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On Asplund functions

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Abstract. A class of convex functions where the sets of subdifferentials behave like the unit ball of the dual space of an Asplund space is found. These functions, which we called Asplund functions also possess some stability properties. We also give a sufficient condition for a function to be an Asplund function in terms of the upper-semicontinuity of the subdifferential map.

Keywords: Fréchet differentiability, convex functions, Asplund spaces

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Introduction

It is known that a Banach space is an Asplund space if and only if B_{X^*} is dentable if and only if (B_{X^*}, w^*) is fragmentable by norm, and if and only if every separable subspace Y of X has separable dual Y^*. It is also known that being an Asplund space is a three-space property.

The purpose of this note is to present a functional version of this theory. We study these equivalent conditions in a certain class of functions which may be defined on a non-Asplund space. For instance, suppose g is a continuous convex function defined on an Asplund space Y and $T : X \to Y$ is a bounded linear map. Then regardless of X, the function defined by $f = g \circ T$ is a generically Fréchet differentiable convex function. The function f and all convex functions bounded above by f belong to the class which we want to consider. These functions exhibit properties similar to those of continuous convex functions defined on an Asplund space.

In Section 1, we present a theorem that consists of several equivalent conditions that are well known in the Asplund space version. We call a continuous convex function an *Asplund function* if it satisfies any of these conditions. As a consequence, a Banach space is an Asplund space if and only if its norm is an Asplund function.

In Section 2, we show that the property of being an Asplund function is stable under restriction and taking quotient to a subspace. However, it does not enjoy a three-space like property. Nevertheless, a condition of a subspace is given to ensure a function is Asplund whenever its restriction to that subspace is Asplund. Further, we modify the proof of a theorem in [C-P] to yield a sufficient condition for a function to be Asplund.

Some related problems were studied in [T_1] and [T_2]. In contrast to these two articles, we do not attempt to establish any approximation theorem in this paper, for we know that Asplund spaces in general may not even admit a Gâteaux differentiable norm.

Notation and preliminaries

We will use the standard notation in the theory of convex functions on a Banach space and Banach space theory. Given a continuous convex function f on a Banach space X, the subdifferential of f at a point x is defined by $\partial f(x) = \{x^* \in X^* : x^*(y-x) \leq f(y) - f(x) \text{ for all } y \in X\}$. The Legendre-Fenchel conjugate f^* of f is defined by $f^*(x^*) = \sup\{(x^*, x) - f(x) : x \in X\}$ for all $x \in X$. The function f^* is w^*-lower semicontinuous, i.e., $\{x^* \in X^* : f^*(x^*) \leq r\}$ is w^*-closed for all $r \in \mathbb{R}$. A continuous convex function is said to be generically Fréchet differentiable if it is Fréchet differentiable on a dense G_δ set. Given a bounded convex set $A \subset X^*$, the indicator function $\delta_A(\cdot)$ is a convex function that takes values zero on A and $+\infty$ elsewhere. The function $\delta_A(\cdot)$ is w^*-lower semicontinuous if and only if A is w^*-closed. A slice of $A \subset X^*$ is a set of the form $S(A, x^{**}, \alpha) = \{x^* \in X^* : x^{**}(x^*) > \sup x^{**}(A) - \alpha\}$, for some $x^{**} \in X^{**}$ and $\alpha > 0$. If $x^{**} \in X$, then $S(A, x^{**}, \alpha)$ is called a w^*-slice. We say that a set F is dentable (w^*-dentable) if for every $\varepsilon > 0$ every bounded subset of F has slices (w^*-slices) of diameter less than ε. Let (Z, τ) be a topological space and ρ be a metric on Z that is not necessarily related to the topology of Z. The space Z is said to be fragmentable by ρ if every non-empty subset of Z admits relatively open sets of arbitrarily small ρ-diameter. We refer the readers to [J-N-R] for the theory of fragmentability. An infinite tree in a set $A \subset X$ is a sequence $\{x_n\}$ in A such that $x_n = \frac{1}{2}(x_{2n} + x_{2n+1})$ for each n. An infinite tree such that $\|x_{2n} - x_{2n+1}\| > 2\varepsilon$ for all n is called an infinite ε-tree. Unless otherwise stated, all topological notions in the dual space refer to the norm topology. We refer to [Ph] and [D-G-Z] for all other unexplained notions and results. We also refer the reader to [Y] for an excellent introduction to the theory of Asplund spaces.

1. Asplund functions

In this section, we establish using known techniques some equivalent definitions of an Asplund function. These properties will be used in the subsequent sections. We find the following fact useful:

Lemma 1. *If f is a continuous convex function defined on X, then we have the following inclusions: $\partial f(X) \subset \operatorname{dom} f^* \subset \overline{\partial f(X)}^{\|\cdot\|}$.*

The first inclusion is clear. For the second inclusion, we apply Ekeland's variational principle as in [F] (see also the proof of (iii) $\Rightarrow$ (i) in [T_1, Theorem 1]).

The main result in this section is the following:

Theorem 2. *Let f be a continuous convex function defined on a Banach space X. Then the following assertions are equivalent.*

(1) *If h is a continuous convex function on X such that $h \leq f$ on X, then h is generically Fréchet differentiable on X.*
(2) *For each positive integer n, every bounded subset of (C_n, w^*), where $C_n = \{x^* \in X^* : f^*(x^*) \leq n\}$, is fragmentable by the norm.*
(3) *For each separable subspace Y of X, the set $\mathrm{dom}\,(f_{\restriction Y})^*$ is separable.*
(4_a) *Every compact subset of $(\mathrm{dom}\ f^*, w^*)$ is fragmentable by the norm.*
(4_b) *Every compact subset of $(\mathrm{dom}\ f^*, w^*)$ is w^*-dentable.*
(5) *For every $\varepsilon > 0$, no w^*-compact subset of $\mathrm{dom}\ f^*$ contains an ε-tree.*
(6) *Every compact subset of $(\mathrm{dom}\ f^*, w^*)$ is dentable.*

If moreover, f is bounded on bounded sets, then the above conditions are also equivalent to:

(7) *let Y be a separable subspace of X. If h is a continuous convex function on X such that $h \leq f$ then there is a selector s for $\partial h_{\restriction Y}$ such that $s(Y) = \{s(y) : y \in Y\}$ is separable.*

Proof: (1) $\Rightarrow$ (2). Indeed, for otherwise, there exists a bounded w^*-closed subset A of C_n that is not fragmentable by the norm. The function $h(\cdot) = ((\delta_A(\cdot) + n)^*)_{\restriction X}$ is bounded above by f and it can be checked that h is nowhere Fréchet differentiable (cf., for instance, [Ph, 2.18]).

(2) $\Rightarrow$ (3). Let Y be a separable subspace of X. By Lemma 1, to show the separability of $\mathrm{dom}\,(f_{\restriction Y})^*$, it suffices to establish the separability of $\partial f_{\restriction Y}(Y)$. Let $R : X^* \to Y^*$ be the restriction map. The map R is w^* to w^* continuous. By Hahn-Banach theorem, we have

$$R\partial f(y) = \partial f_{\restriction Y}(y) \quad \text{for all} \quad y \in Y. \tag{1}$$

For each positive integer n, we define the sets C_n^Y and H_n as follows:

$$C_n^Y = \left\{ y^* \in Y^* : f^*_{\restriction Y}(y^*) \leq n \right\}, \quad \text{and}$$
$$H_n = \partial f_{\restriction Y}(Y) \cap C_n^Y .$$

We note that

$$\begin{aligned} \bigcup_{n=1}^{\infty} H_n = \bigcup_{n=1}^{\infty} (\partial f_{\restriction Y}(Y) \cap C_n^Y) &= \partial f_{\restriction Y}(Y) \cap \Big(\bigcup_{n=1}^{\infty} C_n^Y \Big) \\ &= \partial f_{\restriction Y}(Y) \cap \mathrm{dom}\ f^*_{\restriction Y} = \partial f_{\restriction Y}(Y). \end{aligned} \tag{2}$$

We claim that $H_n \subset R(C_n)$ for each $n \in \mathbb{N}$. Indeed, let $y^* \in H_n$, then $y^* \in \partial f_{\restriction Y}(y)$ for some $y \in Y$. According to (1), we can find a $\hat{y}^* \in \partial f(y)$ such that $R(\hat{y}^*) = y^*$.

We have

$$\begin{aligned} f^*(\hat{y}^*) &= (\hat{y}^*, y) - f(y) \\ &= (y^*, y) - f_{\restriction Y}(y) \\ &= f^*_{\restriction Y}(y^*) \le n. \end{aligned}$$

Therefore $\hat{y}^* \in C_n$, and hence $y^* = R(\hat{y}^*) \in R(C_n)$. From the claim and (2) we have $\partial f_{\restriction Y}(Y) \subset \bigcup_{n=1}^{\infty} R(C_n)$. Suppose $\partial f_{\restriction Y}(Y)$ is not separable, then there exists an integer N such that $R(C_N)$ is not separable. Therefore there exists a $k \in \mathbb{N}$ such that $R(C_N \cap kB_{X^*})$ is not separable. For simplicity, we write $C = C_N \cap kB_{X^*}$.

Note that $R(C)$ is a w^*-compact subset of Y^*. Since Y is separable and $R(C)$ is a nonseparable subset of Y^*, by the arguments as in [Ph, 2.19], we obtain an uncountable set $A \subset R(C)$ and $\varepsilon > 0$ such that any w^*-open subset of A contains two distinct points x^* and y^* such that $\|x^* - y^*\| > \varepsilon$.

Now we follow the proof of [Ph, 5.4]; let $A_1 \subset C$ be a minimal w^*-compact set such that $R(A_1) = \overline{A}^{w^*}$. If U is a non-empty relatively w^*-open subset of A_1, then $A_1 \backslash U$ is w^*-compact and $A_2 = R(A_1 \backslash U)$ is a proper w^*-compact subset of $\overline{A}^{w^*}$ (since A_1 is minimal). Thus $A \backslash A_2$ is a non-empty w^*-open subset of A and it contains two distinct points which are at least ε far apart. Therefore there exist x^* and y^* in U such that $\|x^* - y^*\| > \varepsilon$, contradicting the assumption that C is norm fragmentable.

(3) $\Rightarrow$ (1). According to the proof of [Ph, Theorem 2.11], $f_{\restriction Y}$ is generically differentiable for each separable subspace $Y \subset X$. By the separable reduction theorem in [Gi] (see also [Pr]), f is generically Fréchet differentiable.

(2) $\Rightarrow$ (4_a). We first note that (dom f^*, w^*) is a countable union of norm fragmentable (compact) subsets. Indeed dom $f^* = \bigcup_{n,k \in N} C_{n,k}$, where $C_{n,k} = C_n \cap kB_{X^*}$. Each $C_{n,k}$ is norm fragmentable by assumption. Hence (dom f^*, w^*) is σ-fragmentable by the norm. Consequently by [J-N-R, Theorem 3.1], every compact subset is norm fragmentable.

(4_b) $\Rightarrow$ (2). Given a bounded subset B of C_n, there exists a $k \in \mathbb{N}$ such that $B \subset C_n \cap kB_{X^*}$. Since $C_n \cap kB_{X^*}$ is w^*-compact, B admits slices of small diameter.

(4_a) $\Leftrightarrow$ (4_b). It is clear that (4_b) $\Rightarrow$ (4_a). The proof of (4_a) $\Rightarrow$ (4_b) is identical to that of [N-Ph, Lemma 3].

(4_b) $\Rightarrow$ (6). This is clear as every w^*-slice is a (weak) slice.

(6) $\Rightarrow$ (5). From the definition of a tree, every slice of an infinite ε-tree has diameter at least ε.

(5) $\Rightarrow$ (4_a). It is enough to follow [Du-N] or the proof of [Ph, 5.6].

Clearly (3) $\Rightarrow$ (7).

Finally, suppose f is bounded on bounded sets. We shall show (7) $\Rightarrow$ (3). Let Y be a given separable subspace of X. For each n define a convex function on Y^* by

$$g_n(y^*) = \begin{cases} (f_{\restriction Y})^*(y^*) & \text{if } y^* \in C_n^Y = \{z^* \in \operatorname{dom}(f_{\restriction Y})^* \mid (f_{\restriction Y})^*(z^*) \le n\} \\ \infty & \text{otherwise.} \end{cases}$$

Then g_n is a w^*-lower semicontinuous function such that $g_n \ge (f_{\restriction Y})^*$ and $\operatorname{dom} g_n = C_n^Y$. Note that $\bigcup_{n=1}^{\infty} C_n^Y = \operatorname{dom}(f_{\restriction Y})^*$, hence it suffices to show that C_n^Y is separable for all n. Let f_n be a continuous convex function on Y such that $(f_n)^* = g_n$, then $f_n \le f_{\restriction Y}$. The function f_n may be extended to a convex function on X that is bounded above by f. (For instance, the convex hull of the epigraphs of f_n and f is the epigraph of a required extension of f_n.) By the hypothesis, there is a selector s of ∂f_n such that $s(Y)$ is separable. Put $B = s(Y)$. Without loss of generality, assume that $f(0) < 0$. Therefore $g_n(y^*) > 0$ for all $y^* \in Y^*$. Let $\gamma = \inf\{g_n(y^*) | y^* \in B\} \ge 0$. To establish the separability of C_n^Y, it is sufficient to show that $C_n^Y \subset \overline{conv}^{\|\cdot\|} B$. Suppose this is not the case, let $y_0^* \in C_n^Y \setminus \overline{conv}^{\|\cdot\|} B$. By the separation theorem, there exists $z \in Y^{**}$, $\alpha, \beta \in \mathbb{R}$ such that

$$z(y_0^*) > \beta > \alpha > z(y^*)$$

for all $y^* \in B$. By scaling z, α and β if necessarily, we may assume that $\frac{\beta - \alpha}{2} > g_n(y_0^*) - \gamma$. Let $E = \{y \in Y : \|y\| \le \|z\|,\ (y, y_0^*) > \beta\}$.

Let $\{y_k^*\}_{k \ge 1}$ be a countable dense subset of B. Now, for every positive integer n let $y_n \in E$ be such that

$$|(z - y_n, y_k^*)| < \frac{1}{n} \quad \text{for} \quad k = 1, 2, \ldots, n.$$

Then for each k we have

$$\lim_{n \to \infty} (z - y_n, y_k^*) = 0.$$

As $\{y_n\}$ is bounded, $\lim_{n \to \infty}(z - y_n, y^*) = 0$ for each $y^* \in B$. For each $y \in Y$, define a function h_y on B by

$$h_y(y^*) = (y^*, y) - g_n(y^*).$$

For each $k \in \mathbb{N}$, let $h_k = h_{y_k}$. By the boundedness of the function f, the sequence $\{h_k\}$ is uniformly bounded on B.

Note that for any $y = \sum_{k=1}^{\infty} \lambda_k y_k$, where $\lambda_k \ge 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$, we have

$s(y) \in B$ and

$$\begin{aligned}\sum_{k=1}^{\infty} \lambda_k h_k (s(y)) &= \sum_{k=1}^{\infty} \lambda_k \{(s(y), y_k) - g_n (s(y))\} \\ &= h_y (s(y)) \\ &= f_n (y) \\ &= \sup \{(y^*, y) - g_n (y^*) : y^* \in \operatorname{dom} g_n\} \\ &= \sup \left\{ \left(y^*, \sum_{k=1}^{\infty} \lambda_k y_k \right) - g_n (y^*) : y^* \in C_n^Y \right\} \\ &= \sup \left\{ \sum_{k=1}^{\infty} \lambda_k h_k (y^*) : y^* \in C_n^Y \right\}.\end{aligned}$$

Since $z(y^*) < \alpha$, we have $\limsup_k h_k(y^*) \leq \alpha - g_n(y^*)$ for all $y^* \in B$. Consequently

$$\sup \left\{ \limsup_k h_k (y^*) : y^* \in B \right\} \leq \alpha - \gamma.$$

But by Simons' inequality ([S]), there is a function h,

$$h = \sum_{k=1}^{N} \rho_k h_k,$$

where $\rho_k \geq 0$ and $\sum_{k=1}^{N} \rho_k = 1$, such that

$$\sup \left\{ h(y^*) : y^* \in C_n^Y \right\} \leq \frac{\alpha + \beta}{2} - \gamma.$$

On the other hand,

$$\begin{aligned}h(y_0^*) &= \sum_{k=1}^{N} \rho_k h_k (y_0^*) \\ &= \left(y_0^*, \sum_{k=1}^{N} \rho_k y_k \right) - g_n (y_0^*) > \beta - g_n (y_0^*)\end{aligned}$$

and hence $\beta - g_n(y_0^*) < \frac{\alpha+\beta}{2} - \gamma$. Therefore $\frac{\beta-\alpha}{2} < g_n(y_0^*) - \gamma$ and this contradiction shows that (7) implies (3).

□

Definition 3. Let f be a convex function on a Banach space X, we say that f is an Asplund function if f satisfies any of conditions (1) to (6).

Corollary 4. *If f is an Asplund function, then every w^*-compact subset of* $\operatorname{dom} f^*$ *is w^*-sequentially compact.*

Proof: Let K be a w^*-compact subset of $\operatorname{dom} f^*$. By Theorem 2, K is w^*-dentable, therefore by [St, 3.4], K is w^*-sequentially compact.

$\square$

The following theorem is a consequence of Theorem 2.

Theorem 5. *For a Banach space $(X, \|\cdot\|)$, the following are equivalent:*

(i) *X is an Asplund space;*
(ii) *every continuous convex function on X is an Asplund function;*
(iii) *$\|\cdot\|$ is an Asplund function.*

Remark 1. In [Gi-Sc], the authors gave a sufficient condition for a continuous convex function defined on a open subset A of a Banach space to be generically Fréchet differentiable. They showed that such a function ϕ is generically Fréchet differentiable if for every separable subspace Y where $A \cap Y \neq \emptyset$, $\partial\phi_{\restriction Y}(A \cap Y)$ is separable. This follows from our Theorem 2 when $A = X$.

2. Stability of Asplund functions

Definition 6. Let f be a continuous convex function defined on a Banach space X and let M be a subspace of X. The *quotient function* $\tilde{f}_M$ induced by f is a continuous convex function on the quotient space X/M defined by

$$\tilde{f}_M\,(\hat{x}) = \inf\{f(x+m) : m \in M\},$$

where $\hat{x}$ denotes the coset $x + M$. If f is a norm, $\tilde{f}_M$ is precisely the quotient norm.

Given a subspace M of a Banach space X, the dual space of X/M is isometrically isomorphic to $M^\perp$, the isomorphism is given by $\Phi : M^\perp \to (X/M)^*$, where $\Phi(x^*)(\hat{x}) = x^*(x)$, $x^* \in M^\perp$, $x \in X$. We shall see in the following lemma that the above identification also behaves well in a non-linear situation.

Lemma 7. *Under the above notation, we have* $\Phi(\operatorname{dom} f^* \cap M^\perp) = \operatorname{dom}(\tilde{f}_M)^*$.

Proof: Let $x^* \in \operatorname{dom} f^* \cap M^\perp$ and let $\varphi = \Phi(x^*)$. We need to verify that $\varphi \in \operatorname{dom}(\tilde{f}_M)^*$. To this end, let $\hat{x} \in (X/M)$ and let $\varepsilon > 0$. Pick an $m \in M$ such that $\tilde{f}_M(\hat{x}) \geq f(x+m) - \varepsilon$. Then

$$\begin{aligned}
\varphi(\hat{x}) - \tilde{f}_M\,(\hat{x}) &\leq \varphi(\hat{x}) - f(x+m) + \varepsilon \\
&= \Phi x^*(\hat{x}) - f(x+m) + \varepsilon \\
&= x^*(x+m) - f(x+m) + \varepsilon \leq f^*(x^*) + \varepsilon.
\end{aligned}$$

Therefore $\sup\{\varphi(\hat{x}) - \tilde{f}_M(\hat{x}) : \hat{x} \in (X/M)\} \leq f^*(x^*)$, which means that $\varphi \in \operatorname{dom}(\tilde{f}_M)^*$. To see the reverse inclusion, let $\psi \in \operatorname{dom}(\tilde{f}_M)^*$. Let $y^* \in X^*$ be such that $\Phi(y^*) = \psi$. Then clearly $y^* \in M^\perp$. Now, given any $x \in X$,

$$\begin{aligned} y^*(x) - f(x) &\leq y^*(x) - \tilde{f}_M(\hat{x}) \\ &= \psi(\hat{x}) - \tilde{f}_M(\hat{x}) \leq (\tilde{f}_M)^*(\psi). \end{aligned}$$

□

Proposition 8. *Under the above notation, suppose f is an Asplund function, then so is $\tilde{f}_M$.*

Proof: Let K be a w^*-compact subset of $\operatorname{dom}(\tilde{f}_M)^*$. Since Φ is w^* to w^* continuous, $\Phi^{-1}(K)$ is a w^*-compact subset of $\operatorname{dom} f^*$, and thus it is norm fragmentable by Theorem 2. Therefore K is also norm fragmentable, as Φ is an isometric isomorphism. □

Proposition 9. *Suppose f is an Asplund function, then so is $f_{\restriction M}$.*

Proof: Note that, if $Y \subset M \subset X$, then $(f_{\restriction M})_{\restriction Y} = f_{\restriction Y}$; so Theorem 2 applies. □

At this point, one who is familiar with the theory of Asplund spaces may conjecture that an Asplund function admits a three-space like property, i.e., if $f_{\restriction M}$ and $\tilde{f}_M$ are both Asplund functions, then so is the function f. However, we shall see in the following example that such a trivial generalization does not hold.

Example 10. Let $X = \ell_1 \oplus c_0$, $M = c_0$ and $\|\cdot\|$ be a nowhere Fréchet differentiable norm on ℓ_1. Let $T : c_0 \to \ell_1$ be defined by $T(x_i) = \left(\frac{x_i}{2^i}\right)$, then $T(c_0)$ is norm dense in ℓ_1.

Now we define a real valued function on X as follows:

$$f(x, y) = \|x - Ty\| \quad \text{for} \quad x \in \ell_1 \quad \text{and} \quad y \in c_0.$$

It is easy to see that f is a continuous convex function and it is nowhere Fréchet differentiable (and thus not an Asplund function). The restriction $f_{\restriction M} = f_{\restriction c_0}$ is an Asplund function, as c_0 is an Asplund space. The quotient function $\tilde{f}_M$ is the null function. Indeed, let $\widehat{(x, y)} \in X/M$, then

$$\begin{aligned} \tilde{f}_M\widehat{(x, y)} &= \inf\{f(x, y + m) : m \in c_0\} \\ &= \inf\{\|x - T(y + m)\| : m \in c_0\} \\ &= 0, \end{aligned}$$

as $T(c_0)$ is dense in ℓ_1. □

From the above example, we understand that a stricter condition must be imposed upon the quotient function in order to obtain a three-space like property for the Asplund functions.

Proposition 11. *Let f be a continuous convex function on a Banach space X and M be a subspace of X. Suppose that X/M is an Asplund space and that $f_{\restriction M}$ is an Asplund function, then f is an Asplund function.*

Before we proceed on with the proof, we first establish the separable version of the proposition.

Lemma 12. *Let X be a separable Banach space, suppose that* $\operatorname{dom}(f_{\restriction M})^*$ *and $M^\perp$ are both separable, then* $\operatorname{dom} f^*$ *is also separable.* (*Equivalently, if $f_{\restriction M}$ is an Asplund function and $(X/M)^*$ is separable, then f is an Asplund function.*)

Proof: Let $R : X^* \to M^* = (X^*/M^\perp)$ be the restriction map. It is easy to check that $R\partial f(M) = \partial f_{\restriction M}(M)$. Let $\{x_k^* : k \in \mathbb{N}\}$ be a countable set in $\partial f(M)$ such that $\{R(x_k^*) : k \in \mathbb{N}\}$ is dense in $\partial f_{\restriction M}(M)$ (and thus dense in $\operatorname{dom}(f_{\restriction M}^*)$). Let $\{m_n^\perp : n \in \mathbb{N}\}$ be a countable dense set of $M^\perp$.

Given $x^* \in \operatorname{dom} f^*$ and $\varepsilon > 0$, there is an $R(x_k^*)$ such that $\|R(x^*) - R(x_k^*)\| < \varepsilon/3$. This means that there is an $m_k^\perp \in M^\perp$ such that $\|x^* - x_k^* + m_k^\perp\| < \varepsilon$. Hence $\operatorname{dom} f^*$ lies in the closed linear span of $\{x_k^* : k \in \mathbb{N}\} \cup \{m_n^\perp : n \in \mathbb{N}\}$. □

Proof of Proposition 11: Let S be a separable subspace of X. According to Theorem 2, it suffices to show that $\operatorname{dom}(f_{\restriction S})^*$ is separable. To this end, we follow the arguments from [Y] to obtain a separable subspace Z of X that contains S with the additional property that $M+Z$ is closed and $Z/(Z\cap M) \simeq (M+Z)/M \subset X/M$. So $Z/(Z \cap M)$ is also Asplund. Hence $(Z/(Z \cap M))^* = (Z \cap M)^\perp$ is separable. Since $Z \cap M$ is a subspace of M and $f_{\restriction M}$ is Asplund, $f_{\restriction Z\cap M}$ is also Asplund. Therefore by Lemma 12, $f_{\restriction Z}$ is an Asplund function. According to Proposition 9, $f_{\restriction S}$ is also Asplund, as S is a subspace of Z. □

As noted in Section 1, a Banach space X is an Asplund space if and only if X admits a norm that is an Asplund function. Hence a consequence of Proposition 11 is the following:

Corollary 13 ([N-P, Theorem 14])**.** *Let X be a Banach space and Y be a subspace of X. If both X and X/Y are Asplund spaces, then so is X.*

Definition 14 ([C-P], [Gi-Gr-Si])**.** Let f be a continuous convex function on X. We say that f is quite smooth at x if for every weak neighbourhood W of $0 \in X^*$, there exists a $\delta > 0$ such that

$$\partial f(y) \subset \partial f(x) + W$$

whenever $y \in B(x, \delta)$. We say that f is quite smooth on X if it is quite smooth at each point of X.

Proposition 15. *Let f be a convex function defined on a separable Banach space X. Suppose that f is bounded on bounded sets of X. If every convex continuous function $h \leq f$ is quite smooth, then* $\operatorname{dom} f^*$ *is separable.*

Proof: Our proof is a slight modification of the proof of [C-P, Theorem 1.2]. As in the proof of Theorem 2, it suffices to show that for each n, $C_n = \{x^* :$

$f^*(x^*) \le n\}$ is separable. The function f_n defined as in the proof of Theorem 2 [(7) $\Rightarrow$ (3)] is a continuous convex function on X that is bounded above by f and dom $f_n^* = C_n$. By our hypothesis, f_n is quite smooth. For simplicity, we denote f_n by f.

According to Mazur's theorem, there exists a countable dense set $\{x_k\}$ in X such that f is Gâteaux differentiable at each x_k. For each $k \in \mathbb{N}$, we write $x_k^* = f'(x_k)$ and $F = \{x_k^* : k \in \mathbb{N}\}$. To show that dom $f^* = C_n$ is separable, it suffices to show that dom $f^* \subset \overline{conv}^{\|\cdot\|}F$. Suppose that this is not the case; then there exists $y_0^* \in$ dom $f^* \backslash \overline{conv}^{\|\cdot\|}F$. By the separation theorem, there exists $z^{**} \in X^{**}$ such that

$$z^{**}(y_0^*) > \beta > \alpha > \sup\{z^{**}(x^*) : x^* \in F\}$$

for some $\beta > \alpha$. By scaling the functional z^{**}, α and β, we may assume that $\beta - \alpha > f^*(y_0^*) - \gamma + 1$, where $\gamma = \inf\{f^*(x^*) : x^* \in \text{dom } f^*\} > -\infty$. Let $E = \{x \in X : \|x\| < \|z^{**}\|, (y_0^*, x) > \beta\}$. As in the proof of Theorem 2 [(3) $\Rightarrow$ (7)], we may construct a sequence $\{y_n\}$ in E such that

$$(3) \qquad y_n(x^*) \to z^{**}(x^*) \;\text{ as }\; n \to \infty \;\text{ for all }\; x^* \in F \cup \{y_0^*\}.$$

Note that $\{y_n\}$ as a subset of E is bounded. Now define a sequence of bounded functions h_n on dom f^* by

$$h_n(x^*) = (y_n, x^*) - f^*(x^*).$$

We note that $h_n(x^*) \le f(y_n)$ for all $x^* \in$ dom f^* and for all $n \in \mathbb{N}$. Since f is bounded on bounded sets and $\{y_n : n \in \mathbb{N}\}$ is norm bounded, the sequence $\{h_n\}$ is uniformly bounded on dom f^*.

Let $\varepsilon = \frac{1}{2\|z^{**}\|}$ and let B_ε denote the set $conv(F + \varepsilon B_{X^*})$. We claim that $B_\varepsilon \cap \partial f(x) \neq \emptyset$ for each $x \in X$. Assume on the contrary that there is some $x_0 \in X$ such that the two convex sets B_ε and $\partial f(x_0)$ are disjoint. As B_ε has a non-empty interior, we apply the separation theorem to find an $x^{**} \in S_{X^{**}}$ such that

$$\sup_{b \in B_\varepsilon} x^{**}(b) \le \inf_{x^* \in \partial f(x_0)} x^{**}(x^*).$$

Hence for each $k \in \mathbb{N}$, we have

$$x^{**}(x_k^*) + \varepsilon \le \inf_{x^* \in \partial f(x_0)} x^{**}(x^*).$$

Consequently,

$$(4) \qquad x^{**}(x^* - x_k^*) \ge \varepsilon \;\text{ for each }\; x^* \in \partial f(x_0) \;\text{ and each }\; k \in \mathbb{N}.$$

Now we use the fact that f is quite smooth at x_0 to obtain a $\delta > 0$ such that $\partial f(y) \subset \partial f(x_0) + W$ whenever $\|y - x_0\| < \delta$, where $W = \{x^* \in X^* :$

$|x^{**}(x^*)| < \varepsilon\}$. According to (4), $\partial f(x_k) \nsubseteq \partial f(x_0) + W$ for all $k \in \mathbb{N}$. Therefore $||x_k - x_0|| > \delta$ for each $k \in \mathbb{N}$, contradicting the density of $\{x_k : k \in \mathbb{N}\}$ and hence our claim holds.

Suppose $\{\lambda_k\}$ is a sequence of positive real numbers such that $\sum_{k=1}^{\infty} \lambda_k = 1$, let $y = \sum_{k=1}^{\infty} \lambda_k y_k \in X$, and $y^* \in B_\varepsilon \cap \partial f(y)$. It is clear that

$$\begin{aligned}\sum_{k=1}^{\infty} \lambda_k h_k(y^*) &= (y, y^*) - f^*(y^*) \\ &= f(y) \\ &= \sup_{x^* \in \mathrm{dom}\ f^*} \{(y, x^*) - f^*(x^*)\} \\ &= \sup_{x^* \in \mathrm{dom}\ f^*} \sum_{k=1}^{\infty} \lambda_k h_k(x^*).\end{aligned}$$

Therefore B_ε is a boundary of dom f^* in the sense of [Go]. From (3), we have $\limsup(y_n, x^*) \le \alpha$, for each $x^* \in F$. Therefore $\limsup(y_n, x^*) \le \alpha + \frac{1}{2}$ for each $x^* \in F + \varepsilon B_X$. Hence,

$$\limsup h_n(x^*) \le \alpha + \frac{1}{2} - f^*(x^*) \le \alpha + \frac{1}{2} - \gamma$$

for each $x^* \in B_\varepsilon$ (here we use the convexity of the function $\limsup h_n(\cdot)$). Now by Simons' inequality (cf., [Go], [S]), we get a function $g \in conv\{h_n\}$ such that

$$\sup_{x^* \in \mathrm{dom}\ f^*} g(x^*) \le \alpha + \frac{3}{4} - \gamma.$$

But on the other hand, we have $h_n(y_0^*) = (y_n, y_0^*) - f^*(y_0^*) > \beta - f^*(y_0^*)$ for each $n \in \mathbb{N}$, which means that $g(y_0^*) > \beta - f^*(y_0^*)$. As a result, we get $\beta - f^*(y_0^*) < \alpha + \frac{3}{4} - \gamma$, a contradiction.

In conclusion, we have dom $f^* \subset \overline{conv}^{\|\cdot\|} F$, which means that $C_n = \mathrm{dom}\ f^*$ is separable. □

Using the fact that the restriction of a quite smooth convex function to a subspace is also quite smooth, and Proposition 15, we have the following theorem.

Theorem 16. *Let f be a convex function defined on a Banach space X such that f is bounded on bounded sets of X. Suppose all continuous convex functions bounded above by f is quite smooth, then f is Asplund.*

Proof: Let Y be a separable subspace of X. The restriction of f on Y is also a quite smooth convex function. According to Proposition 15, $f_{\restriction Y}$ is an Asplund function and thus dom $\left(f_{\restriction Y}\right)^*$ is separable. According to Theorem 2, f is an Asplund function. □

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