On infinite dimensional uniform smoothness of Banach spaces

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Abstract. An infinite dimensional counterpart of uniform smoothness is studied. It does not imply reflexivity, but we prove that it gives some l_p-type estimates for finite dimensional decompositions, weak Banach-Saks property and the weak fixed point property.

Keywords: Banach space, nearly uniform smoothness, finite dimensional decomposition, Banach-Saks property, fixed point property

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Introduction

The notion of nearly uniform convexity for Banach spaces was introduced in [8]. It is an infinite dimensional counterpart of the classical uniform convexity. Independently an equivalent property appeared in [7]. The dual property was studied in [17] and [14]. The authors of the first paper called it noncompactly uniform smoothness (NUS in short). In the second paper the name nearly uniform smoothness was used. Let us recall that uniformly smooth spaces are NUS and NUS implies reflexivity.

In [17] the authors introduced also a weak version of NUS called NUS*. For reflexive spaces NUS* is equivalent to NUS. The space c_0 is an example of a NUS* space which is not NUS. NUS* was further considered in [1] and [2]. However the authors of these papers used the name NUS instead of NUS* which may lead to some confusion. In this paper we will follow the terminology of [17].

In [14] a characterization of NUS was given. We use a similar idea to find a characterization of NUS*. It allows us to establish some properties of NUS* spaces. For instance we prove that every finite dimensional decomposition in such a space with the decomposition constant close to one has a blocking which satisfies l_p-type estimates. Analyzing the special case of the space c_0, we show that every finite dimensional decomposition in this space with the decomposition constant less than $\frac{3}{2}$ is shrinking. We also prove that NUS* spaces have the weak Banach-Saks property and the weak fixed point property.

1. Basic definitions

Let X be a Banach space. By B_X and S_X we denote its closed unit ball and unit sphere respectively. Let us take an element $x \in S_X$ and a positive scalar δ. We put

$$S^*(x,\delta) = \{x^* \in B_{X^*} : x^*(x) \geq 1-\delta\}.$$

Let A be a bounded subset of X. Its Kuratowski measure of noncompactness $\alpha(A)$ is defined as the infimum of all numbers $d > 0$ such that A may be covered by finitely many set of diameters smaller than d (compare to [3]). Now we can recall a definition from [17]. A Banach space X is said to be NUS* provided that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in S_X$, then

$$\alpha(S^*(x,\delta)) \leq \epsilon.$$

In the sequel we will also need basic facts concerning finite dimensional decompositions. Let (X_n) be a sequence of finite dimensional subspaces of a Banach space X. It is called a finite dimensional decomposition (FDD in short) of X if each element $x \in X$ has a unique expansion

$$x = \sum_{n=1}^{\infty} x_n,$$

where $x_n \in X_n$ for every n. Then we put $S_n x = x_n$. This formula gives us a bounded linear projection S_n of X onto X_n. Moreover,

$$c = \sup\left\{\left\|\sum_{k=1}^{n} S_k\right\| : n \in \mathbb{N}\right\}$$

is finite (see [13] p. 47). The constant c is called the decomposition constant of (X_n). Clearly $c \geq 1$. Let x be a nonzero element of the space X. The element x is said to be a block of (X_n) if the set $D = \{n : S_n x \neq 0\}$ is finite. The interval

$$\operatorname{ran}(x) = \{n : \min D \leq n \leq \max D\}$$

is called the range of the block x. We say that (x_n) is a sequence of successive blocks if each x_n is a block and $\max \operatorname{ran}(x_k) < \min \operatorname{ran}(x_{k+1})$ for every k.

Let us fix $p \geq 1$. We say that an FDD (X_n) satisfies p-estimates provided that there is a constant C such that

$$\left\|\sum_{k=1}^{n} y_k\right\| \leq C\left(\sum_{k=1}^{n} \|y_k\|^p\right)^{\frac{1}{p}}$$

for all finite sequences of blocks $y_1, \ldots, y_n$ with pairwise disjoint ranges. In case $p = \infty$ one should replace the right hand side expression by $\max_{1\leq k\leq n} \|y_k\|$.

A sequence of subspaces (Y_k) is called a blocking of an FDD (X_n) if there exists an increasing sequence of integers (n_k) such that $n_1 = 0$ and

$$Y_k = X_{n_k+1} + \cdots + X_{n_{k+1}}$$

for every k. The blocking (Y_k) is also an FDD and its decomposition constant does not exceed the decomposition constant of (X_n).

Let (X_n) be an FDD of a subspace Y of a Banach space X. In this case we say that (X_n) is an FDD in X. An FDD (X_n) in a space X is said to be shrinking if the sequence $(S_n^*(Y^*))$ is an FDD of the space dual to the closed linear span Y of $\bigcup_{n=1}^{\infty} X_n$. If an FDD (X_n) has a blocking (Y_k) which satisfies p-estimates with $p > 1$, then (X_n) is shrinking (compare to [11]).

Using an idea from [14], one can easily obtain the following result.

Theorem 1.1. *Let (X_n) be an FDD in a Banach space X. Assume that there is a constant $d < 2$ such that if (x_n) is a sequence of successive blocks in B_X, then*

$$\|x_1 + x_m\| \le d$$

for some $m > 1$. Then for every $p > 1$ such that $d^p < 2$ there exists a blocking (Y_k) of (X_n) which satisfies p-estimates. In particular (X_n) is shrinking.

Having a sequence of nonzero elements (x_n) of a space X, one can consider the sequence (X_n), where X_n is the subspace spanned by the element x_n. The sequence (x_n) is a basic sequence if the corresponding sequence (X_n) is an FDD in X. The decomposition constant of (X_n) is called the basic constant of (x_n).

2. Main results

Theorem 2.1. *A Banach space X is NUS* if and only if for every $\epsilon > 0$ there exists $\eta > 0$ such that if $0 < t < \eta$ and (x_n) is a sequence in B_X, then*

$$\|x_1 + t(x_m - x_n)\| \le 1 + \epsilon t$$

for some integers $n > m > 1$.

Proof: Let us assume that a Banach space X has the following property. There is a constant $\epsilon > 0$ such that for every $\eta > 0$ there exist a positive number $t < \eta$ and a sequence (x_n) in B_X for which

$$\|x_1 + t(x_m - x_n)\| > 1 + \epsilon t \tag{1}$$

whenever $n > m > 1$. Clearly we can assume that $\epsilon < 2$.

Let us take $\delta \in (0, 1)$. Our assumption gives us a positive number $t < \frac{\delta}{2-\epsilon}$ and a sequence (x_n) in B_X which satisfies condition (1). Let (n_k^1) be the sequence of all positive integers. We choose a norm-one functional x_1^* with $x_1^*(x_1 + t(x_2 - x_3)) = \|x_1 + t(x_2 - x_3)\|$. There exists an increasing sequence of positive integers (n_k^2) such that $|x_1^*(x_{n_i^2} - x_{n_j^2})| < \frac{\epsilon}{2}$ for all i, j. Now we take a functional $x_2^* \in S_{X^*}$ for which $x_2^*(x_1 + t(x_{n_2^2} - x_{n_3^2})) = \|x_1 + t(x_{n_2^2} - x_{n_3^2})\|$ and a subsequence (n_k^3) of (n_k^2) with $|x_2^*(x_{n_i^3} - x_{n_j^3})| < \frac{\epsilon}{2}$ for all i, j.

Proceeding in this way, we obtain a sequence (x_n^*) in S_{X^*} and a family of increasing sequences $(n_k^i)_{k\geq 1}$ such that

$$x_k^*(x_1 + t(x_{n_2^k} - x_{n_3^k})) = \|x_1 + t(x_{n_2^k} - x_{n_3^k})\| \tag{2}$$

and

$$|x_k^*(x_{n_2^i} - x_{n_3^i})| < \frac{\epsilon}{2}$$

for every k and $i > k$.

From (1) and (2) it follows that $1 + \epsilon t < x_k^*(x_1) + 2t$ for each k. Consequently

$$x_k^*(x) > 1 - (2 - \epsilon)t > 1 - \delta,$$

where $x = \frac{1}{\|x_1\|} x_1$. Moreover (1) and (2) show that $1 + \epsilon t < 1 + t x_k^*(x_{n_2^k} - x_{n_3^k})$. Hence $\epsilon < x_k^*(x_{n_2^k} - x_{n_3^k})$ for every k. It follows that if $i < j$ then

$$\|x_i^* - x_j^*\| \geq \frac{1}{2}(x_j^* - x_i^*)(x_{n_2^j} - x_{n_3^j}) > \frac{\epsilon}{4}\,.$$

We therefore see that $\alpha(S^*(x,\delta)) \geq \frac{\epsilon}{4}$. This shows that the space X is not NUS*.

Let us in turn assume that a Banach space X is not NUS*. Then there exists $\epsilon > 0$ such that for every $\delta > 0$ one can find an element $x_1 \in S_X$ for which

$$\alpha(S^*(x_1, \delta)) > \epsilon. \tag{3}$$

Given $\eta > 0$, we put $t = \frac{\eta}{2}$. By our assumption there is $x_1 \in S_X$ for which estimate (3) holds with $\delta = \frac{\epsilon t}{32}$. Consequently, we can pick a sequence (x_n^*) in B_{X^*} so that $x_n^*(x_1) \geq 1 - \delta$ for every n and $\|x_m^* - x_n^*\| \geq \frac{\epsilon}{2}$ whenever $m \neq n$.

Let x^* be a weak* cluster point of the set $\{x_n^*\}$. We can assume that $\|y_n^*\| \geq \frac{\epsilon}{4}$, where $y_n^* = x_n^* - x^*$ for each $n \geq 1$. So there is an element $y_n \in S_X$ such that $y_n^*(-y_n) > \frac{7\epsilon}{32}$.

We put $n_1 = 1$. Since zero is a weak* cluster point of $\{y_n^*\}$, one can choose $n_2 > n_1$ so that $|y_{n_2}^*(y_{n_1})| < \frac{\epsilon}{8}$. Continuing this inductive procedure, we obtain an increasing sequence (n_k) of positive integers such that $|y_{n_k}^*(y_{n_i})| < \frac{\epsilon}{8}$ if $1 \leq i < k$. Moreover, passing to a subsequence again, we can assume that $|x^*(y_{n_i} - y_{n_j})| < \frac{\epsilon}{32}$ for all i, j.

Let us now set $x_k = y_{n_k}$ for $k = 2, 3, \ldots$. If $i < j$ then

$$\begin{aligned}
\|x_1 + t(x_i - x_j)\| &> x_{n_j}^*(x_1) + t x_{n_j}^*(x_i - x_j) \\
&\geq 1 - \delta + t y_{n_j}^*(x_i) - t y_{n_j}^*(x_j) + t x^*(x_i - x_j) \\
&\geq 1 - \delta + \frac{\epsilon}{16} t \\
&= 1 + \frac{\epsilon}{32} t\,.
\end{aligned}$$

□

Clearly a closed subspace of a NUS* space has the same property. From Theorem 2.1 it follows in particular that also quotient spaces of a NUS* space are NUS*.

Given a Banach space X and a scalar $t \geq 0$ we put

$$R_X(t) = \sup\{\liminf_{n\to\infty} \|x_1 + tx_n\|\},$$

where the supremum is taken over all weakly null sequences (x_n) in B_X. Clearly $R_X(t) \geq 1$ and one can easily show that $\frac{1}{t}(R_X(t)-1)$ is a nondecreasing function of $t > 0$.

In [14] NUS Banach spaces were characterized. Now we can obtain a similar result for NUS* spaces.

Theorem 2.2. *A Banach space X is NUS* if and only if X does not contain an isomorphic copy of l_1 and*

$$\lim_{t\to 0} \frac{1}{t}(R_X(t) - 1) = 0.$$

Proof: Let X be a NUS* space. By Theorem 2.1 for every $\epsilon > 0$ there is $t > 0$ such that if (y_n) is a sequence in B_X, then

$$\|y_1 + t(y_i - y_j)\| \leq 1 + \frac{\epsilon}{2}t$$

for some $j > i > 1$.

Let t_1 correspond to $\epsilon = 1$. We put $\gamma = \frac{t_1}{1+2t_1}$. If X contained an isomorphic copy of l_1, there would exist a sequence (y_n) in B_X such that

$$(1-\gamma)\sum_{k=1}^{m} |a_k| \leq \left\|\sum_{k=1}^{m} a_k y_k\right\|$$

for all real scalars $a_1, \dots, a_m$ (see [10]). By our assumption we obtain integers $j > i > 1$ for which $\|y_1 + t_1(y_i - y_j)\| \leq 1 + \frac{1}{2}t_1$. But

$$\|y_1 + t_1(y_i - y_j)\| \geq (1-\gamma)(1+2t_1) = 1 + t_1$$

which is a contradiction.

Let us now suppose that

$$\lim_{t\to 0} \frac{1}{t}(R_X(t) - 1) > 0.$$

Then there exists a constant $\epsilon > 0$ such that for each $t > 0$ we can find a weakly null sequence (x_n) in B_X with

$$\|x_1 + tx_n\| > 1 + \epsilon t \tag{4}$$

for every $n > 1$. But

$$\|x_1 + tx_m\| \le \liminf_{n\to\infty} \|x_1 + t(x_m - x_n)\|$$

for every $m > 1$. Consequently, we can choose an increasing sequence (n_k) such that

$$\|x_1 + tx_{n_i}\| \le \|x_1 + t(x_{n_i} - x_{n_j})\| + \frac{\epsilon}{4}t$$

whenever $i < j$. Therefore, Theorem 2.1 gives us an index i for which

$$\|x_1 + tx_{n_i}\| \le 1 + \frac{\epsilon}{2}t.$$

This contradicts (4).

Let us assume in turn that a space X does not contain an isomorphic copy of l_1 and

$$\lim_{t\to 0} \frac{1}{t}(R_X(t) - 1) = 0.$$

We take an arbitrary sequence (y_n) in B_X. By the well known theorem of Rosenthal [16] we can assume that (y_n) is weakly Cauchy. Then $(y_n - y_{n+1})$ is a weakly null sequence in $2B_X$. Our assumption shows for every $\epsilon > 0$ there is $\eta > 0$ such that if $0 < t < \eta$ then there exists $m > 1$ with

$$\|y_1 + t(y_m - y_{m+1})\| \le 1 + \epsilon t.$$

In view of Theorem 2.1 this implies that X is NUS*. □

Let X be a Banach space with an FDD (X_n). From Theorem 2.2 it follows that if (X_n) satisfies p-estimates with $p > 1$, then X is NUS* in some equivalent norm (see [15]). Theorem 2.2 gives us also the next corollary.

Corollary 2.3. *Let X be a NUS* Banach space. Then there exists $p > 1$ such that every shrinking FDD (X_n) in X has a blocking (Y_k) which satisfies p-estimates.*

PROOF: Let (X_n) be a shrinking FDD in the space X. If (x_n) is a sequence of successive blocks of (X_n) such that $x_n \in B_X$ for every n, then (x_n) is weakly null (compare to [13] p. 8). From Theorem 2.2 it follows that there is a positive constant $t < 1$ which does not depend on the sequence, such that $\|x_1 + tx_m\| < 1 + \frac{1}{2}t$ for some $m > 1$. Hence

$$\|x_1 + x_m\| \le \|x_1 + tx_m\| + 1 - t < d,$$

where $d = 2 - \frac{t}{2}$. By Theorem 1.1, this gives us the conclusion of the corollary. □

In [18] it was proved that a Banach space X is reflexive if and only if every basic sequence in X is shrinking. Our next result shows that in NUS* spaces basic sequences with small basic constants are shrinking.

Theorem 2.4. *Let X be a NUS* Banach space. Then there exists a constant $M > 1$ such that if (X_n) is an FDD in X with the decomposition constant less than M, then (X_n) is shrinking.*

Proof: Let X be a NUS* space. By Theorem 2.2 there is a positive number $t < 1$ such that

$$\frac{1}{t}(R_X(2t) - 1) < 1.$$

We put $M = (1+t)(R_X(2t))^{-1}$. Let (X_n) be an FDD in X with the decomposition constant $c < M$. We consider a sequence of successive blocks (x_n) such that $x_n \in B_X$ for every n. Passing to a subsequence we can assume that $(x_n - x_{n+1})$ converges weakly to zero. Then

$$\liminf_{n\to\infty} \|x_1 + tx_n\| \leq c \liminf_{n\to\infty} \|x_1 + t(x_n - x_{n+1})\| \leq cR_X(2t).$$

Hence

$$\|x_1 + x_m\| \leq cR_X(2t) + 1 - t < 2.$$

Now the conclusion follows from Theorem 1.1. □

We have actually shown that each FDD (X_n) with the decomposition constant $c < M$ has a blocking which satisfies some p-estimates, where $p > 1$ depends on c. Corollary 2.3 gives us the following improvement of this result.

Corollary 2.5. *Let X be a NUS* Banach space. Then there exist constants $M, p > 1$ such that each FDD in X with the decomposition constant less than M admits a blocking with p-estimates.*

Let us assume that (X_n) is an FDD of a Banach space X. Using an argument from [12] one can show that if (X_n) satisfies some p-estimates then each shrinking FDD in a quotient space of X has a blocking which satisfies p-estimates, too.

Examples.

1. In [17] it was observed that the space c_0 is NUS*. In case $X = c_0$ we can put $t = \frac{1}{2}$ in the proof of Theorem 2.4. This gives $M = \frac{3}{2}$. Therefore, each FDD in c_0 with the decomposition constant less than $\frac{3}{2}$ has a blocking with ∞-estimates.

Let (Z_n) be a sequence of finite dimensional Banach spaces. Using the same idea, one can actually extend this result to the case of a quotient space of

$$\left(\sum_{n=1}^{\infty} Z_n\right)_{c_0}.$$

It is not clear if $\frac{3}{2}$ is the greatest possible value of the constant M for the space c_0. Considering the summing basis of c_0 (see [4, p. 74]), we obtain the estimate $M \leq 2$.

2. Let J be James' space (see [9]). We consider the space J with the following equivalent norm:

$$\|(x_n)\| = \sup \left(\sum_{i=1}^{m} (x_{n_{2i-1}} - x_{n_{2i}})^2 + 2(x_{n_{2m+1}})^2 \right)^{\frac{1}{2}},$$

where $(x_n) \in J$ and the supremum is taken over all sequences $1 \leq n_1 < n_2 < \cdots < n_{2m+1}$.

Let $x = (x_n)$, $y = (y_n)$ be elements of J. It is easy to check that if there is $k > 1$ such that $x_n = 0$ for all $n > k$ and $y_n = 0$ for all $n \leq k$, then

$$\|x + y\|^2 \leq \|x\|^2 + 2\|y\|^2.$$

It follows that the FDD corresponding to the standard basis of J satisfies 2-estimates. Moreover one can easily show that $R_J(t) = (1 + 2t^2)^{\frac{1}{2}}$ for every $t \geq 0$. From Theorem 2.2 we now see that J is NUS*. For this space we can put $M = \frac{3}{2\sqrt{2}}$. Let us also mention that J is isomorphic to J^{**}, but it is not reflexive. Therefore by a result from [17] J is not a dual space.

Let us recall that a Banach space X has the weak Banach-Saks property if each sequence (x_n) converging weakly to x admits a subsequence (x_{n_k}) whose arithmetic means $\frac{1}{n}\sum_{k=1}^{n} x_{n_k}$ tend to x in norm.

Remark 2.6. *If a Banach space X is NUS*, then X has the weak Banach-Saks property.*

Proof: Let M, p be the constants occurring in Theorem 2.5. Clearly it suffices to prove that each weakly null sequence (x_n) has a subsequence (x_{n_k}) such that the means $\frac{1}{n}\sum_{k=1}^{n} x_{n_k}$ tend to zero. This is obviously true if (x_n) converges to zero in norm. Therefore we can assume that (x_n) is a weakly null sequence which does not converge in norm. Then it has a subsequence (x_{n_k}) which is a basic sequence with the basic constant less than M (see [13] p. 5). From Corollary 2.5 it follows that, passing to a subsequence again, we can assume that there is a constant C for which

$$\left\| \frac{1}{m} \sum_{k=1}^{m} x_{n_k} \right\| \leq C \left(\sum_{k=1}^{m} \left\| \frac{1}{m} x_{n_k} \right\|^p \right)^{\frac{1}{p}}$$

for every m. But the sequence (x_n) is bounded. Therefore

$$\left\| \frac{1}{m} \sum_{k=1}^{m} x_{n_k} \right\| \leq C_1 m^{\frac{1}{p} - 1}$$

for some constant C_1. Since $p > 1$, this shows that the means $\frac{1}{n}\sum_{k=1}^{n} x_{n_k}$ converge to zero. $\square$

Let X be a Banach space. In [5] a coefficient $R(X)$ was defined. In our notation $R(X) = R_X(1)$. Next, in [6] it was proved that if $R(X) < 2$, then the space X has the fixed point property for nonexpansive self-mappings of weakly compact convex sets. Clearly $R(X) \leq R_X(t) + 1 - t$ for every $t \in (0,1)$. Therefore if

$$\lim_{t \to 0} \frac{1}{t}(R_X(t) - 1) < 1$$

then $R(X) < 2$. In particular, we obtain the following result.

Remark 2.7. *Let K be a nonempty weakly compact convex subset of a NUS* Banach space X. Every nonexpansive mapping $T : K \to K$ has a fixed point.*

References

[1] Banaś J., *Compactness conditions in the geometric theory of Banach spaces*, Nonlinear Anal. **16** (1990), 669–682.

[2] Banaś J., Fraczek K., *Conditions involving compactness in geometry of Banach spaces*, Nonlinear Anal. **20** (1993), 1217–1230.

[3] Banaś J., Goebel K., *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[4] van Dulst D., *Reflexive and Superreflexive Banach Spaces*, Mathematisch Centrum, Amsterdam, 1978.

[5] García Falset J., *Stability and fixed points for nonexpansive mappings*, Houston J. Math **20** (1994), 495–505.

[6] García Falset J., *The fixed point property in Banach spaces with NUS-property*, preprint.

[7] Goebel K., Sękowski T., *The modulus of noncompact convexity*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A **38** (1984), 41–48.

[8] Huff R., *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. **10** (1980), 743–749.

[9] James R.C., *Bases and reflexivity of Banach spaces*, Ann. of Math. **52** (1950), 518–527.

[10] James R.C., *Uniformly non-square Banach spaces*, Ann. of Math. **80** (1964), 542–550.

[11] James R.C., *Super-reflexive spaces with bases*, Pacific J. Math. **41** (1972), 409–419.

[12] Johnson W.B., Zippin M., *On subspaces of quotients of $(\sum G_n)_{l_p}$ and $(\sum G_n)_{c_0}$*, Israel J. Math. **13** (1972), 311–316.

[13] Lindenstrauss J., Tzafriri L., *Classical Banach Spaces I. Sequence Spaces*, Springer-Verlag, New York, 1977.

[14] Prus S., *Nearly uniformly smooth Banach spaces*, Boll. U.M.I. (7) **3-B** (1989), 507–521.

[15] Prus S., *Banach spaces and operators which are nearly uniformly convex*, to appear.

[16] Rosenthal H.P., *A characterization of Banach spaces containing l_1*, Proc. Nat. Acad. Sci. (USA) **71** (1974), 2411–2413.

[17] Sękowski T., Stachura A., *Noncompact smoothness and noncompact convexity*, Atti. Sem. Mat. Fis. Univ. Modena **36** (1988), 329–338.

[18] Zippin M., *A remark on bases and reflexivity in Banach spaces*, Israel J. Math. **6** (1968), 74–79.

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