.10 is satisfied for invariable $\mathcal{P}(t) \equiv \mathcal{P}$. This special case was investigated in [17, Section 3.2].

Example 1.12. The domains $\mathcal{P}(t)$, $t \in I_T$, are assumed to satisfy the following conditions:

(C_1) There is a domain $\mathcal{P}_* \subset \mathbb{R}^N$ of the C^∞-class with $\mathcal{P}_* \subseteq \bigcap_{t\in I_T} \mathcal{P}(t)$ and $\partial G \subseteq \partial\mathcal{P}_*$.

(C_2) For each $t \in I_T$ exists a C^∞-isomorphism $\varphi(t) : \mathcal{P}(t) \longleftrightarrow \mathcal{P}_{**}$, where $\mathcal{P}_{**} \subset \mathbb{R}^N$ is a C^∞-domain. The Jacobi determinants of $\varphi(t)$, $t \in I_T$, are uniformly bounded.

Owing to (C_1), the application of Lemma 1.8 at the domain $\mathcal{P}_*$ yields the inequality

$$\|u\|_{p,\partial G} \le \|u\|_{p,\partial\mathcal{P}_*} \le c\|u\|_{W_p^{1/p+\delta}(\mathcal{P}_*)} \le c\|u\|_{W_p^{1/p+\delta}(\mathcal{P}(t))}, \quad \forall\, t \in I_T,\ \forall\, u \in W_p^{1/p+\delta}(\mathcal{P}(t)),$$

where c does not depend on t. On the basis of (C_2) it can be proved that the set of constants $\{c(t)\}_{t\in I_T}$ occurring in (6) is also bounded.

Corollary 1.13. *Let p_* be an arbitrary, but fixed real number with $1 \le p_* < 2(N-1)/(N-2)$. Then there exists some $\theta \in (0,1)$, such that the inequalities*

(E_1) $\|u\|_{p,\partial G} \le c\|u\|_{1,2}^{\theta}\|u\|_{\gamma,\mathcal{P}(t)}^{1-\theta}$, $\gamma > 1$, $\forall\, p \in [1,p_*]$,

(E_2) $\|u\|_{p,\partial G}^{2\sigma} \le c\epsilon\|u\|_{1,2}^2 + c\epsilon^{-c}\|u\|_{\frac{1+\beta}{\beta}\gamma,[\psi(t,\cdot)]}^{2\frac{\sigma-\sigma\theta}{1-\sigma\theta}}$, $\gamma > 1$, $\forall\, \sigma \in (0,1]$, $\forall\, \epsilon > 0$, $\forall\, p \in [1,p_*]$,

hold for all $t \in I_T$ and $u \in W_2^1(G) \cap L_\infty(G)$.

Proof: With consideration to Assumption 1.10, our assertion (E_1) easily follows from Lemma 1.8 and Lemma 1.9 (cf. [17, Folgerung 1.22]). Applying Young's inequality as well as formula (3) to the right hand side of (E_1), we obtain the estimate (E_2). □

Corollary 1.14. *Let λ be an arbitrary, but fixed real number with $\lambda > \lambda_* := (1+\beta)/(2\beta)$. Then,*

$$\Big|\|u\|_{p,[\psi(t',\cdot)]}^p - \|u\|_{p,[\psi(t'',\cdot)]}^p\Big| \le c\left(\epsilon\Big\||u|^{\frac{p-2}{2}}u\Big\|_{1,2}^2 + \epsilon^{-c}\|u\|_{\lambda p,[\psi(t',\cdot)+\psi(t'',\cdot)]}^p\right)|t'-t''|, \quad \forall\, t',t'' \in I_T,\ \forall\, p \ge 2,\ \forall\, \epsilon > 0,$$

holds for all $u \in W_2^1(G) \cap L_\infty(G)$.

Proof: Recalling the assumption $\psi \in C^{0,1}(I_T, L_\kappa(G))$ we obtain

$$\Big|\|u\|_{p,[\psi(t',\cdot)]}^p - \|u\|_{p,[\psi(t'',\cdot)]}^p\Big| = \left|\int_{\mathcal{P}(t')\cup\mathcal{P}(t'')}[\psi(t',x)-\psi(t'',x)]|u(x)|^p\,\mathrm{d}x\right|$$

$$\le \big\|\psi(t',\cdot)-\psi(t'',\cdot)\big\|_\kappa \|u\|_{p\kappa',\mathcal{P}(t')\cup\mathcal{P}(t'')}^p \le c|t'-t''|\,\Big\||u|^{\frac{p-2}{2}}u\Big\|_{2\kappa',\mathcal{P}(t')\cup\mathcal{P}(t'')}^2, \quad \forall\, t',t'' \in I_T.$$

Since $1/[\psi(t',\cdot)+\psi(t'',\cdot)]$ belongs to $L_\beta(\mathcal{P}(t')\cup\mathcal{P}(t''))$, an application of Corollary 1.13 (E_2) (with $\psi(t',\cdot)+\psi(t'',\cdot)$ instead of $\psi(t,\cdot)$) to the right hand side yields the asserted estimate. □

Throughout the remainder of this paper, we shall continue denoting the real number $(1+\beta)/(2\beta)$ by λ_*.

Lemma 1.15. *The inequality* $\|u\|_{1,2}^2 \le c\left(\|u\|_{\nabla,2}^2 + \|u\|_{2\lambda_*,[\psi(t,\cdot)]}^2\right)$ *holds for all* $t\in I_T$ *and all functions* $u\in W_2^1(G)\cap L_\infty(G)$.

Proof: Because the sets $\mathcal{P}(t)$ are assumed to be nonempty subdomains of G, the functionals

$$\mathcal{F}_t(u) := \left\{\|u\|_{\nabla,2}^2 + \left|\int_{\mathcal{P}(t)} u(x)\,\mathrm{d}x\right|^2\right\}^{\frac{1}{2}},\ t\in I_T,$$

define norms on $W_2^1(G)$, which are equivalent to $\|\cdot\|_{1,2}$ (cf. e.g. [3, 5.11.2 Theorem]). Thus, we obtain $\|u\|_{1,2}^2 \le c(t)\ \mathcal{F}_t(u)^2$, $\forall\, t\in I_T$. Using the Hölder continuity of meas$[\mathcal{P}(t)]$ (cf. Remark 1.4), it can be proved that the set of constants $\{c(t)\}_{t\in I_T}$, occurring in these estimates is bounded. Hence, the application of (3) to the right hand side of $\mathcal{F}_t(u)^2 \le \|u\|_{\nabla,2}^2 + \|u\|_{1,\mathcal{P}(t)}^2$ yields the assertion. □

Our next lemma provides a compactness criterion which shall be used to derive convergence properties of the Rothe approximations in Lebesgue spaces.

Lemma 1.16. *Let γ be a real number with $1/2<\gamma\le 1$. Then, a sequence $\{u_n\}_{n=1}^\infty \subset L_2(I_T, W_2^1(G))$ is relatively compact in $L_{2\gamma}(\mathcal{P}_T)$, if it satisfies the conditions*

(C_1) $\|u_n\|_{L_2(I_T,W_2^1(G))}\le c$, $\forall\, n\in\mathbb{N}$, *and*

(C_2) $\displaystyle\int_0^T \|u_n(t+\epsilon,\cdot)-u_n(t,\cdot)\|_{2\gamma,\mathcal{P}(t)}^2\,\mathrm{d}t \le c\epsilon$, $\forall\,\epsilon\in(0,T)$, $\forall\, n\in\mathbb{N}$.

Proof: The basic idea of the proof may be outlined as follows: Using the assumptions (C_1), (C_2), as well as Hölder's inequality, we can show that

$$\int_0^T\int_{\mathcal{P}(t)} |v_n(t+\epsilon,x+y)-v_n(t,x)|^{2\gamma}\,\mathrm{d}x\,\mathrm{d}t \rightrightarrows 0 \text{ as } (\epsilon,y)\longrightarrow(0,0),\ v_n := \chi_{\mathcal{P}_T}u_n,$$

where $\chi_{\mathcal{P}_T}$ denotes the characteristic function of the cylindrical set $\mathcal{P}_T$. Due to Kolmogoroff's compactness criterion, this uniform convergence, and $\|u_n\|_{2\gamma,\mathcal{P}_T}\le c$, $\forall\, n\in\mathbb{N}$, imply the relative compactness of $\{u_n\}_{n=1}^\infty$ in $L_{2\gamma}(\mathcal{P}_T)$.

The details may be adapted from [5, Lemma 2.24], or [17, Lemma 1.41], where the special cases $\{\gamma=1,\mathcal{P}(t)\equiv G\}$ and $\{\mathcal{P}(t)\equiv G\}$ respectively, were considered. □

Within the scope of the Rothe method, the evolution Problem 1.1 is approximated by a sequence of elliptic equations with linear Neumann boundary conditions. An application of the outlined L_∞-technique requires the solution of these discretized problems in $W_q^1(G) \hookrightarrow C(\bar{G})$, $q > N$. Using our assumptions on G, and the uniformly elliptic operator A, the following existence result can be proved:

Theorem 1.17. *Let g_* be an element of $L_r(\partial G)$, $r > N-1$. The nonnegative function $\psi_* \in L_\kappa(G)$, $\kappa > N/2$, is supposed to satisfy $\|\psi_*\|_1 > 0$. Moreover, we assume $u_* \in L_\infty(G)$.*

Then there are real numbers $h_* > 0$ and $q > N$, such that the elliptic boundary value problem

$$\psi_* \frac{u - u_*}{h} + Au = 0, \quad -\left.\frac{\partial u}{\partial \nu_A}\right|_{\partial G} = g_*$$

has a unique weak solution $u \in W_q^1(G) \hookrightarrow C(\bar{G})$, provided that $0 < h < h_*$.

Proof: Our proof may be outlined as follows: Using sequences of C^∞-functions which converge to ψ_* or g_* respectively, we approximate the given problem. With the aid of a Fredholm alternative, proved by F.E. Browder (cf. [1, Corollary to Theorem 5]), these "smoothed" elliptic problems can be solved in $W_p^2(G)$, $p = 2N/(N-2)$. Consequently, the required continuity of the desired solution, i.e., $u \in W_q^1(G)$ with $q > N$, is guaranteed by the restriction $N \leq 5$.

By means of a priori estimates it can be shown that the solutions of the "smoothed" problems approach a weak solution to the given elliptic problem in $W_q^1(G)$. The use of the underlying results, proved by M. Schechter (cf. [14, Theorem 6.1]), requires the assumption $a_{ik} \in C^\infty(\bar{G})$.

We refer to [17, Section 1.4] for the details of the derivation. □

In the proof of Theorem 2.4 we shall use the following weak maximum-minimum principle, which was proved in [4, Theorem 8.1].

Lemma 1.18. *Let the coefficients $a_{ik}(x)$ of the uniformly elliptic operator A be measurable bounded functions on a domain $\Omega \subseteq \mathbb{R}^N$. Then, a weak solution $u \in W_2^1(\Omega)$ to $Au = 0$ in the sense of $(u, v)_{A,\Omega} = 0$, $\forall v \in C_0^1(\Omega)$, satisfies the maximum-minimum principle*

$$\operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \leq \operatorname*{ess\,sup}_{x\in\partial\Omega} |u(x)|.$$

2. Construction and local boundedness of the approximations

The Rothe method is based on a semidiscretization of the given problem with respect to the time variable. For that purpose, we subdivide $I_T = [0, T]$ into n subintervals

$$[t_{i-1}, t_i]\,, \quad t_i = t_i^{(n)} := ih_n, \quad h_n := T/n, \quad i = 1, \ldots, n.$$

Then, for each $n \geq 1$, Problem 1.1 may be approximated by the sequence of linear, elliptic boundary value problems

$$\psi_i \delta u_i + A u_i = 0 \text{ on } G, \quad -\frac{\partial u_i}{\partial \nu_A} = g_i \text{ on } \partial G, \quad i = 1, \ldots, n,$$
$$\text{where } \delta u_i := \frac{u_i - u_{i-1}}{h_n}, \quad \psi_i = \psi(t_i, \cdot), \quad g_i = g(t_i, \cdot, u_{i-1}),$$

and u_0 is given by $u_0(x) := U_0^*(x)$. In weak formulation this discretized problem reads as follows:

Problem 2.1. Let u_0 be defined as $u_0(x) := U_0^*(x)$. Find functions $u_i \in W_q^1(G) \cap B_R[C(\partial G)]$, $i = 1, \ldots, n$, such that the equations

$$(\psi_i \delta u_i, v) + (u_i, v)_A = -(g_i, v)_{\partial G}, \quad i = 1, \ldots, n, \tag{7.i}$$

are satisfied for all $v \in V(G) := W_{q'}^1(G) \cap L_{r'}(\partial G) \cap L_{\kappa'}(G)$.

According to the locally formulated assumptions on the boundary function g, a solution of the discretized Problem 2.1 must be sought in the closed ball $B_R[C(\partial G)]$. On the other hand, known existence results from the elliptic equation theory cannot be applied under such a restriction. For this reason, we first consider a slightly modified discretized problem, where the use of

$$g^R(t, x, \xi) := \begin{cases} g(t, x, \xi), & \text{if } |\xi| \leq R \\ g(t, x, R \text{ sign}(\xi)), & \text{if } |\xi| > R \end{cases}$$

enables us to apply the local assumptions on g globally.

Problem 2.2. Let v_0 be defined by $v_0(x) := U_0^*(x)$. Find functions $v_i \in W_q^1(G) \cap C(\partial G)$, $i = 1, \ldots, n$, such that

$$(\psi_i \delta v_i, v) + (v_i, v)_A = -\left(g_i^R, v\right)_{\partial G}, \quad \forall\, v \in V(G). \tag{8.i}$$

As long as the subdivision of the time interval is sufficiently fine, i.e., $\forall\, h_n \leq h_{n*}$, the "extended" discretized Problem 2.2 may be solved on the basis of Theorem 1.17. According to that existence result, there is a unique solution $u_i \in W_q^1(G)$, $q > N$, to the linear elliptic equation (8.i), provided that the previous function $u_{i-1} \in C(\bar{G})$ is already known. Starting with $i = 1$, this iterative procedure yields:

Lemma 2.3. *Assuming that the subdivision of I_T is sufficiently fine, i.e., $\forall\, n \geq n_*$, the "extended" discretized equations* (8.i), $i = 1, \ldots, n$, *have unique solutions $v_i \in W_q^1(G)$, $q > N$.*

Since the functions v_i are continuous on ∂G, they fulfil the original discretized equations (7.i), provided that they belong to the closed ball $B_R[C(\partial G)]$. Using this basic idea, a local existence result for the discretized Problem 2.1 can be proved.

Theorem 2.4. *There exist a time $T_* \in (0,T]$ and a natural number n_*, such that the following statements are valid:*

(S_1) *For all subdivisions of I_T with $n \geq n_*$, the discretized equations (7.i) are uniquely solvable in $W^1_q(G)$, $q > N$, providing the corresponding t_i satisfies $t_i \leq T_*$.*

(S_2) *The solutions $u_i \in W^1_q(G) \hookrightarrow C(\bar{G})$ of (7.i), $t_i \leq T_*$, fulfil the estimate*

$$\max_{t_i \leq t} \|u_i\|_{C(\bar{G})} \leq \exp(ct^c) \left\{ \|U_0\|^2_{\infty,\mathcal{P}(0)} + ct \right\}^{\alpha} \leq R, \qquad \forall\, t \in I_{T_*} := [0, T_*],\ \forall\, n \geq n_*,$$

where $\alpha \in \mathbb{R}$ belongs to $(0, 1/2]$, and takes on the value $1/2$ if $\|U_0\|_{\infty,\mathcal{P}(0)} > 0$.

Proof: The basic idea of our proof consists in showing, that up to a certain point $T_* \in (0,T]$ the solutions $v_i \in W^1_q(G)$ of the "extended" discretized Problem 2.2 belong to $B_R[C(\bar{G})] \subset B_R[C(\partial G)]$, and thus satisfy the corresponding original discretized equations (7.i) as well. For that purpose, we consider the integral relations (8.i). Since v_i particularly belongs to $W^1_2(G) \cap L_\infty(G)$, $|v_i|^{p-2}v_i$, $p \geq 2$, is an element of $W^1_2(G) \hookrightarrow V(G)$ (cf. [9, Lemma 2]) and may be employed as a test function:

$$(\psi_i \delta v_i, |v_i|^{p-2}v_i) + (v_i, |v_i|^{p-2}v_i)_A = -\left(g^R_i, |v_i|^{p-2}v_i\right)_{\partial G},\ i = 1, \ldots, n,\ \forall\, p \geq 2.$$

The application of

$$\begin{aligned}\left(\psi_i(v_i - v_{i-1}), |v_i|^{p-2}v_i\right) &= \|v_i\|^p_{p,[\psi_i]} - (\psi_i v_{i-1}, |v_i|^{p-2}v_i) \\ &\geq \|v_i\|^p_{p,[\psi_i]} - \|v_{i-1}\|_{p,[\psi_i]} \|v_i\|^{\frac{p}{p'}}_{p,[\psi_i]} \\ &\geq \frac{1}{p}\|v_i\|^p_{p,[\psi_i]} - \frac{1}{p}\|v_{i-1}\|^p_{p,[\psi_i]},\ \forall\, p \geq 2,\end{aligned}$$

and (1) to the left hand side of this equation yields

$$\|v_i\|^p_{p,[\psi_i]} - \|v_{i-1}\|^p_{p,[\psi_i]} + ch_n\|w_i\|^2_{\nabla,2} \leq -ph_n\left(g^R_i, |v_i|^{p-2}v_i\right)_{\partial G}, \qquad i = 1, \ldots, n,\ \forall\, p \geq 2, \tag{9}$$

where $w_i \in W^1_2(G) \cap L_\infty(G)$ is defined as $w_i := |v_i|^{\frac{p-2}{2}} v_i$. According to Corollary 1.14, the two estimates

$$\begin{aligned}\|v_i\|^p_{p,[\psi_i]} &\geq \|v_i\|^p_{p,[\psi_{i+1}]} - c\epsilon h_n \|w_i\|^2_{1,2} - c\epsilon^{-c} h_n \|v_i\|^p_{\lambda p,[\psi_i+\psi_{i+1}]}, \\ -\|v_{i-1}\|^p_{p,[\psi_i]} &\geq -\|v_{i-1}\|^p_{p,[\psi_{i-1}]} - c\epsilon h_n \|w_{i-1}\|^2_{1,2} - c\epsilon^{-c} h_n \|v_{i-1}\|^p_{\lambda p,[\psi_{i-1}+\psi_i]}, \\ &\qquad \forall\, p \geq 2,\ \forall\, \epsilon > 0,\end{aligned}$$

hold for an arbitrary, but fixed real number $\lambda \in (\lambda_*, 1]$. Applying them separately to formula (9), we obtain both

$$
\begin{aligned}
&\|\upsilon_i\|^p_{p,[\psi_{i+1}]} - \|\upsilon_{i-1}\|^p_{p,[\psi_i]} + ch_n\|w_i\|^2_{\nabla,2} \\
&\quad \le -ph_n\left(g_i^R, |\upsilon_i|^{p-2}\upsilon_i\right)_{\partial G} + c\epsilon h_n\|w_i\|^2_{1,2} \\
&\qquad + \ c\epsilon^{-c}h_n\|\upsilon_i\|^p_{\lambda p,[\psi_i+\psi_{i+1}]}, \ i=1,\ldots,n, \ \forall\, p\ge 2, \ \forall\,\epsilon>0,
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
&\|\upsilon_i\|^p_{p,[\psi_i]} - \|\upsilon_{i-1}\|^p_{p,[\psi_{i-1}]} + ch_n\|w_i\|^2_{\nabla,2} \\
&\quad \le -ph_n\left(g_i^R, |\upsilon_i|^{p-2}\upsilon_i\right)_{\partial G} + c\epsilon h_n\|w_{i-1}\|^2_{1,2} \\
&\qquad + \ c\epsilon^{-c}h_n\|\upsilon_{i-1}\|^p_{\lambda p,[\psi_{i-1}+\psi_i]}, \ i=1,\ldots,n, \ \forall\, p\ge 2, \ \forall\,\epsilon>0.
\end{aligned}
$$

The sum of these two inequalities reads as follows:

$$
\begin{aligned}
&\|\upsilon_i\|^p_{p,[\psi_i+\psi_{i+1}]} + ch_n\|w_i\|^2_{\nabla,2} \\
&\quad \le \|\upsilon_{i-1}\|^p_{p,[\psi_{i-1}+\psi_i]} + ph_n\left|\left(g_i^R, |\upsilon_i|^{p-2}\upsilon_i\right)_{\partial G}\right| \\
&\qquad + c\epsilon h_n\left(\|w_{i-1}\|^2_{1,2} + \|w_i\|^2_{1,2}\right) \\
&\qquad + \ c\epsilon^{-c}h_n\left(\|\upsilon_{i-1}\|^p_{\lambda p,[\psi_{i-1}+\psi_i]} + \|\upsilon_i\|^p_{\lambda p,[\psi_i+\psi_{i+1}]}\right), \\
&\qquad\qquad i=1,\ldots,n, \ \forall\, p\ge 2, \ \forall\,\epsilon>0.
\end{aligned} \tag{11}
$$

As a consequence of Corollary 1.13 (E_2) and our assumption $r > N-1$, the estimate

$$
\begin{aligned}
&\|w\|^{\frac{2}{p'}}_{2\frac{r'}{p'},\partial G} \le c\epsilon\|w\|^2_{1,2} + c\epsilon^{-c}\|w\|^{2\frac{1-\theta}{p'-\theta}}_{2\lambda,[\psi(t,\cdot)]}, \ \theta\in(0,1), \\
&\qquad\qquad \forall\, w\in W_2^1(G)\cap L_\infty(G), \ \forall\,\epsilon>0, \ \forall\, t\in I_T,
\end{aligned}
$$

holds for all $p \ge 2r/(r+1)$. Thus, we have the inequality

$$
\begin{aligned}
&\left|\left(g_i^R, |\upsilon_i|^{p-2}\upsilon_i\right)_{\partial G}\right| \le \left\|g_i^R\right\|_{r,\partial G}\left\||\upsilon_i|^{p-1}\right\|_{r',\partial G} \le c\|w_i\|^{\frac{2}{p'}}_{2\frac{r'}{p'},\partial G} \\
&\qquad \le c\epsilon\|w_i\|^2_{1,2} + c\epsilon^{-c}\|\upsilon_i\|^{\varrho(p)p}_{\lambda p,[\psi_i]}, \ \varrho(p) := \frac{1-\theta}{p'-\theta}, \\
&\qquad\qquad i=1,\ldots,n, \ \forall\, p\ge 2, \ \forall\,\epsilon>0.
\end{aligned} \tag{12}
$$

Using Lemma 1.15, its application to the right hand side of (11) yields

$$\begin{aligned}
&\|\upsilon_i\|^p_{p,[\psi_i+\psi_{i+1}]} + ch_n\|w_i\|^2_{\nabla,2} \\
&\quad\le \|\upsilon_{i-1}\|^p_{p,[\psi_{i-1}+\psi_i]} + c\epsilon ph_n\|w_i\|^2_{1,2} + c\epsilon^{-c}ph_n\|\upsilon_i\|^{\varrho(p)p}_{\lambda p,[\psi_i]} \\
&\qquad + c\epsilon h_n\left(\|w_{i-1}\|^2_{1,2} + \|w_i\|^2_{1,2}\right) \\
&\qquad + c\epsilon^{-c}h_n\left(\|\upsilon_{i-1}\|^p_{\lambda p,[\psi_{i-1}+\psi_i]} + \|\upsilon_i\|^p_{\lambda p,[\psi_i+\psi_{i+1}]}\right) \\
&\quad\le \|\upsilon_{i-1}\|^p_{p,[\psi_{i-1}+\psi_i]} + c\epsilon ph_n\left(\|w_{i-1}\|^2_{\nabla,2} + \|w_i\|^2_{\nabla,2}\right) \\
&\qquad + c\epsilon^{-c}ph_n\|\upsilon_i\|^{\varrho(p)p}_{\lambda p,[\psi_i+\psi_{i+1}]} \\
&\qquad + \ c\epsilon^{-c}h_n\left(\|\upsilon_{i-1}\|^p_{\lambda p,[\psi_{i-1}+\psi_i]} + \|\upsilon_i\|^p_{\lambda p,[\psi_i+\psi_{i+1}]}\right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad i=1,\dots,n,\ \forall\, p\ge 2,\ \forall\,\epsilon>0.
\end{aligned}$$

We sum up these estimates for $i = 2, \dots, j$, $j \in \{2, \dots, n\}$, and obtain

$$\begin{aligned}
&\|\upsilon_j\|^p_{p,[\psi_j+\psi_{j+1}]} + ch_n\sum_{i=2}^{j}\|w_i\|^2_{\nabla,2} \\
&\quad\le \|\upsilon_1\|^p_{p,[\psi_1+\psi_2]} + c\epsilon ph_n\sum_{i=1}^{j}\|w_i\|^2_{\nabla,2} \\
&\qquad + \ c\epsilon^{-c}ph_n\sum_{i=1}^{j}\left(\|\upsilon_i\|^p_{\lambda p,[\psi_i+\psi_{i+1}]} + \|\upsilon_i\|^{\varrho(p)p}_{\lambda p,[\psi_i+\psi_{i+1}]}\right),\ \forall\, p\ge 2.
\end{aligned}$$

Now the both formulas (9) and (10) are considered for the case when $i = 1$. In virtue of (12) and Lemma 1.15, their sum may be estimated as follows:

$$\begin{aligned}
&\|\upsilon_1\|^p_{p,[\psi_1+\psi_2]} + ch_n\|w_1\|^2_{\nabla,2} \\
&\quad\le \|\upsilon_0\|^p_{p,[2\psi_1]} + 2ph_n\left|\left(g_1^R, |\upsilon_1|^{p-2}\upsilon_1\right)_{\partial G}\right| + c\epsilon h_n\|w_1\|^2_{1,2} \\
&\qquad + c\epsilon^{-c}h_n\|\upsilon_1\|^p_{\lambda p,[\psi_1+\psi_2]} \\
&\quad\le \|\upsilon_0\|^p_{p,[2\psi_1]} + c\epsilon ph_n\|w_1\|^2_{\nabla,2} + c\epsilon^{-c}ph_n\left(\|\upsilon_1\|^p_{\lambda p,[\psi_1+\psi_2]} + \|\upsilon_1\|^{\varrho(p)p}_{\lambda p,[\psi_1+\psi_2]}\right).
\end{aligned}$$

Consequently, from the previous inequality, we find

$$\|\upsilon_j\|^p_{p,[\psi_j+\psi_{j+1}]} + ch_n \sum_{i=1}^{j} \|w_i\|^2_{\nabla,2}$$
$$\leq \|\upsilon_0\|^p_{p,[2\psi_1]} + c\epsilon p h_n \sum_{i=1}^{j} \|w_i\|^2_{\nabla,2}$$
$$+ \ c\epsilon^{-c} p h_n \sum_{i=1}^{j} \left(\|\upsilon_i\|^p_{\lambda p,[\psi_i+\psi_{i+1}]} + \|\upsilon_i\|^{\varrho(p)p}_{\lambda p,[\psi_i+\psi_{i+1}]} \right),$$
$$j = 1,\ldots,n,\ \forall\, p \geq 2,\ \forall\, \epsilon > 0.$$

By choosing $\epsilon := \delta/p$ with a sufficiently small $\delta > 0$, we obtain
(13)

$$\|\upsilon_j\|^p_{p,[\psi_j+\psi_{j+1}]} \leq \|\upsilon_0\|^p_{p,[2\psi_1]} + cp^c h_n \sum_{i=1}^{j} \left(\|\upsilon_i\|^p_{\lambda p,[\psi_i+\psi_{i+1}]} + \|\upsilon_i\|^{\varrho(p)p}_{\lambda p,[\psi_i+\psi_{i+1}]} \right)$$
$$\leq 2\|\psi\|_{C(I_T,L_1(G))} \|U_0^*\|^p_\infty + cp^c h_n \sum_{i=1}^{j} \left(\|\upsilon_i\|^p_{\lambda p,[\psi_i+\psi_{i+1}]} + \|\upsilon_i\|^{\varrho(p)p}_{\lambda p,[\psi_i+\psi_{i+1}]} \right)$$
$$\leq M\|U_0^*\|^p_\infty + cp^c t_j \left(\max_{i\leq j} \|\upsilon_i\|^p_{\lambda p,[\psi_i+\psi_{i+1}]} + \max_{i\leq j} \|\upsilon_i\|^{\varrho(p)p}_{\lambda p,[\psi_i+\psi_{i+1}]} \right),$$
$$M := 2\|\psi\|_{C(I_T,L_1(G))},\ j = 1,\ldots,n,\ \forall\, p \geq 2,$$

which leads to

$$\max_{t_i\leq t} \|\upsilon_i\|^p_{p,[\psi_i+\psi_{i+1}]}$$
$$\leq M\|U_0^*\|^p_\infty + cp^c t \left(\max_{t_i\leq t} \|\upsilon_i\|^p_{\lambda p,[\psi_i+\psi_{i+1}]} + \max_{t_i\leq t} \|\upsilon_i\|^{\varrho(p)p}_{\lambda p,[\psi_i+\psi_{i+1}]} \right),$$
$$\forall\, t \in I_T,\ \forall\, p \geq 2.$$

On the basis of this inequality, the norm $\|\upsilon_i\|_{p,[\psi_i+\psi_{i+1}]}$, $p \geq 2$, may be estimated by $\|\upsilon_i\|_{2,[\psi_i+\psi_{i+1}]}$. For that purpose, we consider the sequence $p_k := 2\lambda^{-k}$, $k = 0,1,2,\ldots$. Then, using the notations

$$m_k(n,t) := M^{-1/p_k} \max_{t_i\leq t} \|\upsilon_i\|_{p_k,[\psi_i+\psi_{i+1}]},\ \ \varrho_k := \varrho(p_k),$$

the previous inequality can be written in the form

$$m_k(n,t) \leq \left\{ \|U_0^*\|^{p_k}_\infty + M^{(1-\lambda)/\lambda} cp_k^c t\, m^{p_k}_{k-1}(n,t) + M^{(\rho_k-\lambda)/\lambda} cp_k^c t\, m^{\rho_k p_k}_{k-1}(n,t) \right\}^{1/p_k}$$
$$\leq \left\{ \|U_0^*\|^{p_k}_\infty + cp_k^c t \left[m^{p_k}_{k-1}(n,t) + m^{\rho_k p_k}_{k-1}(n,t) \right] \right\}^{1/p_k},\ k = 1,2,\ldots,\ \forall\, t \in I_T.$$

As it was shown in [12, Proof of Theorem 3.1] or [17, Hilfssatz 2.13], carrying out this recursion yields

$$m_k(n,t) \le \exp(ct^c) \max\left\{\|U_0^*\|_\infty, m_0(n,t)\right\}^{2\alpha}, \ k=1,2,\ldots, \ \forall\, t \in I_T, \tag{14}$$

where $\alpha \in \mathbb{R}$ belongs to $(0,1/2]$ and takes on the value $1/2$ if $\|U_0^*\|_\infty > 0$. Now the expression $m_0(n,t)$ will be estimated on the basis of formula (13), which is considered for the case where $p=2$. Owing to this inequality, the following holds

$$\begin{aligned}\|\upsilon_j\|^2_{2,[\psi_j+\psi_{j+1}]} &\le M\|U_0^*\|^2_\infty + ch_n \sum_{i=1}^{j} \left(\|\upsilon_i\|^2_{2,[\psi_i+\psi_{i+1}]} + \|\upsilon_i\|^{2\varrho(2)}_{2,[\psi_i+\psi_{i+1}]}\right) \\ &\le M\|U_0^*\|^2_\infty + ct_j + ch_n \sum_{i=1}^{j} \|\upsilon_i\|^2_{2,[\psi_i+\psi_{i+1}]}, \ j=1,\ldots,n.\end{aligned}$$

By means of Gronwall's Lemma in the discrete form (cf. [6, Lemma 1.3.19]) we consequently obtain

$$\|\upsilon_j\|^2_{2,[\psi_j+\psi_{j+1}]} \le (1+ch_n)\left(M\|U_0^*\|^2_\infty + ct_j\right)\exp(ct_{j-1}), \ j=1,\ldots,n,$$

so that $m_0(n,t)$ may be estimated by

$$\begin{aligned}m_0(n,t) = M^{-\frac{1}{2}} \max_{t_i\le t} \|\upsilon_i\|_{2,[\psi_i+\psi_{i+1}]} \le \left[(1+ch_n)\left(\|U_0^*\|^2_\infty + ct\right)\exp(ct)\right]^{\frac{1}{2}}, \\ \forall\, t \in I_T.\end{aligned}$$

Therefore, from (14) it results

$$\begin{aligned}M^{-\frac{1}{p_k}} \max_{t_i\le t} \|\upsilon_i\|_{p_k,[\psi_i+\psi_{i+1}]} &\le \exp\left(ct^c\right)\left[(1+ch_n)\left(\|U_0^*\|^2_\infty + ct\right)\exp(ct)\right]^{\alpha} \\ &\le \exp\left(ct^c\right)\left(\|U_0^*\|^2_\infty + ct\right)^{\alpha}, \ \forall\, k\in\mathbb{N}, \ \forall\, t\in I_T.\end{aligned}$$

Since the right hand side of this inequality does not depend on p_k, and

$$\begin{aligned}\lim_{p\to\infty} \|u\|_{p,[\psi(t',\cdot)+\psi(t'',\cdot)]} = \|u\|_{\infty,\mathcal{P}(t')\cup\mathcal{P}(t'')}, \\ \forall\, u \in L_\infty(\mathcal{P}(t')\cup\mathcal{P}(t'')), \ \forall\, t',t'' \in I_T,\end{aligned}$$

taking the limit as $p_k \to \infty$ yields

$$\begin{aligned}\max_{t_i\le t} \|\upsilon_i\|_{C\left(\overline{\mathcal{P}(t_i)\cup\mathcal{P}(t_{i+1})}\right)} &= \max_{t_i\le t} \|\upsilon_i\|_{\infty,\mathcal{P}(t_i)\cup\mathcal{P}(t_{i+1})} \\ &\le \exp\left(ct^c\right)\left(\|U_0^*\|^2_\infty + ct\right)^{\alpha}, \ \forall\, t \in I_T.\end{aligned}$$

Moreover, as $\partial\mathcal{E}(t_i)$ is contained in $\overline{\mathcal{P}(t_i)\cup\mathcal{P}(t_{i+1})}$, according to the weak maximum-minimum principle formulated in Lemma 1.18, we obtain

$$\max_{t_i\le t}\|v_i\|_{C(\bar G)}\le \max_{t_i\le t}\|v_i\|_{C\left(\overline{\mathcal{P}(t_i)\cup\mathcal{P}(t_{i+1})}\right)}.$$

Thus, our assumption $\|U_0\|_{\infty,\mathcal{P}(0)}=\|U_0^*\|_\infty<R$ enables us to fix up a point $T_*\in(0,T]$ such that

$$\max_{t_i\le t}\|v_i\|_{C(\bar G)}\le \exp\left(ct^c\right)\left(\|U_0^*\|_\infty^2+ct\right)^\alpha\le R,\ \forall\, t\in[0,T_*].$$

So the functions v_i defined on $t_i\times G$, $t_i\le T_*$, belong to $B_R[C(\partial G)]$ and, consequently, satisfy the corresponding (original) discretized equations (7.i).

Since any solution of (7.i) fulfills the “extended” discretized equation (8.i) as well, its uniqueness follows from Lemma 2.3. So our proof is complete. □

Theorem 2.4 guarantees the weak solvability of the discretized equations (7.i) up to the point $T_*\in(0,T]$, which does not depend upon the subdivision of the time interval I_T. Throughout the remainder of this paper, the greatest $i\in\mathbb{N}$ with $t_i=ih_n\le T_*$ will be denoted by $i_*=i_*(n)$. By piecewise linear or constant extension of the solutions u_i, $i\le i_*(n)$, respectively, for each $n\ge n_*$ we obtain the Rothe approximations

$$u^{(n)}(t,x):=\begin{cases} u_{i-1}(x)+(t-t_{i-1})\,\delta u_i(x) & \forall\, t\in[t_{i-1},t_i],\ 1\le i\le i_*\\ u_{i_*}(x)+(t-t_{i_*})\,\delta u_{i_*}(x) & \forall\, t\in[t_{i_*},T_*]\end{cases},$$

$$\bar u^{(n)}(t,x):=\begin{cases} U_0(x) & \forall\, t\in[-h_n,0]\\ u_i(x) & \forall\, t\in(t_{i-1},t_i],\ 1\le i\le i_*\ ,\\ u_{i_*}(x) & \forall\, t\in[t_{i_*},T_*]\end{cases}$$

which are defined on $Q_{T_*}:=I_{T_*}\times G$. Owing to Theorem 2.4 they satisfy the estimates

(15)
$$\left\|u^{(n)}(t,\cdot)\right\|_{C(\bar G)}\le \exp\left(ct^c\right)\left(\|U_0\|_{\infty,\mathcal{P}(0)}^2+ct\right)^\alpha\le R,\ \forall\, t\in[0,T_*-h_n],$$

(16)
$$\left\|\bar u^{(n)}(t,\cdot)\right\|_{C(\bar G)}\le \exp\left(ct^c\right)\left(\|U_0\|_{\infty,\mathcal{P}(0)}^2+ct\right)^\alpha\le R,\ \forall\, t\in[-h_n,T_*].$$

Moreover, we introduce the functions

$$\bar g^{(n)}(t,x):=\begin{cases} g_0(x)=g(0,x,U_0(x)), & t=0\\ g_i(x), & \forall\, t\in(t_{i-1},t_i],\ 1\le i\le i_*\ ,\\ g_{i_*}(x), & \forall\, t\in[t_{i_*},T_*]\end{cases}$$

$$\bar\psi^{(n)}(t,x):=\begin{cases} \psi_0(x), & t=0\\ \psi_i(x), & \forall\, t\in(t_{i-1},t_i],\ 1\le i\le i_*\ ,\\ \psi_{i_*}(x), & \forall\, t\in[t_{i_*},T_*]\end{cases}$$

so that the solved discretized equations (7.i), $i \le i_*(n)$, may be extended to

$$\left(\bar{\psi}^{(n)}(t,\cdot)\frac{\partial u^{(n)}}{\partial t}(t,\cdot), v(t,\cdot)\right) + \left(\bar{u}^{(n)}(t,\cdot), v(t,\cdot)\right)_A$$
$$= -\left(\bar{g}^{(n)}(t,\cdot), v(t,\cdot)\right)_{\partial G}, \ \forall\, t \in I_{T_*}, \ \forall\, v \in V(Q_{T_*}).$$

By integrating this formula over I_{T_*} the following statement results:

Approximation Scheme 2.5. *For all $n \ge n_*$, the functions $u^{(n)}$ and $\bar{u}^{(n)}$ fulfil the integral relation*

$$(17) \quad -\left(u^{(n)}, \frac{\partial}{\partial t}(\psi v)\right)_{Q_{T_*}} - (\psi(0,\cdot)U_0, v(0,\cdot)) + \left(\left(\bar{\psi}^{(n)} - \psi\right)\frac{\partial u^{(n)}}{\partial t}, v\right)_{Q_{T_*}}$$
$$+ \int_{I_{T_*}} \left(\bar{u}^{(n)}(t,\cdot), v(t,\cdot)\right)_A \mathrm{d}t = -\left(\bar{g}^{(n)}, v\right)_{\Gamma_{T_*}}, \ \forall\, v \in V(Q_{T_*}).$$

3. The convergence of the approximations to a solution

By a limit process in Approximation Scheme 2.5 we will show that subsequences of $\{u^{(n)}\}_{n=1}^\infty$ and $\{\bar{u}^{(n)}\}_{n=1}^\infty$ actually approach a weak solution to Problem 1.1. The derivation of appropriate convergence statements requires various a priori estimates which are based on the following lemma:

Lemma 3.1. *For all subdivisions of I_T with $n \ge n_*$, the solutions $u_i \in W_q^1(G)$ of the discretized equations* (7.i), *$i \le i_*(n)$, satisfy the estimates*

$$(\mathrm{E}_1) \qquad h_n \sum_{t_i \le T_*} \|u_i\|_{1,2}^2 \le c, \ \ h_n^2 \sum_{t_1 \le t_i \le T_*} \|\delta u_i\|_{2,[\psi_i]}^2 \le c, \qquad \textit{and}$$

$$(\mathrm{E}_2) \qquad h_n \sum_{t_j \le T_* - t_k} \|u_{j+k} - u_j\|_{2,[\psi_j]}^2 \le ckh_n = ct_k, \ \forall\, k \in \{0,1,\ldots,i_*(n)\}.$$

Proof: In order to show the estimates stated in (E_1) are satisfied, we first consider the discretized equations (7.i), $i \le i_*(n)$, with $v = u_i$ as test functions. The application of

$$(\psi_i(u_i - u_{i-1}), u_i) = \frac{1}{2}\left(\|u_i\|_{2,[\psi_i]}^2 - \|u_{i-1}\|_{2,[\psi_i]}^2 + \|u_i - u_{i-1}\|_{2,[\psi_i]}^2\right)$$

and (1) to the left hand side of this formula yields

$$\|u_i\|_{2,[\psi_i]}^2 - \|u_{i-1}\|_{2,[\psi_i]}^2 + \|u_i - u_{i-1}\|_{2,[\psi_i]}^2 + ch_n\|u_i\|_{\nabla,2}^2 \le -2h_n(g_i, u_i)_{\partial G},$$
$$i = 1, \ldots, i_*(n).$$

Going through the same steps which led from (9) to (10) (with $p = 2$ and $\lambda = 1$) and using Lemma 1.15, we obtain

$$\begin{aligned}&\|u_i\|^2_{2,[\psi_{i+1}]} + h_n^2\|\delta u_i\|^2_{2,[\psi_i]} + ch_n\|u_i\|^2_{\nabla,2}\\ &\quad\le \|u_{i-1}\|^2_{2,[\psi_i]} + 2h_n\,|(g_i,u_i)_{\partial G}| + c\epsilon h_n\|u_i\|^2_{\nabla,2} + c\epsilon^{-c}h_n\|u_i\|^2_{2,[\psi_i+\psi_{i+1}]},\\ &\hspace{20em} i = 1,\dots,i_*(n),\ \forall\,\epsilon > 0.\end{aligned}$$

Since the functions u_i, $i \le i_*(n)$, belong to the closed ball $B_R[C(\bar G)]$, the integrals $(g_i,u_i)_{\partial G}$ are uniformly bounded. Consequently we obtain the inequality

$$\begin{aligned}&\|u_i\|^2_{2,[\psi_{i+1}]} + h_n^2\|\delta u_i\|^2_{2,[\psi_i]} + ch_n\|u_i\|^2_{\nabla,2}\\ &\qquad\le \|u_{i-1}\|^2_{2,[\psi_i]} + ch_n + c\epsilon h_n\|u_i\|^2_{\nabla,2} + c\epsilon^{-c}h_n\|u_i\|^2_{2,[\psi_i+\psi_{i+1}]},\\ &\hspace{20em} i = 1,\dots,i_*(n),\ \forall\,\epsilon > 0,\end{aligned}$$

which will be summed up for $i = 1,\dots,j$, $j \in \{1,\dots,i_*(n)\}$:

$$\begin{aligned}&\|u_j\|^2_{2,[\psi_{j+1}]} + h_n^2\sum_{i=1}^{j}\|\delta u_i\|^2_{2,[\psi_i]} + ch_n\sum_{i=1}^{j}\|u_i\|^2_{\nabla,2}\\ &\hspace{10em}\le \|u_0\|^2_{2,[\psi_1]} + c\epsilon h_n\sum_{i=1}^{j}\|u_i\|^2_{\nabla,2} + c(\epsilon)t_j,\ \forall\,\epsilon > 0.\end{aligned}$$

Our assertion (E_1) follows from this estimate by choosing $\epsilon > 0$ sufficiently small.

The proof of (E_2) is also based on the discretized equations (7.i), $i \le i_*(n)$. Using $v = u_{j+k} - u_j$, $0 \le j \le j+k \le i_*$, as test function, we sum them up for $i = j+1,\dots,j+k$. In view of the identity

$$\begin{aligned}&\sum_{i=j+1}^{j+k}\big(\psi_i(u_i-u_{i-1}),u_{j+k}-u_j\big)\\ &\quad= \sum_{i=j+1}^{j+k}\big(\psi_iu_i-\psi_{i-1}u_{i-1},u_{j+k}-u_j\big) - \sum_{i=j+1}^{j+k}\big((\psi_i-\psi_{i-1})u_{i-1},u_{j+k}-u_j\big)\\ &\quad= \big(\psi_{j+k}u_{j+k}-\psi_ju_j,u_{j+k}-u_j\big) - \sum_{i=j+1}^{j+k}\big((\psi_i-\psi_{i-1})u_{i-1},u_{j+k}-u_j\big)\\ &\quad= \big\|u_{j+k}-u_j\big\|^2_{2,[\psi_j]} + \big((\psi_{j+k}-\psi_j)u_{j+k},u_{j+k}-u_j\big)\\ &\qquad - \sum_{i=j+1}^{j+k}\big((\psi_i-\psi_{i-1})u_{i-1},u_{j+k}-u_j\big),\end{aligned}$$

we have

$$\|u_{j+k}-u_j\|^2_{2,[\psi_j]} + h_n \sum_{i=j+1}^{j+k} (u_i, u_{j+k}-u_j)_A = -h_n \sum_{i=j+1}^{j+k} (g_i, u_{j+k}-u_j)_{\partial G}$$
$$+ \sum_{i=j+1}^{j+k} \big((\psi_i-\psi_{i-1})u_{i-1}, u_{j+k}-u_j\big) - \big((\psi_{j+k}-\psi_j)u_{j+k}, u_{j+k}-u_j\big)\,.$$

Since the functions u_i, $i \le i_*(n)$, belong to $B_R[C(\bar{G})]$, an application of Hölder's inequality to the right hand side leads to

$$\|u_{j+k}-u_j\|^2_{2,[\psi_j]} \le h_n \sum_{i=j+1}^{j+k} \big|(g_i, u_{j+k}-u_j)_{\partial G}\big|$$
$$+ \sum_{i=j+1}^{j+k} \big|\big((\psi_i-\psi_{i-1})u_{i-1}, u_{j+k}-u_j\big)\big|$$
$$+ \big|\big((\psi_{j+k}-\psi_j)u_{j+k}, u_{j+k}-u_j\big)\big| + h_n \sum_{i=j+1}^{j+k} \big|(u_i, u_{j+k}-u_j)_A\big|$$
$$\le ch_n \sum_{i=j+1}^{j+k} \|g_i\|_{r,\partial G} + c \sum_{i=j+1}^{j+k} \|\psi_i-\psi_{i-1}\|_\kappa + c\|\psi_{j+k}-\psi_j\|_\kappa$$
$$+ h_n \sum_{i=j+1}^{j+k} \big|(u_i, u_{j+k}-u_j)_A\big|\,.$$

In virtue of the local boundedness of $g(\cdot,\cdot,\xi)$, and the assumption $\psi \in C^{0,1}(I_T, L_\kappa(G))$, it follows that

$$\|u_{j+k}-u_j\|^2_{2,[\psi_j]} \le ckh_n + h_n \sum_{i=j+1}^{j+k} \big|(u_i, u_{j+k}-u_j)_A\big|$$
$$\le ckh_n + ch_n \sum_{i=j+1}^{j+k} \|u_i\|_{\nabla,2}\|u_{j+k}-u_j\|_{\nabla,2}$$
$$\le ckh_n + ckh_n\|u_{j+k}\|^2_{\nabla,2} + ckh_n\|u_j\|^2_{\nabla,2} + ch_n \sum_{i=j+1}^{j+k} \|u_i\|^2_{\nabla,2}\,.$$

Summing up this formula for $j = 0, 1, \ldots, i_*(n) - k$, we obtain

$$\sum_{j=0}^{i_*-k} \|u_{j+k} - u_j\|^2_{2,[\psi_j]}$$

$$\le ck(i_* - k + 1)h_n + ckh_n \sum_{j=0}^{i_*} \|u_j\|^2_{\nabla,2} + ch_n \sum_{j=0}^{i_*-k} \sum_{i=j+1}^{j+k} \|u_i\|^2_{\nabla,2} \,.$$

So, with consideration to

$$\sum_{j=0}^{i_*-k} \sum_{i=j+1}^{j+k} \|u_i\|^2_{\nabla,2} = \sum_{j=0}^{i_*-k} \sum_{i=1}^{k} \|u_{i+j}\|^2_{\nabla,2} = \sum_{i=1}^{k} \sum_{j=0}^{i_*-k} \|u_{i+j}\|^2_{\nabla,2}$$

$$\le \sum_{i=1}^{k} \sum_{j=0}^{i_*} \|u_j\|^2_{\nabla,2} \le k \sum_{j=0}^{i_*} \|u_j\|^2_{\nabla,2} \,,$$

we have the inequality

$$\sum_{j=0}^{i_*-k} \|u_{j+k} - u_j\|^2_{2,[\psi_j]} \le ck + ckh_n \sum_{j=0}^{i_*} \|u_j\|^2_{\nabla,2} \,,$$

which proves our assertion (E_2) since (E_1) guarantees the boundedness of $h_n \sum_{j=0}^{i_*} \|u_j\|^2_{\nabla,2}$. □

Corollary 3.2. *Let $\nu = 2\beta/(1+\beta)$. Then for all $n \ge n_*$ the functions $u^{(n)} \in C\left(I_{T_*}, C\left(\bar{G}\right)\right)$ and $\bar{u}^{(n)} \in L_\infty\left(I_{T_*}, C\left(\bar{G}\right)\right)$ satisfy the estimates*

(E_1) $\displaystyle\int_{I_{T_*}} \left\|u^{(n)}(t,\cdot) - \bar{u}^{(n)}(t,\cdot)\right\|^2_{\nu,\mathcal{P}(t)} \,\mathrm{d}t \le ch_n,$

(E_2) $\displaystyle\int_{I_{T_*}} \left\|\bar{u}^{(n)}(t,\cdot) - \bar{u}^{(n)}(t-h_n,\cdot)\right\|^2_{\nu,\mathcal{P}(t)} \,\mathrm{d}t \le ch_n,$

(E_3) $\displaystyle\int_0^{T_*-\epsilon} \left\|\bar{u}^{(n)}(t+\epsilon,\cdot) - \bar{u}^{(n)}(t,\cdot)\right\|^2_{\nu,\mathcal{P}(t)} \,\mathrm{d}t \le c\epsilon,\ \forall\, \epsilon \in (0,T_*),$

(E_4) $\displaystyle\int_{I_{T_*}} \left\|u_t^{(n)}(t,\cdot)\right\|^2_{\nu,\mathcal{P}(t)} \,\mathrm{d}t \le ch_n^{-1},$

(E_5) $\displaystyle\left\|u^{(n)}\right\|_{L_2(I_{T_*},W_2^1(G))} \le c,\ \ \left\|\bar{u}^{(n)}\right\|_{L_2(I_{T_*},W_2^1(G))} \le c.$

Proof: Let $t \in (0,T_*)$ be an arbitrary point of time, which belongs to the subinterval $(t_{j-1}, t_j]$. Then, owing to formula (3) with $p = 2$, Corollary 1.14 with

$\lambda = 1$, and Theorem 2.4 the following holds:

$$\begin{aligned}
\|u_{j+k} - u_j\|^2_{\nu,\mathcal{P}(t)} &\le c\|u_{j+k} - u_j\|^2_{2,[\psi(t,\cdot)]} \\
&\le c\|u_{j+k} - u_j\|^2_{2,[\psi_j]} + ch_n\|u_{j+k} - u_j\|^2_{1,2} \\
&\quad + ch_n\|u_{j+k} - u_j\|^2_{2,[\psi(t,\cdot)+\psi_j]} \\
&\le c\|u_{j+k} - u_j\|^2_{2,[\psi_j]} + ch_n\left(1 + \|u_j\|^2_{1,2} + \|u_{j+k}\|^2_{1,2}\right).
\end{aligned} \tag{18}$$

Since an arbitrary, but fixed real number $\epsilon \in (0, T_*)$ may be expressed as $\epsilon = t_{k-1} + \epsilon'$, where $k = k(n)$ depends on the subdivision n, and $\epsilon' \in \mathbb{R}$ satisfies the condition $0 < \epsilon' \le h_n$, in virtue of Lemma 3.1 we obtain

$$\begin{aligned}
&\int_0^{T_*-\epsilon} \left\|\bar{u}^{(n)}(t+\epsilon,\cdot) - \bar{u}^{(n)}(t,\cdot)\right\|^2_{\nu,\mathcal{P}(t)} \,\mathrm{d}t \\
&\le \sum_{j=1}^{i_*-k+1} \int_{t_{j-1}}^{t_j-\epsilon'} \|u_{j+k-1} - u_j\|^2_{\nu,\mathcal{P}(t)} \,\mathrm{d}t + \sum_{j=1}^{i_*-k} \int_{t_j-\epsilon'}^{t_j} \|u_{j+k} - u_j\|^2_{\nu,\mathcal{P}(t)} \,\mathrm{d}t \\
&\le \sum_{j=1}^{i_*-k+1} \int_{t_{j-1}}^{t_j-\epsilon'} \|u_{j+k-1} - u_j\|^2_{2,[\psi_j]} \,\mathrm{d}t + \sum_{j=1}^{i_*-k} \int_{t_j-\epsilon'}^{t_j} \|u_{j+k} - u_j\|^2_{2,[\psi_j]} \,\mathrm{d}t \\
&\quad + c\cdot\vartheta(k-1)\cdot h_n \sum_{j=1}^{i_*} \int_{t_{j-1}}^{t_j-\epsilon'} \left(1 + \|u_j\|^2_{1,2}\right) \mathrm{d}t + ch_n \sum_{j=1}^{i_*} \int_{t_j-\epsilon'}^{t_j} \left(1 + \|u_j\|^2_{1,2}\right) \mathrm{d}t \\
&\le c(k-1)(h_n - \epsilon') + ck\epsilon' + c[\vartheta(k-1)(h_n - \epsilon') + \epsilon'] \le c\epsilon,
\end{aligned}$$

with $\vartheta(0) := 0$ and $\vartheta(i) := 1 \; \forall\, i \in \mathbb{N},\ i \ge 1$.

Thus the assertion (E_3) is proved. Now, setting $k = 1$, we sum up (18) for $j = 1, \dots, i_*(n)$. With Lemma 3.1, this gives the estimate

$$h_n^2 \sum_{j=1}^{i_*} \|\delta u_j\|^2_{\nu,\mathcal{P}(t)} \le ch_n^2 \sum_{j=1}^{i_*} \|\delta u_j\|^2_{2,[\psi_j]} + ch_n \sum_{j=1}^{i_*} \left(1 + \|u_j\|^2_{1,2}\right) \le c,$$

which, in view of the definition of the functions $u^{(n)}$, $\bar{u}^{(n)}$, leads to the assertions (E_1), (E_2) and (E_4). Since (E_5) immediately follows from Lemma 3.1, our proof is complete. □

Now the main result of this paper can be formulated:

Theorem 3.3. *There are subsequences* $\{u^{(n_k)}\}_{k=1}^\infty \subseteq \{u^{(n)}\}_{n=n_*}^\infty$, $\{\bar{u}^{(n_k)}\}_{k=1}^\infty \subseteq \{\bar{u}^{(n)}\}_{n=n_*}^\infty$, *for which the following convergence properties hold:*

(C_0) $\{\bar{u}^{(n_k)}\}_{k=1}^\infty$ *is weakly convergent in* $L_2(I_{T_*}, W_2^1(G))$ *to a function* u.

(C_1) *Let p_* be an arbitrary, but fixed real number with $1 \le p_* < \infty$. Then both subsequences approach the restriction of $u \in L_2(I_{T_*}, W_2^1(G))$ to $\mathcal{P}_{T_*}$ in $L_{p_*}(\mathcal{P}_{T_*})$.*

(C_2) *Let p_* be an arbitrary, but fixed real number with $1 \le p_* < \infty$. Then the subsequences converge in $L_{p_*}(\Gamma_{T_*})$ to u.*

The limit function $u \in L_2(I_{T_*}, W_2^1(G))$ is a weak solution to the parabolic-elliptic initial boundary value Problem 1.1 in the sense of Definition 1.7.

PROOF: Our proof is subdivided in two sections. First the asserted convergence statements will be shown. On the basis of these properties the weak solvability of our Problem 1.1 can be proved by means of a limit process in the Approximation Scheme 2.5.

(a) According to Corollary 3.2 (E_5) the sequence $\{\bar{u}^{(n)}\}_{n=n_*}^{\infty}$ is bounded in $L_2(I_{T_*}, W_2^1(G))$. Thus, there is a subsequence $\{\bar{u}^{(n_k)}\}_{k=1}^{\infty}$, having the convergence property (C_0).

For simplicity's sake, the indices $\{n_k\}_{k=1}^{\infty}$ will be retained in all the subsequences throughout the remainder of this proof.

The derivation of our assertion (C_1) is based on the compactness criterion formulated in Lemma 1.16. Because of Corollary 3.2 (E_3), (E_5) its application leads to the following statement:

There exists a subsequence $\{\bar{u}^{(n_k)}\}_{k=1}^{\infty}$ which is convergent in $L_\nu(\mathcal{P}_{T_*})$ to a function υ. In view of (C_0) we may show by standard arguments that υ is the restriction of $u \in L_2(I_{T_*}, W_2^1(G))$ to $\mathcal{P}_{T_*}$. Due to Corollary 3.2 (E_1) $\{u^{(n_k)}\}_{k=1}^{\infty}$ tends to the same limit $u \in L_\nu(\mathcal{P}_{T_*})$.

On the basis of Lebesgue's theorem (on majorized convergence) these results can be extended to $L_{p_*}(\mathcal{P}_{T_*})$, $1 \le p_* < \infty$, as follows: As a consequence of their convergence in $L_\nu(\mathcal{P}_{T_*})$, $\{u^{(n_k)}\}_{k=1}^{\infty}$ and $\{\bar{u}^{(n_k)}\}_{k=1}^{\infty}$ contain subsequences $\{u^{(n_k)}(t,x)\}_{k=1}^{\infty}$, $\{\bar{u}^{(n_k)}(t,x)\}_{k=1}^{\infty}$, which tend to $u(t,x)$ pointwise almost everywhere on $\mathcal{P}_{T_*}$ (cf. e.g. [3, 2.8.1 Theorem (ii)]). Moreover, according to (16), the limit element u belongs to the closed ball $B_R[L_\infty(\mathcal{P}_{T_*})]$. Now we can see that the following conditions are satisfied:

(a) Almost everywhere on $\mathcal{P}_{T_*}$ the functions $\left|u^{(n_k)}(t,x) - u(t,x)\right|^{p_*}$, $\left|\bar{u}^{(n_k)}(t,x) - u(t,x)\right|^{p_*}$ are integrable and tend to zero as $k \to \infty$.

(c) According to the formulas (15), (16) they can be bounded by a constant almost everywhere on $\mathcal{P}_{T_*}$.

Therefore, the application of Lebesgue's theorem leads to

$$\lim_{k\to\infty} \left\|u^{(n_k)} - u\right\|_{p_*,\mathcal{P}_{T_*}}^{p_*} = \int_{\mathcal{P}_{T_*}} \lim_{k\to\infty} \left|u^{(n_k)}(t,x) - u(t,x)\right|^{p_*} \, dx \, dt = 0,$$

$$\lim_{k\to\infty} \left\|\bar{u}^{(n_k)} - u\right\|_{p_*,\mathcal{P}_{T_*}}^{p_*} = 0.$$

The convergence property (C_2) can be derived with the aid of Corollary 1.13 (E_1). According to the interpolation inequality which was formulated there, the functions $u^{(m,n)} := u^{(m)} - u^{(n)}$ satisfy

$$\left\|u^{(m,n)}(t,\cdot)\right\|_{\gamma_0,\partial G} \le c\left\|u^{(m,n)}(t,\cdot)\right\|_{1,2}^{1-\theta}\left\|u^{(m,n)}(t,\cdot)\right\|_{\gamma,\mathcal{P}(t)}^{\theta},\ \theta\in(0,1),\ \gamma>1,\quad \forall\, t\in I_{T_*},$$

where γ_0 is an arbitrary, but fixed real number with $1 < \gamma_0 < 2(N-1)/(N-2)$. Integrating this formula over I_{T_*}, we obtain

$$(19)\quad \left\|u^{(m,n)}\right\|_{L_2(I_{T_*},L_{\gamma_0}(\partial G))} \le \left(\left\|u^{(m)}\right\|_{L_2(I_{T_*},W_2^1(G))} + \left\|u^{(n)}\right\|_{L_2(I_{T_*},W_2^1(G))}\right)^{1-\theta} \times \times\left\{\int_{I_{T_*}}\left\|u^{(m,n)}(t,\cdot)\right\|_{\gamma,\mathcal{P}(t)}^2\right\}^{\frac{\theta}{2}}.$$

Therefore, (C_1) implies that $\left\{u^{(n_k)}\right\}_{k=1}^{\infty} \subset \left\{u^{(n)}\right\}_{n=n_*}^{\infty}$ approaches u in $L_2(I_{T_*},L_{\gamma_0}(\partial G))$, for Corollary 3.2 ($E_5$) guarantees the boundedness of this sequence in $L_2(I_{T_*},W_2^1(G))$. Analogously, we derive the same result for $\left\{\bar{u}^{(n)}\right\}_{n=n_*}^{\infty}$.

Now these convergence properties may be extended to $L_{p_*}(\Gamma_{T_*})$, $1\le p_*<\infty$, in the same way as in the proof of (C_1). On Γ_{T_*} (instead of $\mathcal{P}_{T_*}$) we duplicate the appropriate argumentation which is based on an application of Lebesgue's theorem, and obtain (C_2). From (16), it follows that $u\in B_R[L_\infty(\Gamma_{T_*})]$.

(b) Now it remains to show that for the subsequence $\{n_k\}_{k=1}^{\infty}\subseteq\{n\}_{n=n_*}^{\infty}$, Approximation Scheme 2.5 approaches the integral relation (4), and therefore, the function u weakly solves Problem 1.1. For that purpose, we have to derive two additional convergence properties.

Because of the boundedness of $\left\{\bar{g}^{(n)}\right\}_{n=n_*}^{\infty}$ in $L_\infty(I_{T_*},L_r(\partial G))$, it contains a subsequence $\left\{\bar{g}^{(n_k)}\right\}_{k=1}^{\infty}$ which tends to a function $\phi\in L_\infty(I_{T_*},L_r(\partial G))$ in the w^*-topology. In order to show that $\phi(t,x)$ equals $g(t,x,u(t,x))$ almost everywhere on Γ_{T_*}, we consider

$$\int_{I_{T_*}}\left\|\bar{u}^{(n)}(t-h_n,\cdot)-u(t,\cdot)\right\|_{2,\partial G}^2\,\mathrm{d}t \le 2\int_{I_{T_*}}\left\|\bar{u}^{(n)}(t-h_n,\cdot)-\bar{u}^{(n)}(t,\cdot)\right\|_{2,\partial G}^2\,\mathrm{d}t + 2\left\|\bar{u}^{(n)}-u\right\|_{2,\Gamma_{T_*}}^2.$$

Analogously to the formula (19), the first summand of the right hand side may be estimated by

$$\int_{I_{T_*}} \left\| \bar{u}^{(n)}(t-h_n,\cdot) - \bar{u}^{(n)}(t,\cdot) \right\|_{2,\partial G}^2 \mathrm{d}t$$
$$\leq c \left\{ \int_{I_{T_*}} \left\| \bar{u}^{(n)}(t-h_n,\cdot) - \bar{u}^{(n)}(t,\cdot) \right\|_{\frac{1}{\lambda},\mathcal{P}(t)}^2 \mathrm{d}t \right\}^{\theta}, \ \theta \in (0,1).$$

Therefore, with consideration to Corollary 3.2 (E_2), and the convergence property (C_1), we have

$$\int_{I_{T_*}} \left\| \bar{u}^{(n_k)}(t-h_n,\cdot) - u(t,\cdot) \right\|_{2,\partial G}^2 \mathrm{d}t \longrightarrow 0 \ \text{ as } \ n_k \to \infty.$$

Based on our assumptions on the function g, it may be easily shown that the Nemyckii operator $\mathcal{G}_*(v,u)(t,x) := g^R(v(t),x,u(t,x))$ defines a continuous mapping $\mathcal{G}_* : L_2(I_{T_*}) \times L_2(\Gamma_{T_*}) \longrightarrow L_r(\Gamma_{T_*})$ (cf. [17, Folgerung 1.28] or [18, Proposition 26.6]). Consequently, the subsequence $\{\bar{g}^{(n_k)}\}_{k=1}^{\infty}$ converges to $g(\cdot,\cdot,u)$ in $L_r(\Gamma_{T_*})$. By means of standard arguments this implies that $\phi(t,x)$ is equal to $g(t,x,u(t,x))$ almost everywhere on Γ_{T_*}, and thus

$$(20) \quad \int_{I_{T_*}} \left(\bar{g}^{(n_k)}(t,\cdot) - g(t,\cdot,u(t,\cdot)), v(t,\cdot) \right)_{\partial G} \mathrm{d}t \longrightarrow 0, \ \forall\, v \in L_1(I_{T_*}, L_{r'}(\partial G)).$$

Moreover, in virtue of the $\psi \in C^{0,1}(I_T, L_\kappa(G))$ and Corollary 3.2 (E_4) where $\nu > \kappa'$, we obtain

$$\left(\left(\bar{\psi}^{(n)} - \psi \right) u_t^{(n)}, v \right)_{\mathcal{P}_{T_*}}$$
$$\leq \int_{I_{T_*}} \left\| \bar{\psi}^{(n)}(t,\cdot) - \psi(t,\cdot) \right\|_{\kappa,\mathcal{P}(t)} \left\| u_t^{(n)}(t,\cdot) \right\|_{\nu,\mathcal{P}(t)} \|v(t,\cdot)\|_{\infty,\mathcal{P}(t)} \,\mathrm{d}t$$
$$\leq c h_n^{1/2} \left\{ \int_{T_*} \|v(t,\cdot)\|_{\infty,\mathcal{P}(t)}^2 \,\mathrm{d}t \right\}^{1/2} \longrightarrow 0, \quad \forall\, v \in L_2(I_{T_*}, L_\infty(\mathcal{P}(t))).$$

Now the convergence properties (C_1), (C_0), (20), and (21) enable us to carry out the limit process $n_k \to \infty$ in Approximation Scheme 2.5 for the subsequence $\{n_k\}_{k=1}^{\infty} \subseteq \{n\}_{n=n_*}^{\infty}$ and test functions $v \in V(Q_{T_*}) \cap L_2(I_{T_*}, L_\infty(\mathcal{P}(t)))$. Since $V(Q_{T_*}) \cap L_2(I_{T_*}, L_\infty(\mathcal{P}(t)))$ is dense in $V(Q_{T_*})$ this shows that the function u satisfies the integral equation (4), and, therefore, weakly solves the parabolic-elliptic Problem 1.1.

□

Remark 3.4. Uniqueness of a weak solution to Problem 1.1 can be proved by standard arguments, if, in addition to Assumption 1.6, $g(t,x,\xi)$ is Lipschitz-continuous with respect to (t,ξ).

Even without this additional assumption uniqueness can be shown, provided the solution is more regular than guaranteed by Theorem 3.3. Namely, if $u(t,\cdot)$ exists for all $t \in I_{T_*}$ in the sense of traces, then the space $V(Q_T)$ of test functions (cf. Definition 1.7) may be extended to $\tilde{V}(Q_T)$ by removing the restriction $v(T,x) \equiv 0$. Now the basic idea of the proof of uniqueness can be outlined as follows:

Let u_1, u_2 be weak solutions and $u = u_1 - u_2$. For almost all $t_0 \in I_{T_*}$ with $u(t_0,\cdot) \in W_2^1(G)$ we solve the Dirichlet problem

$$(22)\quad -(\psi\, v)_t + Av = 0 \ \text{ on } \ Q_{t_0}, \qquad v = 0 \ \text{ on } \ \Gamma_{t_0}, \qquad v(t_0,x) = R_\varepsilon u(t_0,x),$$

where $R_\varepsilon u$ is an appropriate approximation of u with $R_\varepsilon u|_{\partial G} = 0$. Employing the resulting weak solution $v \in \tilde{V}(Q_{t_0})$ as test function in (4), we arrive at

$$(\psi(t_0,\cdot)\, u(t_0,\cdot)\, , R_\varepsilon u(t_0,\cdot)) = 0\ .$$

It follows that $u(t_0,\cdot)|_{\mathcal{P}(t_0)} = 0$ as $\varepsilon \to 0$. In view of the weak maximum principle Lemma 1.18 we finally obtain $u(t_0,\cdot) = 0$ in G for almost all $t_0 \in I_{T_*}$, i.e. we have uniqueness.

Note that a weak solution to problem (22) exists if ψ_t is sufficiently small. This topic shall be addressed in a forthcoming paper of the first author.

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