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Smooth graphs

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Abstract. A graph G on ω_1 is called $<\omega$*-smooth* if for each uncountable $W \subset \omega_1$, G is isomorphic to $G[W \setminus W']$ for some finite $W' \subset W$. We show that in various models of ZFC if a graph G is $<\omega$-smooth, then G is necessarily trivial, i.e. either complete or empty. On the other hand, we prove that the existence of a non-trivial, $<\omega$-smooth graph is also consistent with ZFC.

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1. Introduction

Answering a question of R. Jamison, H.A. Kierstead and P.J. Nyikos proved in [3]: *if the uncountable induced subgraphs of an uncountable n-uniform hypergraph are pairwise isomorphic, then the hypergraph must be either empty or complete.* In this note we investigate how many uncountable subgraphs of a graph G on ω_1 can be isomorphic to G provided that it is *non-trivial*, i.e. it is not complete nor empty. As a corollary of [1, Theorem 4.2] we can get the following positive result: the existence of a non-trivial graph on ω_1 which embeds into each of its uncountable subgraphs is consistent with ZFC. To formulate this and the forthcoming results precisely we need the following definition.

Definition 1.1. A graph G on ω_1 is called κ*-smooth* ($<\kappa$*-smooth*) if for each uncountable $W \subset \omega_1$, G is isomorphic to $G[W \setminus W']$ for some $W' \subset W$ with $|W'| \le \kappa$ ($|W'| < \kappa$).

Fact 1.2. *If a graph G on ω_1 is n-smooth for some $n \in \omega$, then G is complete or empty.*

Proof: Pick ordinals $x_0, x_1, \ldots, x_n$ from ω_1 by finite induction such that for each $j \le n$ we have

$$x_j \in \bigcap_{i<j} G(x_i) \quad \text{and} \quad \left|\bigcap_{i\le j} G(x_i)\right| = \omega_1.$$

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If we cannot find a suitable x_j, then taking $W = \bigcap_{i<j} G(x_i)$ we have $|W| = \omega_1$ but $|W \cap G(w)| \leq \omega$ for each $w \in W$. Thus $G[W]$ contains an uncountable induced empty subgraph and so G is empty.

Assume now that we could choose the sequence $\{x_i : i \leq n\}$. Then let $W = \{x_i : i \leq n\} \cup \bigcap_{i \leq n} G(x_i)$. Since G is n-smooth there is $W' \subset W$, $|W'| \leq n$ such that $G \cong G[W \setminus W']$. Fix $i < n+1$ such that $x_i \notin W'$. Since $x_i \in W \setminus W'$, $W \subset G(x_i) \cup \{x_i\}$ and $G \cong G[W \setminus W']$ it follows that there is $w \in \omega_1$ such that $\omega_1 \subset G(w) \cup \{w\}$ and so for each uncountable $V \subset \omega_1$ there is $v \in V$ such that $|V \setminus G(v)| \leq n$. Thus G contains an uncountable complete subgraph and so G is complete. $\square$

On the other hand, in [1, Theorem 4.2] it was shown that $\diamondsuit^+$ implies that there is a Suslin tree $\mathcal{T} = \langle \omega_1, \prec \rangle$ such that for each uncountable $X \subset \omega_1$ there is a countable $X' \subset X$ such that $\mathcal{T} \cong \mathcal{T} \restriction (X \setminus X')$. Thus the comparability graph of $\mathcal{T}$ is ω-smooth and clearly non-trivial. However, the question whether a $<\omega$-smooth graphs on ω_1 is necessarily trivial was left open. This gap will be filled up here: we show that (i) in different models of ZFC every $<\omega$-smooth graph on ω_1 is complete or empty, (ii) the existence of a non-trivial, $<\omega$-smooth graph G on ω_1 is consistent with ZFC.

The following question however remains unanswered:

Problem 1. *Is there a non-trivial, ω-smooth or just ω_1-smooth graph on ω_1 (in ZFC)?*

We use the standard set-theoretical notation throughout, cf. [2]. For a graph G, $V(G)$ denotes the set of vertices of G, $E(G)$ the family of edges of G. If $H \subset V(G)$, $G[H]$ denotes the induced subgraphs of G on H. Given $x \in V(G)$ put $G(x) = \{y \in V(G) : \{x, y\} \in E(G)\}$. If G and H are graphs we write $G \cong H$ if G and H are isomorphic.

If G and G' are graphs, $\mathrm{Iso_p}(G, G')$ denotes the family of isomorphisms between finite induced subgraphs of G and G'.

If q is a function let $\mathrm{supp}\,(q) = \mathrm{dom}\,(q) \cup \mathrm{ran}\,(q)$.

For a cardinal κ we denote by $\mathcal{C}_\kappa$ the standard poset $\langle \mathrm{Fn}(\kappa, 2; \omega), \supseteq \rangle$ which adds κ Cohen reals to the ground model.

2. Models without non-trivial $<\omega$-smooth graphs

Lemma 2.1. *If G is a $<\omega$-smooth graph on ω_1 and G has a — not necessarily spanned — subgraph isomorphic to the bipartite graph $[\omega; \omega_1]$, then G is complete.*

Proof: Fix $A \in [\omega_1]^{\omega}$ and $B \in [\omega_1]^{\omega_1}$ such that $[A, B] \subset E(G)$. Let

$$X = \{\alpha \in \omega_1 : |\omega_1 \setminus G(\alpha)| \leq \omega\}.$$

We show that X is uncountable. Indeed, let $\alpha < \omega_1$. Then for some finite $C \subset A \cup B$ and $D \subset \omega_1 \setminus \alpha$ the graphs $G[(A \cup B) \setminus C]$ and $G[(\omega_1 \setminus \alpha) \setminus D]$ are isomorphic witnessed by a function f. Then $f''(A \setminus C) \subset X$, so $X \not\subset \alpha$, i.e. $|X| = \omega_1$.

Now, by recursion, we can construct a set $Y = \{y_\eta : \eta < \omega_1\} \subset X$ such that $y_\eta \in X \cap \bigcap_{\xi<\eta} G(y_\xi)$. Then $G[Y]$ is complete and so G is also complete which was to be proved. □

Let us remark that the statement of Lemma 2.1 fails for ω-smooth graphs: the comparability graph G of the Suslin tree $\mathcal{T}$ constructed in [1, Theorem 4.2] is non-trivial and ω-smooth, but $[\omega;\omega_1] \subset G$ and $[\omega_1;\omega_1] \subset \overline{G}$.

Let us recall the definition of splitting number $\mathfrak{s}$:

$$\mathfrak{s} = \min\{|\mathcal{A}| : \mathcal{A} \subset \left[\omega\right]^\omega \wedge \forall\, X \in \left[\omega\right]^\omega\ \exists\, A \in \mathcal{A}\ |X \cap A| = |X \setminus A| = \omega\}.$$

Theorem 2.2. *Every $<\omega$-smooth graph on ω_1 is trivial provided* (1) *or* (2) *or* (3) *below hold:*

(1) $\omega_1 < \mathfrak{s}$,
(2) $2^\omega < 2^{\omega_1}$,
(3) *in a model obtained by adding ω_2 Cohen reals to some model V.*

Proof of Theorem 2.2 (1): Assume that G is $<\omega$-smooth. For each $\alpha \in \omega_1$ let $F_\alpha = G(\alpha) \cap \omega$. The family $\mathcal{F} = \{F_\alpha : \alpha < \omega_1\}$ is not a splitting family for $\mathfrak{s} > \omega_1$ so there is an infinite set $B \subset \omega$ such that $B \subset^* F_\alpha$ or $B \subset^* \omega \setminus F_\alpha$ for each $\alpha \in \omega_1$. Then there is $n \in \omega$ and an uncountable $I \subset \omega_1$ such that either $B \setminus n \subset F_\alpha$ for each $\alpha \in I$ or $B \setminus n \in \omega \setminus F_\alpha$ for each $\alpha \in I$. Thus either $[B \setminus n, I] \subset E(G)$ or $[B \setminus n, I] \cap E(G) = \emptyset$, i.e. $[\omega;\omega_1]$ is a subgraph of either G or $\overline{G}$, and so G is trivial by Lemma 2.1. □

Proof of Theorem 2.2 (2): Assume on the contrary, that G is $<\omega$-smooth and non-trivial. By Lemma 2.1, we can choose an uncountable set $A \subset \omega_1 \setminus \omega$ such that $G(\alpha) \cap \omega \neq^* G(\beta) \cap \omega$ for each $\{\alpha,\beta\} \in \left[A\right]^2$.

For each uncountable $X \subset A$ fix a finite set $C_X \subset \omega_1$ and an isomorphism f_X between $G[(\omega \cup X) \setminus C_X]$ and G. Since $2^\omega < 2^{\omega_1}$ there are sets $X, Y \in \left[A\right]^{\omega_1}$ such that $|X \setminus Y| \geq \omega$, $C_X = C_Y$ and $f_X \restriction \omega = f_Y \restriction \omega$. Let $\xi \in X \setminus Y \setminus C_X$. Then $f = f_Y^{-1} \circ f_X$ is an isomorphism between $G[(\omega \cup X) \setminus C_X]$ and $G[(\omega \cup Y) \setminus C_Y]$ such that $f \restriction (\omega \setminus C_X) = \mathrm{id} \restriction (\omega \setminus C_X)$. Taking $\eta = f(\xi)$ we obtain that $G(\xi) \cap (\omega \setminus C_X) = G(\eta) \cap (\omega \setminus C_X)$ which contradicts the choice of A because $\eta \neq \xi$ for $\xi \notin \mathrm{ran}(f)$. □

Proof of Theorem 2.2 (3): Assume that G is a graph on ω_1 in $V^{\mathcal{C}_{\omega_2}}$. Fix $\alpha < \omega_2$ such that $G \in V^{\mathcal{C}_\alpha}$. Since $\mathcal{C}_{\omega_2} = \mathcal{C}_\alpha * \mathcal{C}_{\omega_2 \setminus (\alpha+\omega_1)} * \mathcal{C}_{[\alpha,\alpha+\omega_1)}$, by Lemma 2.1 it is enough to prove the following statement:

Lemma 2.3. *If G is a graph on ω_1, $[\omega;\omega_1] \not\subset G, \overline{G}$, then G is not $<\omega$-smooth in $V^{\mathcal{C}_{\omega_1}}$.*

Proof of Lemma 2.3: Applying Lemma 2.1, we can find an uncountable $A \subset \omega_1 \setminus \omega$ such that $G(\alpha) \cap \omega \neq^* G(\beta) \cap \omega$ for each $\{\alpha, \beta\} \in [A]^2$. If $\mathcal{G}$ is the $\mathcal{C}_{\omega_1}$-generic filter over V, let $X = \{\alpha \in A : \exists\, p \in \mathcal{G}\ p(\alpha) = 1\}$. We show that

$$1_{\mathcal{C}_{\omega_1}} \Vdash \text{“}G \text{ and } G[(\omega \cup \dot{X}) \setminus Y] \text{ are not isomorphic for any } Y \in [\omega_1]^{<\omega}.\text{”}$$

Assume on the contrary that $p \in \mathcal{C}_{\omega_1}$, $Y \in [\omega_1]^{<\omega}$ and $\dot{f}$ is a $\mathcal{C}_{\omega_1}$-name of a function such that

$$p \Vdash \text{“}\dot{f} \text{ is an isomorphism between } G \text{ and } G[(\omega \cup \dot{X}) \setminus Y]\text{”}.$$

Fix $\omega \leq \nu < \omega_1$ such that $\operatorname{dom}(p) \cup Y \subset \nu$, $p \Vdash$ “$\dot{f}''\nu = ((\omega \cup \dot{X}) \setminus Y) \cap \nu$” and $p \Vdash$ “$\dot{f} \restriction \nu \in V[\mathcal{G} \restriction \nu]$.” From now on we work in $V[\mathcal{G} \restriction \nu]$. Let $h = f \restriction \nu$ and $B = h^{-1}(\omega \setminus Y)$. Since $G(\alpha) \cap \omega \neq^* G(\beta) \cap \omega$ for each $\{\alpha, \beta\} \in [A]^2$ it follows that if $\{\zeta, \xi\} \in [\omega_1 \setminus \nu]^2$, then $G(\zeta) \cap B \neq^* G(\xi) \cap B$. Thus for each $\xi \in \omega_1 \setminus \nu$ we have

$$f(\xi) = \alpha \ \text{ iff } \ h''(G(\xi) \cap B) =^* (G(\alpha) \cap \omega).$$

Hence $f \restriction (\omega_1 \setminus \nu)$ can be defined in $V[\mathcal{G} \restriction \nu]$ and so $X \setminus \nu \in V[\mathcal{G} \restriction \nu]$, which is impossible by the choice of X. □

The proof of Theorem 2.2 is complete. □

The following theorem claims that if CH holds in the ground model, then the statement of Lemma 2.3 can be strengthened: we can find a set in the ground model witnessing that G is not $<\omega$-smooth in $V^{\mathcal{C}_{\omega_1}}$.

Theorem 2.4. *If CH holds and G is a graph on ω_1 such that $[\omega; \omega_1] \not\subset G, \overline{G}$, then there is an uncountable subset X of ω_1 such that*

$$V^{\mathcal{C}_{\omega_1}} \models \text{“}G \text{ is not isomorphic to } G[X \setminus Y] \text{ for any } Y \in [\omega_1]^{\omega_1}\text{”}.$$

The proof is quite long and technical, so we omit it.

3. Generic construction of a non-trivial $<\omega$-smooth graph

Theorem 3.1. *If $2^{\omega_1} = \omega_2$, then there is a c.c.c poset P of size ω_2 such that*

$$V^P \models \text{there is a non-trivial, } <\omega\text{-smooth graph } G \text{ on } \omega_1.$$

Proof: We construct $P = \mathcal{C} * P'$ in two steps: in the first step, forcing with $\mathcal{C} = \operatorname{Fn}(\omega_1, 2; \omega)$, we add ω_1-many Cohen reals to V to introduce our desired graph G. Then, in the second step, we add many isomorphisms between certain subgraphs of G to $V^{\mathcal{C}}$ to guarantee $<\omega$-smoothness of G in $V^{\mathcal{C} * P'}$.

To simplify our notation we take $\mathcal{C} = \mathrm{Fn}(\left[\omega_1\right]^2, 2; \omega)$ and define the graph G on ω_1 in $V[\mathcal{G}]$, where $\mathcal{G}$ is the $\mathcal{C}$-generic filter over V, in the straightforward way:

$$\{\alpha,\beta\} \in E(G) \text{ iff } \exists\, p \in \mathcal{G}\ p(\{\alpha,\beta\}) = 1.$$

If $c \in \mathcal{C}$ let $\operatorname{supp} c = \bigcup \operatorname{dom} c$ and $G^c = \left\langle \operatorname{supp} c, c^{-1}\{1\}\right\rangle$. Let us remark that if $c, c' \in \mathcal{C}$, $c \le c'$ and $\operatorname{dom} c' = \left[\operatorname{supp} c'\right]^2$, then $G^{c'}$ is a spanned subgraph of G^c.

To obtain $P' = P_{\omega_2}$ we carry out a finite support iteration of c.c.c posets

$$\langle P_\alpha : \alpha \le \omega_2, Q_\alpha : \alpha < \omega_2\rangle$$

in the following way: in the α^{th} step, we pick an uncountable subset X_α of ω_1 in the intermediate model $V^{\mathcal{C}*P_\alpha}$ and then we try to find a finite set Y_α and c.c.c poset Q_α such that

$$V^{\mathcal{C}*P_\alpha*Q_\alpha} \models \text{“}G \text{ and } G[X_\alpha \setminus Y_\alpha] \text{ are isomorphic witnessed by a function } f_\alpha.\text{”}$$

The poset Q_α will consist of certain isomorphisms between finite subgraphs of G and $G[X_\alpha \setminus Y_\alpha]$, ordered by the reverse inclusion. In other words, we force with certain finite approximations of an isomorphism between G and $G[X_\alpha \setminus Y_\alpha]$.

The problem is the right choice of Q_α because we should meet two contradictory requirements. First, the poset Q_α should satisfy c.c.c and forcing with Q_α cannot introduce an uncountable empty or complete subgraph of G, therefore Q_α cannot contain too many elements. On the other hand, to guarantee that a Q_α-generic filter gives an isomorphism between G and $G[X_\alpha \setminus Y_\alpha]$ we need some density arguments, i.e. certain subsets of Q_α should be dense in Q_α, which involves that Q_α cannot be too small. As it turns out, it will be quite easy to meet the first requirement, the hard part of the proof is how to cope with the second one.

Now assume that P_α is constructed and let us see the induction step.

First, using a bookkeeping function, we pick the set $X_\alpha \in \left[\omega_1\right]^{\omega_1} \cap V^{\mathcal{C}*P_\alpha}$ in such a way that

$$(*) \qquad \{X_\alpha : \alpha < \omega_2\} = \left[\omega_1\right]^{\omega_1} \cap V^{\mathcal{C}*P_{\omega_2}}.$$

To construct the poset Q_α we need the following induction hypotheses. To formulate it we use two notions. A graph G is *strongly non-trivial* provided that each uncountable family of pairwise disjoint, finite subsets of $V(G)$ contains four distinct elements, a, b, c, d such that $[a,b] \subset E(G)$ and $[c,d] \cap E(G) = \emptyset$. If G is a graph, a set $A \subset V(G)$ is called *dense in G* iff for each pair B and B' of disjoint finite subsets of $V(G)$ there is $\alpha \in A$ such that $G(\alpha) \supset B$ and $G(\alpha) \cap B' = \emptyset$.

Induction hypothesis.

(I) $V^{\mathcal{C}*P_\alpha} \models$ “G is strongly non-trivial”,

(II) $V^{\mathcal{C}*P_\alpha} \models$ “$\forall\, X \in \left[\omega_1\right]^{\omega_1}\ \exists\, Y \in \left[X\right]^{<\omega}\ \forall\, \delta < \omega_1\ \exists\, A \in \left[X \setminus \delta\right]^{\omega}$ A is dense in $G[X \setminus Y]$”.

The preservation of the induction hypotheses (I) and (II) during the iteration will be verified later in Lemmas 3.5 and 3.9.

We continue the construction of the poset Q_α. Using (II) fix $Y_\alpha \in \left[X_\alpha\right]^{<\omega}$ and pairwise disjoint countable subsets $\{D_\xi : \xi < \omega_1\}$ of $X_\alpha \setminus Y_\alpha$ which are dense in $G[X_\alpha \setminus Y_\alpha]$.

Let us recall that for each $\beta < \alpha$ in the β^{th} step we already constructed an isomorphism f_β between G and $G[X_\beta \setminus Y_\beta]$. For each $\beta < \alpha$ the set $C_\beta = \{\nu < \omega_1 : f_\beta{}''\nu \subset \nu\}$ is clearly club and C_β belongs to $V^{\mathcal{C}*P_\beta*Q_\beta} \subset V^{\mathcal{C}*P_\alpha}$. Since P_α satisfies c.c.c and $|\alpha| < 2^{\omega_1} = \omega_2$, there is a club set $C \subset \omega_1$ even in V such that $|C \setminus C_\beta| \le \omega$ for each $\beta < \alpha$.

The club set $C = \{\gamma_\nu : \nu < \omega_1\}$ gives a natural partition $\mathcal{A}_\alpha = \{A^\alpha_\nu : \nu < \omega_1\}$ of ω_1 into countable pieces: let $A^\alpha_\nu = [\gamma_\nu, \gamma_{\nu+1})$ for $\nu < \omega_1$. We can thin out C to contain only limit ordinals and in this case every A^α_ν is infinite. Define the map $\mathrm{rk}_\alpha : \omega_1 \to \omega_1$ by the formula $\xi \in A^\alpha_{\mathrm{rk}_\alpha(\xi)}$.

If $\beta < \alpha$, then $|C \setminus C_\beta| \le \omega$ and so all but countably many A^α_ν's are f_β-closed. By shrinking C we can assume every A^α_η contains some D_ξ and so

(i) $A^\alpha_\eta \cap (X_\alpha \setminus Y_\alpha)$ is dense in $G[X_\alpha \setminus Y_\alpha]$.

Since $A^\alpha_\eta \in V$ and infinite, it follows

(ii) A^α_η is dense in G.

For $\eta < \omega_1$ let $O_\eta = [\omega\eta, \omega\eta + \omega)$ and $B^\alpha_\eta = \bigcup\{A^\alpha_\eta : \nu \in O_\eta\}$. Put $\mathcal{B}_\alpha = \left\langle B^\alpha_\eta : \eta < \omega_1 \right\rangle$.

Given two sets Z and W, denote by $\mathrm{Bij_p}(Z, W)$ the family of bijections between finite subsets of Z and W.

If $p \in \mathrm{Bij_p}(\omega_1, X \setminus Y)$, a sequence $\vec{x} = \langle x_0, x_1, \ldots, x_n, \rangle$ of countable ordinals is a *p-loop* iff $n \ge 1$, $x_0 = x_n$ and there is a sequence $\langle k_0, \ldots, k_{n-1}\rangle \in {}^n\{-1, +1\}$ such that

(iii) $\mathrm{rk}_\alpha(x_{i+1}) = \mathrm{rk}_\alpha(p^{k_i}(x_i))$ for each $i < n$,

(iv) there is no $i < n$ such that $\{k_i, k_{i+1}\} = \{-1, +1\}$, $x_{i+1} = p^{k_i}(x_i)$ and $x_{i+2} = p^{k_{i+1}}(x_{i+1})$.

We say that p is *loop-free* if there is no p-loop.

Now we are in the position to define the poset Q_α. We put a finite function $p \in \mathrm{Iso_p}(G, G[X_\alpha \setminus Y_\alpha])$ into Q_α iff

(v) $p'' B_\eta \subset B_\eta$ for each $\eta < \omega_1$,

(vi) p is loop-free.

As promised, Q_α is ordered by the reverse inclusion: $Q_\alpha = \langle Q_\alpha, \supseteq\rangle$.

Let us recall that $\operatorname{supp} p = \operatorname{dom}(p) \cup \operatorname{ran}(p)$ for $p \in Q_\alpha$.

We need to show that Q_α satisfies c.c.c and a Q_α-generic filter gives an isomorphism between G and $G[X_\alpha \setminus Y_\alpha]$. First we prove an auxiliary lemma.

Lemma 3.2. *If $p, q \in \mathrm{Bij_p}(\omega_1, \omega_1)$, $\mathrm{rk}_\alpha{}'' \operatorname{supp} p \cap \mathrm{rk}_\alpha{}'' \operatorname{supp} q = \emptyset$ and $\vec{x} = \langle x_0, \ldots, x_n\rangle$ is a $(p \cup q)$-loop, then $\vec{x}$ is either a p-loop or a q-loop.*

Proof: Assume that $x_0 \in \operatorname{supp} p$. Then $x_0 \notin \operatorname{supp} q$, so $\mathrm{rk}_\alpha(x_1) = \mathrm{rk}_\alpha(p^{k_0}(x_0))$ for some $k_0 \in \{-1, +1\}$. Since $p^{k_0}(x_0) \in \operatorname{supp} p$ we have $\mathrm{rk}_\alpha(x_1) = \mathrm{rk}_\alpha(p^{k_0}(x_0)) \notin \mathrm{rk}_\alpha{}'' \operatorname{supp} q$ and so $x_1 \notin \operatorname{supp} q$. Repeating this argument we yield $\{x_0, \dots, x_n\} \subset \operatorname{supp} p \setminus \operatorname{supp} q$ and so $\vec{x}$ is a p-loop. □

Lemma 3.3. *Q_α satisfies c.c.c.*

Proof: We work in $V^{\mathcal{C}*P_\alpha}$. Assume that $\{q_\xi : \xi < \omega_1\} \subset Q_\alpha$, $c_\xi = \operatorname{supp} q_\xi$ and $r_\xi = \mathrm{rk}_\alpha{}'' c_\xi$. Applying standard Δ-system and counting arguments we can find $I \in [\omega_1]^{\omega_1}$ such that

(1) $\{c_\xi : \xi \in I\}$ forms a Δ-system with kernel c,
(2) $\{r_\xi : \xi \in I\}$ forms a Δ-system with kernel r,
(3) $\mathrm{rk}_\alpha{}'' c = r$,
(4) $\mathrm{rk}_\alpha{}''(c_\xi \setminus c) = r_\xi \setminus r$ for each $\xi \in I$,
(5) $q_\xi \restriction c = q'$ for each $\xi \in I$.

Since G is strongly non-trivial in $V^{\mathcal{C}*P_\alpha}$ by the induction hypothesis (I), there is $\{\xi, \zeta\} \in [I]^2$ such that $[c_\xi \setminus c, c_\zeta \setminus c] \subset E(G)$. We show that $q = q_\xi \cup q_\zeta \in Q_\alpha$. Clearly $q \in \mathrm{Iso_p}(G, G[X_\alpha \setminus Y_\alpha])$ and q satisfies (v). Since $q = q' \cup (q_\xi \setminus q') \cup (q_\zeta \setminus q')$ and the sets $\mathrm{rk}_\alpha{}'' q'$, $\mathrm{rk}_\alpha{}''(q_\xi \setminus q')$ and $\mathrm{rk}_\alpha{}''(q_\zeta \setminus q')$ are pairwise disjoint we have that q satisfies (vi) as well by Lemma 3.2. □

If $\mathcal{G}^{Q_\alpha}$ is the Q_α-generic filter over $V^{\mathcal{C}*P_\alpha}$ let $f_\alpha = \bigcup\{q : q \in \mathcal{G}^{Q_\alpha}\}$.

Lemma 3.4. $V^{\mathcal{C}*P_\alpha*Q_\alpha} \models$ *"f_α is an isomorphism between G and $G[X_\alpha \setminus Y_\alpha]$."*

Proof: We need to prove that $\operatorname{dom}(f_\alpha) = \omega_1$ and $\operatorname{ran}(f_\alpha) = X_\alpha \setminus Y_\alpha$ which follows if for each $\nu \in \omega_1$ and $\mu \in X \setminus Y$ both

$$D_\nu = \{q \in Q_\alpha : \nu \in \operatorname{dom} q\}$$

and

$$R_\mu = \{q \in Q_\alpha : \mu \in \operatorname{ran} q\}$$

are dense in Q_α. Fix $q \in Q_\alpha$. Write $\mathrm{rk}_\alpha(\nu) = \omega\eta + n$. Pick $\omega\eta \le \zeta < \omega\eta + \omega$ such that $(\operatorname{supp} q) \cap A^\alpha_\zeta = \emptyset$. Since $A^\alpha_\zeta \cap (X_\alpha \setminus Y_\alpha)$ is dense in $G[X_\alpha \setminus Y_\alpha]$ we can find $\nu' \in A^\alpha_\zeta \cap (X_\alpha \setminus Y_\alpha)$ such that $\{\nu', q(\xi)\} \in E(G)$ iff $\{\nu, \xi\} \in E(G)$ for each $\xi \in \operatorname{dom} q$. Let $q' = q \cup \{\langle \nu, \nu' \rangle\}$. By the choice of ζ', $\mathrm{rk}_\alpha(\nu') = \zeta \notin \mathrm{rk}_\alpha{}''(\operatorname{supp} q)$, so this extension of q cannot introduce a q'-loop, i.e. $q' \in Q_\alpha$. Thus $q' \in D_\nu$ and $q' \le q$ which was to be proved. The density of R_μ can be verified by a similar argument using the density of A^α_ζ in G. □

The induction step is complete so the theorem is proved provided we can verify the induction hypotheses (I) and (II) in every $V^{\mathcal{C}*P_\gamma}$. First we deal with (I) because it is fairly easy. Checking (II) is the crux of our proof.

Lemma 3.5. *The induction hypothesis* (I) *holds, i.e. G is strongly non-trivial in every $V^{\mathcal{C}*P_\alpha}$.*

Proof: First remark that G is clearly strongly non-trivial in $V^{\mathcal{C}}$. By [1, Lemma 4.10] we can assume that $\alpha = \gamma + 1$ and G is strongly non-trivial in $V^{\mathcal{C}*P_\gamma}$. Working in $V^{\mathcal{C}*P_\alpha}$ assume that $q \Vdash$ "$\{\dot{x}_\xi : \xi < \omega_1\}$ *are pairwise disjoint, finite subsets of* ω_1." For each $\xi < \omega_1$ pick a condition $q_\xi \leq q$ and a finite subset x_ξ of ω_1 such that $q_\xi \Vdash$ "$\dot{x}_\xi = x_\xi$". Since Q_γ satisfies c.c.c, we can assume that the sets x_ξ are pairwise disjoint.

We can also assume that $x_\xi \subset \operatorname{dom} q_\xi$ because in Lemma 3.4 we showed that the sets D_ν are dense in Q_γ.

From now on we can argue as in Lemma 3.3. Let $c_\xi = \operatorname{supp} q_\xi$ and $r_\xi = \mathrm{rk}_\gamma{}'' c_\xi$. We can find $I \in \left[\omega_1\right]^{\omega_1}$ such that $\{c_\xi : \xi \in\}$ forms a Δ-system with kernel c and $\{r_\xi : \xi \in I\}$ forms a Δ-system with kernel r, moreover $\mathrm{rk}_\gamma{}'' c = r$, $\mathrm{rk}_\gamma{}''(c_\xi \setminus c) = r_\xi \setminus r$, $q_\xi \restriction c$ is independent from ξ and $x_\xi \subset c_\xi \setminus c$ for each $\xi \in I$. Write $c'_\xi = c_\xi \setminus c$, $q'_\xi = q_\xi \restriction c'_\xi$, $r'_\xi = r_\xi \setminus r$ and $q' = q_\xi \restriction c$.

Since G is strongly non-trivial in $V^{\mathcal{C}*P_\xi}$ there are $\xi_0, \xi_1, \zeta_0, \zeta_1 \in I$ such that $[c'_{\xi_0}, c'_{\zeta_0}] \subset E(G)$ and $[c'_{\xi_1}, c'_{\zeta_1}] \cap E(G) = \emptyset$. Then $q^i = q_{\xi_i} \cup q_{\zeta_i} \in \mathrm{Iso}_{\mathrm{p}}(G, G[X \setminus Y])$ and q^i clearly satisfies (v). Since $q^i = q' \cup q'_{\xi_i} \cup q'_{\zeta_i}$ and the sets $\mathrm{rk}_\gamma{}'' q'$, $\mathrm{rk}_\gamma{}'' q'_{\xi_i}$ and $\mathrm{rk}_\gamma{}'' q'_{\zeta_i}$ are pairwise disjoint we have that q^i satisfies (vi) as well by Lemma 3.2. Thus

$$q^0 \Vdash [\dot{x}_{\xi_0}, \dot{x}_{\zeta_0}] \subset E(G)$$

and

$$q^1 \Vdash [\dot{x}_{\xi_1}, \dot{x}_{\zeta_1}] \cap E(G) = \emptyset.$$

□

Now we start to work on (II).

Definition 3.6. Assume that $\mathcal{H}$ is a family of functions, $\operatorname{dom}(h) \cup \operatorname{ran}(h) \subset \omega_1$ for each $h \in \mathcal{H}$. A sequence $\vec{x} = \langle x_0, x_1, \ldots, x_n \rangle \in {}^n\omega_1$ is called an *$\mathcal{H}$-loop* if $n \geq 1$, $x_0 = x_n$, and there are sequences $\langle h_0, \ldots, h_{n-1} \rangle \in {}^n\mathcal{H}$ and $\langle k_0, \ldots, k_{n-1} \rangle \in {}^n\{-1, +1\}$ such that

- (vii) $h_i^{k_i}(x_i) = x_{i+1}$ for each $i < n$,
- (viii) there is no $i < n - 1$ such that $h_i = h_{i+1}$ and $\{k_i, k_{i+1}\} = \{-1, +1\}$.

Let $Z \subset \omega_1$. We say that $\mathcal{H}$ *acts loop-free on Z* if

- (ix) Z is h-closed for each $h \in \mathcal{H}$,
- (x) Z does not contain any $\mathcal{H}$-loop.

Definition 3.7. A condition $p = \langle c, q \rangle \in \mathcal{C} * P_\alpha$ is called *determined* iff

1. q is a function, $\operatorname{dom}(q) \in \left[\omega_1\right]^{<\omega}$,
2. $q(\eta)$ is a function for each $\eta \in \operatorname{dom}(q)$,
3. $\bigcup\{\operatorname{supp} q(\eta) : \eta \in \operatorname{dom}(q)\} \subset \operatorname{supp} c$,
4. $\operatorname{dom}(c) = \left[\operatorname{supp} c\right]^2$.

The determined conditions are dense in $\mathcal{C} * P_\alpha$.

Lemma 3.8. *In $V^{\mathcal{C}*P_\alpha}$ for each $J \in [\alpha]^{<\omega}$ there is $\mu < \omega_1$ such that $\{f_\xi : \xi \in J\}$ acts loop-free on $\omega_1 \setminus \mu$.*

Proof: We work in $V[\mathcal{G}]$, where $\mathcal{G}$ is the $\mathcal{C} * P_\alpha$-generic filter over V. The lemma will be proved by induction on $\max J$. Let $\zeta = \max J$ and $J' = J \setminus \{\zeta\}$. Using the inductive hypothesis fix $\mu < \omega_1$ such that

(a) $\mu = \bigcup\{B \in \mathcal{B}_\zeta : B \cap \mu \neq \emptyset\}$,
(b) if $A \in \mathcal{A}_\zeta$ and $A \subset \omega_1 \setminus \mu$, then A is f_ξ-closed for each $\xi \in J'$,
(c) $\{f_\xi : \xi \in J'\}$ acts loop-free on $\omega_1 \setminus \mu$.

Assume on the contrary that $\langle x_0, \dots, x_n\rangle \in {}^{n+1}(\omega_1 \setminus \mu)$ is an $\{f_\xi : \xi \in J\}$-loop witnessed by the sequences $\langle g_i : i < n\rangle \in {}^n\{f_\xi : \xi \in J\}$ and $\langle k_i : i < n\rangle \in {}^n\{-1, +1\}$. Let $M = \{m < n : g_m = f_\zeta\}$. By the induction hypothesis $M \neq \emptyset$. Write $M = \{m_j : j < \ell\}$, $m_0 < \dots < m_{\ell-1}$. Let $y_0 = x_{m_0}$, $y_1 = x_{m_1}$, $\dots$, $y_{\ell-1} = x_{m_{\ell-1}}$ and $y_\ell = x_{m_0}$. Pick a determined condition $\langle c, q\rangle \in \mathcal{G}$ such that $y_j, f_\zeta^{k_{m_j}}(y_j) \in \operatorname{dom}(q(\zeta)) \cap \operatorname{ran}(q(\zeta))$ for each $j < \ell$. We claim that $\langle y_j : j \leq \ell\rangle$ is a $q(\zeta)$-loop witnessed by the sequence $\langle k_{m_j} : j < \ell\rangle$, which contradicts the choice of Q_ζ. Condition (iii) holds because $\operatorname{rk}_\zeta(y_{j+1}) = \operatorname{rk}_\zeta(f_\zeta^{k_{m_j}}(y_j))$ by (b). Assume on the contrary that (iv) fails, i.e. there is $j < \ell$ such that $\{k_{m_j}, k_{m_{j+1}}\} = \{-1, +1\}$, $y_{j+1} = f_\zeta^{k_{m_j}}(y_j)$, $y_{j+2} = f_\zeta^{k_{m_{j+1}}}(y_{j+1})$ and $y_j = y_{j+2}$. Since $f_\zeta^{k_{m_j}}(y_j) = f_\zeta^{k_{m_j}}(x_{m_j}) = x_{m_j+1}$ and $y_{j+1} = x_{m_{j+1}}$, and so $x_{m_j+1} = x_{m_{j+1}}$, by (c) it follows that $m_j + 1 = m_{j+1}$. Similarly, $m_{j+1} + 1 = m_{j+2}$. Thus $x_{m_j} = y_j$, $x_{m_j+1} = y_{j+1}$ and $x_{m_j+2} = y_{j+2}$. So $x_{m_j} = x_{m_j+2}$, $g_{m_j} = g_{m_j+1} = f_\zeta$ and $\{k_{m_j}, k_{m_j+1}\} = \{-1, +1\}$ which contradicts our assumption that $\langle g_i : i < n\rangle$ and $\langle k_i : i < n\rangle$ satisfied (viii). $\square$

Lemma 3.9. *The induction hypothesis* (II) *holds in $V^{\mathcal{C}*P_\alpha}$, i.e.*
$V^{\mathcal{C}*P_\alpha} \models$ "$\forall X \in [\omega_1]^{\omega_1}\ \exists Y \in [X]^{<\omega}\ \forall \delta < \omega_1\ \exists A \in [X \setminus \delta]^\omega$ *A is dense in $G[X \setminus Y]$*".

Proof: Assume that

$$1_{\mathcal{C}*P_\alpha} \Vdash X = \{\dot{x}_\xi : \xi < \omega_1\} \in [\omega_1]^{\omega_1}.$$

Pick determined conditions $p_\xi = \langle c_\xi, q_\xi\rangle \in \mathcal{C} * P_\alpha$ and $x_\xi \in \omega_1$ such that $p_\xi \Vdash$ "$\dot{x}_\xi = x_\xi$". We can assume that $x_\xi \in \operatorname{supp} c_\xi$. Write $J_\xi = \operatorname{dom} q_\xi$ and $Z_\xi = \operatorname{supp}(c_\xi)$.

Now there is $K \in [\omega_1]^{\omega_1}$ such that the conditions $\{p_\xi : \xi \in K\}$ are "*pairwise twins*", i.e.

(1) $\{Z_\xi : \xi \in K\}$ forms a Δ-system with kernel Z,
(2) $\{J_\xi : \xi \in K\}$ forms a Δ-system with kernel J,

(3) $\max Z < \min(Z_\xi \setminus Z) < \max(Z_\xi \setminus Z) < \min(Z_{\xi'} \setminus Z)$ for $\xi < \xi' \in K$,

(4) $|Z_\xi| = |Z_{\xi'}|$ for $\{\xi, \xi'\} \in [K]^2$; denote by $\varphi_{\xi,\xi'}$ the natural bijection between Z_ξ and $Z_{\xi'}$,

(5) $c_{\xi'}((\{\varphi_{\xi,\xi'}(\nu), \varphi_{\xi,\xi'}(\nu')\})) = c_\xi(\{\nu, \nu'\})$ for $\{\nu, \nu',\} \in [Z_\xi]^2$ and $\{\xi, \xi'\} \in [K]^2$,

(6) $q_{\xi'}(\eta) = \{\langle \varphi_{\xi,\xi'}(\nu), \varphi_{\xi,\xi'}(\nu')\rangle : \langle \nu, \nu'\rangle \in q_\xi(\eta)\}$ for $\eta \in J$ and $\{\xi, \xi'\} \in [K]^2$.

Since $\mathcal{B}_\eta$ is a partition of ω_1 into countable pieces for $\eta \in J$, there is a club set $C = \{\gamma_\nu : \nu < \omega_1\} \subset \omega_1$ in $V^{\mathcal{C}*\mathcal{P}_\alpha}$ such that for each $\eta \in J$ and $\nu < \omega_1$ we have

$$[\gamma_\nu, \gamma_{\nu+1}) = \bigcup\{B \in \mathcal{B}_\eta : B \cap [\gamma_\nu, \gamma_{\nu+1}) \neq \emptyset\}.$$

Since $\mathcal{C} * P_\alpha$ is c.c.c we can assume that $C \in V$.

By thinning out K we can assume that if $\xi < \xi' \in K$, then there is $\gamma \in C$ such that $\max(Z_\xi \setminus Z) < \gamma < \min(Z_{\xi'} \setminus Z)$, moreover $\max Z < \min C$.

By Lemma 3.8 fix $\mu \in C$ such that $\delta \le \mu$ and $1_{\mathcal{C}*P_\alpha} \Vdash$ "$\{f_\eta : \eta \in J\}$ *acts loop-free on* $\omega_1 \setminus \mu$".

If $\vec{E} = \langle \eta_0, \ldots, \eta_{n-1}\rangle \in {}^nJ$ and $\vec{k} = \langle k_0, \ldots, k_{n-1}\rangle \in {}^n\{-1, +1\}$ for some $n \in \omega$, then let

$$f_{\langle \vec{E},\vec{k}\rangle} = f_{\eta_{n-1}}^{k_{n-1}} \circ \cdots \circ f_{\eta_0}^{k_0}.$$

If $p = \langle c, q\rangle$ is determined and $J \subset \mathrm{dom}(q)$ we define the *q-approximation of* $f_{\langle \vec{E},\vec{k}\rangle}$, $f^q_{\langle \vec{E},\vec{k}\rangle}$, in the natural way:

$$f^q_{\langle \vec{E},\vec{k}\rangle} = q(\eta_{n-1})^{k_{n-1}} \circ \cdots \circ q(\eta_0)^{k_0}.$$

We say that $f_{\langle \vec{E},\vec{k}\rangle}$ is *irreducible* if there is no $i < n-1$ such that $\eta_i = \eta_{i+1}$ and $\{k_i, k_{i+1}\} = \{-1, +1\}$.

Let $\xi \in K$ be arbitrary. An irreducible $f_{\langle \vec{E},\vec{k}\rangle}$ is *active* iff $\mathrm{dom}\, f^{q_\xi}_{\langle \vec{E},\vec{k}\rangle} \cap (Z_\xi \setminus Z) \neq \emptyset$, i.e. there is a sequence $\vec{x} = \langle x_0, \ldots, x_{n-1}\rangle \in {}^n(Z_\xi \setminus Z)$ such that $x_{i+1} = q_\xi(\eta_i)^{k_i}(x_i)$ for $i < n$. Observe that the definition of activeness above does not depend on the choice ξ because the conditions $\{\langle c_\xi, q_\xi\rangle : \xi \in K\}$ are pairwise twins.

We say that $\vec{x}$ witnesses that $f_{\langle \vec{E},\vec{k}\rangle}$ is active.

Let $K' \in [K]^\omega$, $\dot{A} = \{\langle p_\xi, x_\xi\rangle : \xi \in K'\}$ and $\zeta \in K \setminus K'$. Let $r^* = \langle c^*, q^*\rangle \le p_\zeta$ be a determined condition such that for each active $f_{\langle \vec{E},\vec{k}\rangle}$ and $w \in Z$ the value $f^{q^*}_{\langle \vec{E},\vec{k}\rangle}(w)$ is defined. Let

$$Y = \{f^{r^*}_{\langle \vec{E},\vec{k}\rangle}(w) : f_{\langle \vec{E},\vec{k}\rangle} \text{ is active and } w \in Z\}.$$

Claim. *Y is finite.*

Proof of the claim: Since $\{f_\eta : \eta \in J\}$ acts loop-free on $Z_\zeta \setminus Z$, the elements of a witnessing sequence are pairwise different, so there are only finitely many of them and a witnessing sequence works only for one active $f_{\left\langle \vec{E},\vec{k}\right\rangle}$. So there is only finitely many active $f_{\left\langle \vec{E},\vec{k}\right\rangle}$. $\square$

We show that

$$(\bullet)\qquad r^* \Vdash \dot{A} \text{ is dense in } G[\omega_1 \setminus Y].$$

which completes the proof of Lemma 3.9.

To verify $(\bullet)$ assume that $r' \le r^*$, $r' = \left\langle c', q'\right\rangle$ is determined, $B \in \left[\omega_1 \setminus Y\right]^{<\omega}$ and $b \in {}^B2$.

Pick $\xi \in K$ such that $\operatorname{supp}(c') \cap \operatorname{supp}(c_\xi) = Z$ and $\operatorname{dom}(q') \cap \operatorname{dom}(q_\xi) = J$. To prove $(\bullet)$, it is enough to construct a common extension $p = \langle c, q\rangle$ of $r' = \left\langle c', q'\right\rangle$ and $p_\xi = \left\langle c_\xi, q_\xi\right\rangle$ such that $c(x_\xi, \beta) = b(\beta)$ for each $\beta \in B$.

Let $\operatorname{supp} c = \operatorname{supp} c' \cup \operatorname{supp} c_\xi$. Put $\operatorname{dom} q = \operatorname{dom} q' \cup \operatorname{dom} q_\xi$ and let

$$q(\eta) = \begin{cases} q'(\eta) \cup q_\xi(\eta) & \text{if } \ \eta \in J, \\ q'(\eta) & \text{if } \ \eta \in \operatorname{dom} q' \setminus J, \\ q_\xi(\eta) & \text{if } \ \eta \in \operatorname{dom} q_\xi \setminus J. \end{cases}$$

Put $c^- = c' \cup c_\xi$.

We should define $c \supset c^-$ on the set

$$E = \{\{a, b\} : a \in Z_\xi \setminus Z, b \in \operatorname{supp} c' \setminus Z\}$$

such that every $q(\eta)$ is a partial isomorphism of G, more precisely, $q(\eta) \in \operatorname{Iso}_{\mathrm{p}}(G^c, G^c)$. To do so, observe that if we take

$$E^+ = \{\{a, b\} : a \in Z_\xi \setminus Z, b \in \operatorname{supp} c'\}$$

and for $e \in E^+$ define $a_e = e \cap (Z_\xi \setminus Z)$ and $b_e = e \cap \operatorname{supp} c'$, then $q(\eta) \in \operatorname{Iso}_{\mathrm{p}}(G^c, G^c)$ if and only if $(\dagger)$ below holds:

$$(\dagger)\qquad \text{if } e = \{a_e, b_e\} \in E^+, \text{ then } c\{a_e, b_e\} = c\{q_\xi(\eta)(a_e), q'(\eta)(b_e)\}.$$

Define an equivalence relation $\equiv$ on E^+: $e \equiv e'$ iff $e = e'$ or there is an active $f_{\left\langle \vec{E},\vec{k}\right\rangle}$ such that $a_{e'} = f^{q_\xi}_{\left\langle \vec{E},\vec{k}\right\rangle}(a_e)$ and $b_{e'} = f^{q'}_{\left\langle \vec{E},\vec{k}\right\rangle}(b_e)$.

Claim 3.9.1. *If $e \equiv e'$ and $a_e = a_{e'}$, then $e = e'$.*

Proof of Claim 3.9.1: Assume $e \equiv e'$ and $b_e \neq b_{e'}$. Then there is an active $f_{\langle \vec{E},\vec{k}\rangle}$ such that $a_{e'} = f^{q_\xi}_{\langle \vec{E},\vec{k}\rangle}(a_e)$ and $b_{e'} = f^{q'}_{\langle \vec{E},\vec{k}\rangle}(b_e)$. Since $1 \Vdash$ "$\{f_\eta : \eta \in J\}$ acts freely on $\omega_1 \setminus \mu$" it follows that $a_e \neq f^{q'}_{\langle \vec{E},\vec{k}\rangle}(a_e)$ and so $a_e \neq a_{e'}$. □

Claim 3.9.2. *If $e, e' \in E^+ \cap \mathrm{dom}(c^-)$ and $e \equiv e'$, then $c^-(e) = c^-(e')$.*

Proof of Claim 3.9.2: Fix an active $f_{\langle \vec{E},\vec{k}\rangle}$ such that $a_{e'} = f^{q_\xi}_{\langle \vec{E},\vec{k}\rangle}(a_e)$ and $b_{e'} = f^{q'}_{\langle \vec{E},\vec{k}\rangle}(b_e)$. Since $e, e' \in E^+ \cap \mathrm{dom}(c^-)$ it follows that $e, e' \in \mathrm{dom}(c_\xi)$ and so $b_e, b_{e'} \in Z$. If $b_e \in \mathrm{dom}\, f^{q_\xi}_{\langle \vec{E},\vec{k}\rangle}$, then $b_{e'} = f^{q_\xi}_{\langle \vec{E},\vec{k}\rangle}(b_e)$ and so $c^-(e) = c_\xi(e) = c_\xi(e') = c^-(e')$ because of $p_\xi = \langle c_\xi, q_\xi \rangle \in \mathcal{C} * P_\alpha$. Unfortunately, $b_e \in \mathrm{dom}\, f^{q_\xi}_{\langle \vec{E},\vec{k}\rangle}$ cannot be guaranteed, so we need an additional argument here.

Let $\varphi = \varphi_{\xi,\zeta}$ be the function witnessing that p_ξ and p_ζ are twins.

Since $f_{\langle \vec{E},\vec{k}\rangle}$ is active and $b_e \in Z$ it follows that $f^{q^*}_{\langle \vec{E},\vec{k}\rangle}(b_e)$ is defined and so $f^{q^*}_{\langle \vec{E},\vec{k}\rangle}(b_e) = b_{e'}$. Put $\underline{e} = \varphi'' e$ and $\underline{e}' = \varphi'' e'$. Since $c^-(e) = c_\xi(e) = c_\zeta(\underline{e}) = c^*(\underline{e})$ and $c^-(e') = c_\xi(e') = c_\zeta(\underline{e}') = c^*(\underline{e}')$ it is enough to show that $c^*(\underline{e}) = c^*(\underline{e}')$.

First observe that $b_{\underline{e}} = \varphi(b_e) = b_e$, $b_{\underline{e}'} = \varphi(b_{e'}) = b_{e'}$ and so $b_{\underline{e}'} = f^{q^*}_{\langle \vec{E},\vec{k}\rangle}(b_{\underline{e}})$. Moreover $a_{\underline{e}'} = \varphi(a_{e'}) = \varphi(f^{q_\xi}_{\langle \vec{E},\vec{k}\rangle}(a_e)) = f^{q_\zeta}_{\langle \vec{E},\vec{k}\rangle}(\varphi(a_e)) = f^{q_\zeta}_{\langle \vec{E},\vec{k}\rangle}(a_{\underline{e}}) = f^{q^*}_{\langle \vec{E},\vec{k}\rangle}(a_{\underline{e}})$. Thus using $r^* = \langle c^*, q^* \rangle \leq \langle c_\zeta, q_\zeta \rangle$ we have

$$c^*(a_{\underline{e}'}, b_{\underline{e}'}) = c^*(f^{q^*}_{\langle \vec{E},\vec{k}\rangle}(a_{\underline{e}}), f^{q^*}_{\langle \vec{E},\vec{k}\rangle}(b_{\underline{e}})) = c^*(a_{\underline{e}}, b_{\underline{e}}),$$

which completes the proof of the claim. □

Claim 3.9.3. *If $e \in E^+ \cap \mathrm{dom}(c^-)$ and $e \equiv e'$, then $b_{e'} \in Y$.*

Proof of Claim 3.9.3: Since $e \in E^+ \cap \mathrm{dom}(c^-)$ we have $b_e \in Z$. Fix an active $f_{\langle \vec{E},\vec{k}\rangle}$ such that $a_{e'} = f^{q_\xi}_{\langle \vec{E},\vec{k}\rangle}(a_e)$ and $b_{e'} = f^{q'}_{\langle \vec{E},\vec{k}\rangle}(b_e)$. Since $f_{\langle \vec{E},\vec{k}\rangle}$ is active it follows that $f^{q^*}_{\langle \vec{E},\vec{k}\rangle}(b_e)$ is defined and $f^{q^*}_{\langle \vec{E},\vec{k}\rangle}(b_e) \in Y$. But $f^{q'}_{\langle \vec{E},\vec{k}\rangle}(b_e) = f^{q^*}_{\langle \vec{E},\vec{k}\rangle}(b_e)$ so $b_{e'} \in Y$ which was to be proved. □

By Claims 3.9.1–3.9.3 we can find a condition $c \in \mathcal{C}$ with $\operatorname{supp} c = \operatorname{supp} c' \cup \operatorname{supp} c_\xi$ and $\operatorname{dom} c = \left[\operatorname{supp} c\right]^2$ such that

(a) $c \supset c^- = c' \cup c_\xi$,
(b) $c(e) = c(e')$ whenever $e \equiv e'$,
(c) $c(\{x_\xi, \beta\}) = b(\beta)$ for $\beta \in B$.

Then (†) holds and as we have seen above, $\langle c, q\rangle \in \mathcal{C} * P_\alpha$ and

$$\langle c, q\rangle \Vdash (\forall\, \beta \in B)\ \{x_\xi, \beta\} \in E(G) \ \text{ iff } \ b(\beta) = 1.$$

Thus (•) holds. Hence Lemma 3.9. is proved. □

So we have shown that (II) is preserved during the inductive construction, which was the last step to prove Theorem 3.1. □

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