Uniform stabilization of solutions to a quasilinear wave equation with damping and source terms

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Abstract. In this note we prove the exponential decay of solutions of a quasilinear wave equation with linear damping and source terms.

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega := \Gamma$. In this note we are concerned with the asymptotic behavior of the global solution to the mixed problem

$$(P) \qquad \begin{cases} u_{tt} - (\alpha + 2\beta\|\nabla u\|_2^2)\Delta u + \delta u_t = \mu u^3 \text{ in } \Omega \times \mathbb{R}_+, \\ u = 0 \text{ on } \Gamma \times \mathbb{R}_+, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \text{ in } \Omega. \end{cases}$$

Here $\alpha > 0$, $\beta \geq 0$, $\delta \in \mathbb{R}$, $\mu \in \mathbb{R}$ and Δ is the Laplacian in $\mathbb{R}^3$.

When $\mu = -1$, $\delta > 0$ and $|u|^\alpha u$ $(\alpha \geq 0)$ stands in place of u^3, the existence of a global solution was proved by Hosoya and Yamada [2] for sufficiently small initial data in a more general framework. Their method, however, depending on a kind of monotonocity of the term $|u|^\alpha u$, is not applicable to the case of $\mu = 1$ in its original form (when $\delta = 0$ and $\mu = -1$, see also Hosoya and Yamada [1]). By the construction of the so called stable set and unstable set (see e.g. [6], [7], [8]), quite recently Ikehata [3] has been successful in proving the global existence of solutions to (P) when $\alpha > 0$, $\beta > 0$, $\delta > 0$ and $\mu > 0$. Roughly speaking, he showed that if $\{u_0, u_1\}$ belong to the stable set and are small enough, then (P) admits a global strong solution. Before the statement of his main results, let us first introduce some notations and definitions:

$c(\Omega, p)$ will denote the constant in the Sobolev-Poincaré inequality

$$\|u\|_p \leq c(\Omega, p)\|\nabla u\|_2$$

for $u \in H_0^1(\Omega)$, where $2 \leq p \leq 6$ and $\|u\|_p$ means the usual $L^p(\Omega)$-norm.

We define the energy of the solutions of (P) by the usual formula

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{\alpha}{2}\|\nabla u\|_2^2+\frac{\beta}{2}\|\nabla u\|_2^4-\frac{\mu}{4}\|u\|_4^4.$$

Ikehata [3] has shown the following theorems.

Theorem 1.1 (local existence). *Let $\alpha>0$, $\beta\geq 1$, $\delta\in\mathbb{R}$ and $\mu\in\mathbb{R}$. Then for any $(u_0,u_1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$, there exists a real number $T_m>0$ such that problem (P) admits a unique solution $u(x,t)$ which belongs to the class*

$$C([0,T_m);H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T_m);H_0^1(\Omega))\cap C^2([0,T_m);L^2(\Omega)),$$

and if $T_m<+\infty$ then

$$\lim_{t\uparrow T_m}\{\|\nabla u_t(t,.)\|_2^2+\|\Delta u(t,.)\|_2^2\}=+\infty.$$

Theorem 1.2 (global existence). *Let $\alpha>0$, $\beta\geq 0$, $\delta>0$, and $\mu\in\mathbb{R}$. Assume further that $\mu c(\Omega,4)^4\leq 2\beta$. Then there exists a number $\gamma>0$ such that if $(u_0,u_1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$ with $E(0)\geq\frac{\mu}{4}\|u_0\|_4^4$ satisfy $\|\Delta u_0\|_2^2+\|\nabla u_1\|_2^2<\gamma$, then $T_m=+\infty$.*

As far as we know, there is no result concerning the asymptotic behavior of the global strong solution. Our purpose in this paper is to prove the uniform stabilization of the solution of (P).

Our paper is organized as follows. In Section 2 we give our main result. In Section 3 we prove the main theorem.

2. Statement of the main theorem

Throughout this paper the functions considered are all real valued and the notations for their norms are adopted as usual (e.g. Lions [5]), and for simplicity we shall denote u' and u'' instead of u_t and u_{tt}.

Our main theorem is the following

Main theorem. *Under the hypotheses of Theorem 1.2 we have*

$$E(t)\leq ce^{-\omega t}\quad\text{for all}\quad t\geq 0,$$

where c is a positive constant depending only on the initial energy $E(0)$ in a continuous way, and ω is a positive constant.

Before giving the proof, we recall the following useful lemma.

Lemma 2.1 ([4, Theorem 8.1]). *Let $E:\mathbb{R}_+\to\mathbb{R}_+$ be a non-increasing function and assume that there exists a constant $T>0$ such that*

$$\int_t^{+\infty}E(s)\,ds\leq TE(t),\quad\forall\,t\in\mathbb{R}_+\,.$$

Then

$$E(t)\leq E(0)e^{1-\frac{t}{T}},\quad\forall\,t\geq T.$$

(Observe that the above inequality is also satisfied for $0\leq t\leq T$.)

3. Proof of the main theorem

First, remark that the assumption $\mu c(\Omega, 4)^4 \le 2\beta$ implies

$$\frac{\mu}{2}\|u\|_4^4 \le \beta\|\nabla u\|_2^4 \quad \text{for all} \quad u \in H_0^1(\Omega), \tag{3.1}$$

and then $E(t) \ge 0$ for all $t \ge 0$.

As $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, the solution of (P) has a sufficient regularity to justify all computations that follow. Multiplying the first equation in (P) with u', integrating by parts in $\Omega \times (0, T)$ we obtain easily that

$$E(0) - E(T) = \int_0^T \int_\Omega \delta u'^2 \, dxdt, \quad 0 < T < +\infty;$$

it follows that the energy is non-increasing, locally absolutely continuous and

$$E' = -\int_\Omega \delta u'^2 \, dx \quad \text{a.e. in} \quad \mathbb{R}_+.$$

We have

$$0 = \int_0^T \int_\Omega u(u'' - (\alpha + 2\beta|\nabla u|^2)\Delta u + \delta u' - \mu u^3) \, dxdt$$

$$= \Big[\int_\Omega uu'\Big]_0^T - \int_0^T \int_\Omega 2u'^2 + \int_0^T \int_\Omega (u'^2 + \alpha|\nabla u|^2 + 2\beta|\nabla u|^4 - \mu|u|^4) + \int_0^T \int_\Omega \delta uu',$$

whence

$$\int_0^T \int_\Omega u'^2 + \alpha|\nabla u|^2 + 2\beta|\nabla u|^4 - \mu|u|^4 = -\Big[\int_\Omega uu'\Big]_0^T + \int_0^T \int_\Omega (2u'^2 - \delta uu'),$$

that is, by (3.1)

$$2\int_0^T E(t) \, dt \le -\Big[\int_\Omega uu' \, dx\Big]_0^T + \int_0^T \int_\Omega (2u'^2 - \delta uu') \, dxdt.$$

From now on we denote by c different positive constants.

Using the non-increasing property of E, the Sobolev imbedding $H^1 \subset L^2$, the definition of E and the Cauchy-Schwarz inequality, we have

$$\left|\int_\Omega uu' \, dx\right| \le cE(0).$$

We conclude that

$$2\int_0^T E(t) \, dt \le cE(0) + \int_0^T \int_\Omega (2u'^2 - \delta uu') \, dxdt. \tag{3.2}$$

Now fix an arbitrarily small $\varepsilon > 0$ (to be chosen later), and applying the Young inequality we obtain the following estimates

$$\int_\Omega (2u'^2 - \delta uu')\,dxdt \le c(-E') + cE^{\frac{1}{2}}(-E')^{\frac{1}{2}}$$
$$\le \varepsilon E - c(\varepsilon)E',$$

hence

$$\int_0^T \int_\Omega (2u'^2 - \delta uu')\,dx\,dt \le \varepsilon \int_0^T E(t)\,dt + c(\varepsilon)E(0).$$

By substitution into (3.2) we obtain

$$(2-\varepsilon)\int_0^T E(t)\,dt \le c(\varepsilon)E(0).$$

Choosing $\varepsilon = 1$ we deduce that

$$\int_0^T E(t)\,dt \le cE(0),$$

we may thus complete the proof by applying Lemma 2.1.

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