Extraresolvability and cardinal arithmetic

O.T. Alas, S. Garcia-Ferreira, A.H. Tomita

Abstract. Following Malykhin, we say that a space X is *extraresolvable* if X contains a family $\mathcal{D}$ of dense subsets such that $|\mathcal{D}| > \Delta(X)$ and the intersection of every two elements of $\mathcal{D}$ is nowhere dense, where $\Delta(X) = \min\{|U| : U$ is a nonempty open subset of $X\}$ is the *dispersion character* of X. We show that, for every cardinal κ, there is a compact extraresolvable space of size and dispersion character 2^κ. In connection with some cardinal inequalities, we prove the equivalence of the following statements: 1) $2^\kappa < 2^{\kappa^+}$, 2) $(\kappa^+)^\kappa$ is extraresolvable and 3) $A(\kappa^+)^\kappa$ is extraresolvable, where $A(\kappa^+)$ is the one-point compactification of the discrete space κ^+. For a regular cardinal $\kappa \geq \omega$, we show that the following are equivalent: 1) $2^{<\kappa} < 2^\kappa$; 2) $G(\kappa,\kappa)$ is extraresolvable; 3) $G(\kappa,\kappa)^\lambda$ is extraresolvable for all $\lambda < \kappa$; and 4) there exists a space X such that X^λ is extraresolvable, for all $\lambda < \kappa$, and X^κ is not extraresolvable, where $G(\kappa,\kappa) = \{x \in \{0,1\}^\kappa : |\{\xi < \kappa : x_\xi \neq 0\}| < \kappa\}$ for every $\kappa \geq \omega$. It is also shown that if X is extraresolvable and $\Delta(X) = |X|$, then all powers of X have a dense extraresolvable subset, and λ^κ contains a dense extraresolvable subspace for every cardinal $\lambda \geq 2$ and for every infinite cardinal κ. For an infinite cardinal κ, if $2^\kappa > \mathfrak{c}$, then there is a totally bounded, connected, extraresolvable, topological Abelian group of size and dispersion character equal to κ, and if $\kappa = \kappa^\omega$, then there is an ω-bounded, normal, connected, extraresolvable, topological Abelian group of size and dispersion character equal to κ.

Keywords: extraresolvable, κ-resolvable

Classification: Primary 54A35, 03E35; Secondary 54A25

0. Introduction

In this paper, all topological spaces are assumed to be Tychonoff without isolated points (in the case, when we consider an infinite power the factor space is allowed to have isolated points). The purpose of this article is to continue the study of the extraresolvable spaces. This time we shall focus on the cardinal arithmetic that arises naturally from the theory of extraresolvable spaces.

The class of resolvable spaces was introduced by E. Hewitt in [He]. He called a space *resolvable* if it has two disjoint dense subsets. E. Hewitt also proved in [He] that metric spaces and locally compact spaces are resolvable (see [CG1, Theorem 3.7]). It follows from the Definition that a space X cannot have more than

Research supported by Fundação de Amparo a Pesquisa do Estado de São Paulo (Processo No. 1997/1968-4), and by CONACYT grant number 1030PE.
The second-listed author is pleased to thank the hospitality of the Institute of Mathematics and Statistics of São Paulo University.

$\Delta(X)$-many pairwise disjoint dense subsets. Years later, Ceder [Ce] investigated the spaces that contain k-many pairwise disjoint dense subsets, for a cardinal $\kappa \geq 2$ (in the literature, these spaces are called *κ-resolvable*). In particular, a space X is said to be *maximally resolvable* if X is $\Delta(X)$-resolvable. It is known that metric spaces and locally compact spaces are maximally resolvable (see [CG1] and, for a more general result, see [Py]).

The extraresolvable spaces were introduced by V.I. Malykhin in [Ma], and some examples and topological properties of extraresolvable spaces are given in [GMT], [CG2] and [CG3].

Definition 0.1 (Malykhin)**.** A space X is called extraresolvable if there exists a family $\mathcal{D}$ of dense subsets of X such that $|\mathcal{D}| > \Delta(X)$ and $D \cap D'$ is nowhere dense whenever $D, D' \in \mathcal{D}$ and $D \neq D'$.

It is shown in [GMT] that every countable Frechét-Urysohn space is extraresolvable. Hence, the rational numbers $\mathbb{Q}$ is extraresolvable. It is also proved in [GMT] that the real line $\mathbb{R}$ is $\mathfrak{c}$-resolvable, but it is not extraresolvable. On the other hand, every extraresolvable space is ω-resolvable (see either [GMT] or [Ma]).

In the first section, we give some preliminary results and the basic definitions. One of the main lemmas of the paper is proved in the second section (Lemma 2.1). Using this lemma we show, for every infinite cardinal κ, the equivalence of the conditions 1) $2^\kappa < 2^{\kappa^+}$; 2) $(\kappa^+)^\kappa$ is extraresolvable; and 3) $A(\kappa^+)^\kappa$ is extraresolvable. The third section is devoted to the study of extraresolvable dense subsets of topological products.

1. Preliminaries

Cardinal variables are denoted by the Greek letters α, δ, κ and λ. If $\kappa \geq 2$ and $\alpha \geq \omega$ are cardinal numbers, then we define

$$\log_\kappa(\alpha) = \min\{\lambda : \alpha < \kappa^\lambda\}.$$

We have the following properties:

(1) If we assume GCH, then $\log_\kappa(\kappa) = cf(\kappa)$ for every cardinal κ.

(2) If $\alpha' \leq \alpha$ and $\kappa \leq \kappa'$, then $\log_{\kappa'}(\alpha') \leq \log_\kappa(\alpha)$.

(3) If $\alpha \leq \kappa$, then $\log_\kappa(\alpha) \leq \log_\kappa(\kappa) \leq cf(\kappa) \leq \kappa$.

(4) If $\kappa = \lambda^\alpha$, then $\alpha < \log_\kappa(\kappa)$.

(5) If $\kappa = \lambda^\alpha$ and $\lambda \leq \log_\kappa(\kappa)$, then $\log_\kappa(\kappa)^\alpha = \kappa$.

(6) If $cf(\kappa) = \omega$, then $\log_\kappa(\kappa) = \omega$.

For a set X and an infinite cardinal number α, we put $[X]^\alpha = \{A \subseteq X : |A| = \alpha\}$, the meaning of $[X]^{<\alpha}$ and $[X]^{\leq\alpha}$ should be clear.

If X is a locally compact space, then $A(X)$ will denote the one-point compactification of X. A space X is $< \alpha$*-bounded* if every subset A of X of size

$< \alpha$ has compact closure. An *ω-bounded* space is a $< \omega_1$-bounded space. If X is a space, $x \in X$ and θ is an ordinal, then x^θ will denote the point in X^θ with all its coordinates equal to x. A family $\mathcal{S}$ of nonempty subsets of a space X is said to be a *π-network* if every nonempty open subset of X contains an element of $\mathcal{S}$. The *π-net weight* of X is $\pi - nw(X) = \min\{|\mathcal{S}| : \mathcal{S}$ is a π-network of $X\}$. A *net* for a space X is a collection $\mathcal{N}$ of subsets of X such that every non-empty subset of X is the union of elements of $\mathcal{N}$. The *net weight* of X is defined by $nw(X) = \min\{|\mathcal{N}| : \mathcal{N}$ is a net for $X\}$. For every space X, we have that $\pi - nw(X) \leq nw(X)$. We use the standard notation $d(X)$, $\chi(X)$, $t(X)$, $\pi w(X)$ and $w(X)$ for the density, the character, the tightness, the π-weight and the weight of a space X, respectively. We remark that $\Delta(X^\kappa) = |X|^\kappa$ for every space X and for every infinite cardinal κ.

A.G. El'kin [El] has shown that if $\pi w(X) \leq \Delta(X)$, then X is maximally resolvable, and E.G. Pytkeev [Py] proved that if $t(X) < \Delta(X)$, then X is maximally resolvable; hence, every locally compact space is maximally resolvable: for other results concerning maximal resolvability the reader is referred to [Ce], [CG1], [Pa] and [Py]. El'kin's Theorem can be generalized as follows.

First, we state a cardinal function that extends the dispersion character of a space.

Definition 1.1. Given a family $\mathcal{F}$ of non-empty subsets of X, we denote by $\Delta(\mathcal{F}) = \omega \cdot \min\{|F| : F \in \mathcal{F}\}$.

For every space (X, τ) we have that $\Delta(X) = \Delta(\tau - \{\emptyset\})$ and $\Delta(\mathcal{N}) \leq \Delta(X)$ for all π-network $\mathcal{N}$ of X. If X is a space with uncountable dispersion character and $\mathcal{N}$ is the π-network of X consisting of the singletons, then $\Delta(\mathcal{N}) = \omega < \Delta(X)$.

Next, we slightly modify Lemma 3.5 from [CG1].

Theorem 1.2. *If X has a π-network $\mathcal{N}$ such that*

$$|\mathcal{N}| \leq \Delta(\mathcal{N}),$$

then there is a family $\mathcal{D}$ of pairwise disjoint dense subsets of X such that $D \cap N \neq \emptyset$ for every $D \in \mathcal{D}$ and for every $N \in \mathcal{N}$, and $|\mathcal{D}| = \Delta(\mathcal{N})$.

Two very important relationships between extraresolvable spaces and resolvable spaces were established in [Ma] and [GMT]:

Theorem 1.3. *Let X be a space. Then,*

1. *if X is extraresolvable, then X is ω-resolvable; and*
2. *if $|X|^{nw(X)} = \Delta(X)$ then X is not extraresolvable. Furthermore, if there is a π-base of size at most $\Delta(X)$, then X is maximally resolvable.*

It follows from Theorem 1.3 that $X^{nw(X)}$ is never extraresolvable for any space X. Hence, the real line $\mathbb{R}$ and the remainder $\beta(\omega) - \omega$ of $\beta(\omega)$ are c-resolvable and are not extraresolvable. Observe that if X is extraresolvable, then every finite power of X is extraresolvable as well. In fact, we have:

Theorem 1.4. *If X is extraresolvable, then X^κ is extraresolvable for every cardinal κ with $\Delta(X) = \Delta(X^\kappa)$.*

PROOF: Let X be an extraresolvable space, let $\mathcal{D}$ be a family of dense subsets of X witnessing its extraresolvability and let κ be a cardinal with $\Delta(X) = \Delta(X^\kappa)$. Then, it is clear that $\mathcal{E} = \{D \times X^{\kappa - \{0\}} : D \in \mathcal{D}\}$ is a family of dense subsets of X with pairwise nowhere dense intersection, and $\Delta(X^\kappa) = \Delta(X) < |\mathcal{D}| = |\mathcal{E}|$. Therefore, X^κ is extraresolvable. □

2. Topological spaces

Our first result improves Theorem 3.8 of [GMT].

Lemma 2.1. *Let κ and α be cardinal numbers with $\omega \leq \alpha$. If X satisfies one of the following two lists of conditions:*

1. *$cf(\kappa) = \alpha$;*
2. *X has a π-network $\mathcal{N}$ such that $|\mathcal{N}| \leq \kappa \leq \kappa^{<\alpha} \leq \Delta(\mathcal{N})$;*
3. *every subset of X of size $< \kappa$ is nowhere dense; and*
4. *$\Delta(X) < \kappa^\alpha$,*

or

- $1'$. *X has a π-network $\mathcal{N}$ such that $|\mathcal{N}| \leq \alpha \leq \kappa^{<\alpha} \leq \Delta(\mathcal{N})$;*
- $2'$. *every subset of X of size $< \alpha$ is nowhere dense; and*
- $3'$. *$\Delta(X) < \kappa^\alpha$,*

then X is extraresolvable.

PROOF: First, we assume that the clauses $1-4$ hold. Let $\mathcal{N}$ be a π-network of X with $|\mathcal{N}| \leq \kappa \leq \kappa^\alpha \leq \Delta(\mathcal{N})$. By Theorem 1.2, we may find $\kappa^{<\alpha}$-many pairwise disjoint dense subsets of X such that each one of them meets every element of $\mathcal{N}$. Enumerate these dense subsets as $\{D_s : s \in \kappa^{<\alpha}\}$ and enumerate the π-network $\mathcal{N}$ as $\{E_\nu : \nu < \kappa\}$, we repeat elements if it is necessary. Fix a sequence of infinite ordinals $\{\lambda_\xi : \xi < \alpha\}$ which is cofinal and strictly increasing in κ. For each $s \in \kappa^{<\alpha}$, we choose $N_s \in [D_s]^{\leq |\lambda_{dom(s)}|}$ so that $N_s \cap E_\nu \neq \emptyset$, for every $\nu < \lambda_{dom(s)}$. For each $f \in \kappa^\alpha$ we define $M_f = \bigcup_{\xi<\alpha} N_{f\restriction\xi}$. Then, $\{M_f : f \in \kappa^\alpha\}$ witnesses that X is extraresolvable.

Now, suppose that X satisfies conditions $1'-3'$. In this case, we enumerate the dense subsets as $\{D_s : s \in \kappa^{<\alpha}\}$ and enumerate the π-network $\mathcal{N}$ as $\{E_\nu : \nu < \alpha\}$. For each $s \in \kappa^{<\alpha}$, we choose $x_s \in D_s \cap E_{dom(s)}$. We define $M_f = \{x_{f\restriction\xi} : \xi < \alpha\}$ for each $f \in \kappa^\alpha$. It is not hard to see that $\{M_f : f \in \kappa^\alpha\}$ is the required family. □

It follows from Theorem 2.1 that if $w(X) = \omega \leq \Delta(X) < 2^\omega$, then X has a family $\mathcal{D}$ of dense subsets such that $|\mathcal{D}| = 2^\omega$ and $D \cap F$ is nowhere dense whenever $D, E \in \mathcal{D}$ and $D \neq E$.

Theorem 2.2. *Let X be a space. If $\kappa = |X|^\alpha$ for some infinite cardinal α and $\log_\kappa(\kappa) = d(X) = \pi - nw(X)$, then X^α is extraresolvable.*

Proof: Put $\lambda = \log_\kappa(\kappa)$. Let $\mathcal{S}$ be a π-network of X with $|\mathcal{S}| = \pi - nw(X)$. If $\pi_\xi : X^\alpha \to X$ is the projection map, for each $\xi < \alpha$, then $\mathcal{N} = \{\bigcap_{\xi \in F} \pi_\xi^{-1}(S_\xi) : F \in [\alpha]^{<\omega}, S_\xi \in \mathcal{S}\}$ is a π-network of X^α of cardinality $\leq \lambda$, and each one of its elements has size κ. That is

$$|\mathcal{N}| \leq \lambda \leq \kappa = \kappa^{<\lambda} = \Delta(\mathcal{N}) < \kappa^\lambda.$$

It is clear that every subset of X^α of size $< \lambda$ is nowhere dense and since $\Delta(X^\alpha) = \kappa < \kappa^\lambda$, by Lemma 2.1, X^α is extraresolvable. $\square$

Corollary 2.3. *If $\kappa = \lambda^\alpha$ for some $\lambda \leq \log_\kappa(\kappa)$ and for some infinite cardinal α, then $A(\log_\kappa(\kappa))^\alpha$ and $(\log_\kappa(\kappa))^\alpha$ are extraresolvable. In particular, for every infinite cardinal α, the space $A(\log_{2^\alpha}(2^\alpha))^\alpha$ is a compact extraresolvable space of size and dispersion character 2^α.*

It follows from Corollary 2.3 there exists a compact extraresolvable space X of size and dispersion character $\mathfrak{c}$, and hence, by Example 3.21 of [GMT], for every $\kappa > \mathfrak{c}$ the product space $X \times A(\kappa)$ is a compact extraresolvable space of size κ and dispersion character $\mathfrak{c}$.

Theorem 2.4. *For every infinite cardinal κ, the following are equivalent:*

1. $2^\kappa < 2^{\kappa^+}$;
2. $(\kappa^+)^\kappa$ *is extraresolvable;*
3. $A(\kappa^+)^\kappa$ *is extraresolvable.*

Proof: The proofs of the equivalences (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3) are similar each other. So, we only prove the former one.

(1) $\Rightarrow$ (2). Since $w((\kappa^+)^\kappa) \leq \kappa^+ \leq (\kappa^+)^\kappa = (\kappa^+)^{<\kappa^+} = \Delta((\kappa^+)^\kappa)$, every subset of $(\kappa^+)^\kappa$ of size $< \kappa^+$ is nowhere dense and $\Delta((\kappa^+)^\kappa) = (\kappa^+)^\kappa \leq 2^\kappa < 2^{\kappa^+} = (\kappa^+)^{\kappa^+}$, by Lemma 2.1, $(\kappa^+)^\kappa$ is extraresolvable.

(2) $\Rightarrow$ (1). Suppose that $2^\kappa = 2^{\kappa^+}$. Since $|(\kappa^+)^\kappa|^{w((\kappa^+)^\kappa)} = (\kappa^+)^{\kappa^+} = 2^{\kappa^+} = 2^\kappa = \Delta((\kappa^+)^\kappa) = (\kappa^+)^\kappa$, we must have that $(\kappa^+)^\kappa$ cannot be extraresolvable because of Theorem 1.3, but this is a contradiction. $\square$

From Theorem 2.4, CH implies that ω_1^ω and $A(\omega_1)^\omega$ are extraresolvable, and under the assumption $2^\omega = 2^{\omega_1}$ these spaces cannot be extraresolvable.

The following lemma is taken from [CG2].

Lemma 2.5. *If X is a space such that*

1. $\omega \leq w(X) \leq \Delta(X)$; *and*
2. *every subset of X of size $< \Delta(X)$ is nowhere dense,*

then X is extraresolvable.

This fact implies that if α is an infinite cardinal with $\alpha^\omega = \alpha$, then α^ω and $A(\alpha)^\omega$ are extraresolvable. Hence, $\mathfrak{c}^\omega$ and $A(\mathfrak{c})^\omega$ are extraresolvable.

In the next theorem, we shall consider the following cardinal number:

$$\mathfrak{p} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq [\omega]^\omega \text{ has the strong finite intersection property and there is not } A \in [\omega]^\omega \text{ with } |A - B| < \omega \text{ for all } B \in \mathcal{B}\},$$

where *strong finite intersection property* means that every finite subfamily has infinite intersection.

Theorem 2.6 [$\mathfrak{p} = \mathfrak{c}$]. *If X satisfies that $\omega = |X| \leq \chi(X) < \mathfrak{c}$, then X is extraresolvable.*

Proof: Assume $\mathfrak{p} = \mathfrak{c}$. For every $x \in X$, let $\mathcal{B}_x$ be a local base at x with $|\mathcal{B}_x| \leq \chi(X)$. Fix $x \in X$. Since $\mathcal{B}_x$ has the strong finite intersection property and $|\mathcal{B}_x| < \mathfrak{c}$, there is $A_x \in [\omega]^\omega$ such that $|A_x - B| < \omega$ for all $B \in \mathcal{B}_x$. Put

$$\mathcal{N} = \{E : E \text{ is a cofinite subset of } A_x \text{ for some } x \in X\}.$$

Then, $\mathcal{N}$ is a countable π-network of X consisting of infinite sets and so, by Lemma 2.1, X is extraresolvable. □

3. Subspaces of products

An example of a metric extraresolvable topological group of size and dispersion character $\mathfrak{c}$ is the following.

Example 3.1. *There is a metric extraresolvable topological Abelian group of size and dispersion character $\mathfrak{c}$.*

Proof: Put $\lambda = \log_{\mathfrak{c}}(\mathfrak{c})$. We have that $G = [\lambda]^{<\omega}$ has an Abelian group structure with addition $A + B = (A - B) \cup (B - A)$, for $A, B \in G$. If G is equipped with the discrete topology, by Theorem 2.2, then G^ω is extraresolvable. Thus, G^ω is a metric extraresolvable topological Abelian group with $|G| = \Delta(G) = \mathfrak{c}$. □

We will next show that some Σ products are extraresolvable under some cardinal arithmetic assumptions. We need the following notation and properties:

If $\omega \leq \alpha \leq \kappa$, then the space

$$G(\kappa, \alpha) = \{x \in \{0,1\}^\kappa : |\{\xi < \kappa : x_\xi \neq 0\}| < \alpha\}$$

satisfies that:

(1) $G(\kappa, \alpha)$ is a dense topological subgroup of $\{0,1\}^\kappa$;

(2) $G(\kappa, \alpha)$ is $< cf(\alpha)$-bounded;

(3) $|G(\kappa,\alpha)| = \kappa^{<\alpha}$;
(4) $\Delta(G(\kappa,\alpha)) = \kappa^{<\alpha}$;
(5) $w(G(\kappa,\alpha)) = \kappa$;
(6) if $\alpha < \kappa$, then every subset of $G(\kappa,\alpha)$ of size $< \kappa$ is nowhere dense;
(7) if $\alpha = \kappa$, then every subset of $G(\kappa,\kappa)$ of size $< cf(\kappa)$ is nowhere dense; and
(8) every subset of $G(\kappa, cf(\kappa))$ of size $< \kappa$ is nowhere dense.

Lemma 3.2. *Let $\kappa \geq \alpha \geq \omega$. If $G(\kappa,\alpha)^\lambda$ is extraresolvable for some $\lambda < \kappa$, then $(\kappa^{<\alpha})^\lambda < 2^\kappa$.*

Proof: If $(\kappa^{<\alpha})^\lambda = 2^\kappa$ for $\lambda < \kappa$, then $\Delta(G(\kappa,\alpha)^\lambda) = |G(\kappa,\alpha)|^\lambda = (\kappa^{<\alpha})^\lambda = \kappa^\kappa = 2^\kappa = |G(\kappa,\alpha)^\lambda|^{w(G(\kappa,\alpha)^\lambda)}$. According to Theorem 1.3, $G(\kappa,\alpha)$ cannot be extraresolvable, which is a contradiction. $\square$

The following result is a direct consequence of Lemma 2.1.

Lemma 3.3. *Let κ be an infinite cardinal. If $(\kappa^{<cf(\kappa)})^\lambda < \kappa^{cf(\kappa)}$ for a cardinal $\lambda < cf(\kappa)$, then $G(\kappa, cf(\kappa))^\lambda$ is extraresolvable. In particular, if κ is an infinite strong limit regular (strongly inaccessible) cardinal, then $G(\kappa,\kappa)^\lambda$ is extraresolvable, for every $\lambda < \kappa$, and $G(\kappa,\kappa)$ cannot be extraresolvable.*

Proof: We shall verify that the conditions $1-4$ of Lemma 2.1 hold. Indeed, we have that

$$w(G(\kappa, cf(\kappa))^\lambda) = \kappa \leq \kappa^{<cf(\kappa)} \leq (\kappa^{<cf(\kappa)})^\lambda = \Delta(G(\kappa, cf(\kappa))^\lambda)) < \kappa^{cf(\kappa)},$$

and we pointed out above that every subset of $G(\kappa, cf(\kappa))$ of size $< \kappa$ is nowhere dense. Therefore, $G(\kappa, cf(\kappa))$ is extraresolvable. $\square$

Under GCH, if κ is a regular cardinal, then $G(\kappa, cf(\kappa))$ is extraresolvable. One more application of Lemma 3.3 is that if κ is an infinite cardinal with $cf(\kappa) = \omega$, then $G(\kappa,\omega)$ is extraresolvable. Notice that $G(\omega,\omega)$ is extraresolvable.

Lemma 3.4. *Let $\omega \leq \kappa \leq \alpha$. If X satisfies that X^λ is extraresolvable, for all $\lambda < \kappa$, and X^α is not extraresolvable, then $2^{<\kappa} \leq |X|^{<\kappa} < |X|^\alpha$.*

Proof: Fix $\lambda < \kappa$. Since X^λ is extraresolvable there is a family $\{D_\xi : \xi < |X^\lambda|^+\}$ of dense subsets of X^λ witnessing that X^λ is extraresolvable. Since X^α and $X^\lambda \times X^\alpha$ are homeomorphic and $\{D_\xi \times X^\alpha : \xi < |X^\lambda|^+\}$ is a family of dense subsets of $X^\lambda \times X^\alpha$ with the property that every two elements have nowhere dense intersection, we must have that $|X^\lambda|^+ < |X^\alpha|^+$, and so $|X|^\lambda < |X|^\alpha$. Since $cf(|X|^\alpha) > \alpha \geq \kappa$, $2^{<\kappa} \leq |X|^{<\kappa} < |X|^\alpha$. $\square$

Observe from Lemma 3.2 that if $G(\kappa, cf(\kappa))^\lambda$ is extraresolvable for some infinite cardinal κ and some $\lambda < cf(\kappa)$, then $\kappa^{<cf(\kappa)} < 2^\kappa$. The next two theorems are consequences of Lemmas 3.2, 3.3 and 3.4.

Theorem 3.5. *Let κ be an infinite cardinal such that $G(\kappa, cf(\kappa))^{cf(\kappa)}$ is not extraresolvable. Then the following are equivalent:*

1. $\kappa^{<cf(\kappa)} < \kappa^{cf(\kappa)}$;
2. $G(\kappa, cf(\kappa))$ *is extraresolvable;*
3. $G(\kappa, cf(\kappa))^{\lambda}$ *is extraresolvable for every* $\lambda < cf(\kappa)$;
4. *there exists a space* X *such that* $|X|^{cf(\kappa)} = \kappa^{cf(\kappa)}$, $|X|^{<cf(\kappa)} = \kappa^{<cf(\kappa)}$, X^{λ} *is extraresolvable for all* $\lambda < cf(\kappa)$ *and* $X^{cf(\kappa)}$ *is not extraresolvable.*

PROOF: (1) $\Rightarrow$ (2). This follows from Lemma 3.3.

(2) $\Rightarrow$ (3). We need the following fact which lies in the proof of Lemma 1.26 of [CN]:

($*$) if $\delta \geq \omega$ and $\gamma \geq 2$ are cardinal numbers, then $(\gamma^{<\delta})^{\alpha} = \gamma^{<\delta}$ for all $\alpha < cf(\delta)$.

By assumption, there is a family $\mathcal{D}$ of dense subsets of $G(\kappa, cf(\kappa))$ such that $\kappa^{<cf(\kappa)} < |\mathcal{D}|$ and $D \cap F$ is nowhere dense whenever $D, F \in \mathcal{D}$ and $D \neq E$. It follows from ($*$) that $\Delta(G(\kappa, cf(\kappa))^{\lambda}) = (\kappa^{<cf(\kappa)})^{\lambda} = \kappa^{<cf(\kappa)}$ for every $\lambda < cf(\kappa)$ and hence, the family of dense subsets $\{D \times G(\kappa, cf(\kappa))^{\lambda - \{0\}} : D \in \mathcal{D}\}$ witnesses the extraresolvability of $G(\kappa, cf(\kappa))^{\lambda}$ for every $\lambda < cf(\kappa)$.

(3) $\Rightarrow$ (4). We have that $|G(\kappa, cf(\kappa))| = \kappa^{<cf(\kappa)} = (\kappa^{<cf(\kappa)})^{<cf(\kappa)}$, by Lemma 1.26 from [CN], and $|G(\kappa, cf(\kappa))|^{cf(\kappa)} = \kappa^{cf(\kappa)}$. Then, the space $X = G(\kappa, cf(\kappa))$ satisfies the conditions of clause 4.

(4) $\Rightarrow$ (1). It follows from Lemma 3.4 that $\kappa^{<cf(\kappa)} = |X|^{<cf(\kappa)} < |X|^{cf(\kappa)} = \kappa^{cf(\kappa)}$ □

As an application of Theorem 3.5 and the fact that if κ is an infinite regular cardinal, then $2^{<\kappa} = \kappa^{<\kappa}$ (a proof is available in [CN, p. 18]), we have the following.

Theorem 3.6. *For a regular cardinal* $\kappa \geq \omega$, *the following are equivalent:*

1. $2^{<\kappa} < 2^{\kappa}$;
2. $G(\kappa, \kappa)$ *is extraresolvable;*
3. $G(\kappa, \kappa)^{\lambda}$ *is extraresolvable for all* $\lambda < \kappa$;
4. *there exists a space* X *such that* X^{λ} *is extraresolvable, for all* $\lambda < \kappa$, *and* X^{κ} *is not extraresolvable.*

The implication of (4) $\Rightarrow$ (1) of Theorem 3.6 holds for any cardinal:

Theorem 3.7. *Let* κ *be an infinite cardinal. If* X *is a space such that* X^{λ} *is extraresolvable for every cardinal* $\lambda < \kappa$ *and* X^{κ} *is not extraresolvable, then* $2^{<\kappa} < 2^{\kappa}$.

PROOF: Following the proof of Lemma 3.4, we have that $|X^{\lambda}|^{+} < |X^{\kappa}|^{+}$ for every cardinal $\lambda < \kappa$, and hence $2^{\lambda} < 2^{\kappa}$ for every $\lambda < \kappa$. If $cf(2^{<\kappa}) < cf(\kappa)$, then $2^{<\kappa} = 2^{\lambda}$ for some $\lambda < \kappa$ and hence $2^{<\kappa} < 2^{\kappa}$. If $cf(2^{<\kappa}) = cf(\kappa)$, then $cf(2^{<\kappa}) < cf(2^{\kappa})$ and hence $2^{<\kappa} < 2^{\kappa}$. $\square$

Corollary 3.8 [*GCH*]. *$G(\kappa, cf(\kappa))^{cf(\kappa)}$ is not extraresolvable for every infinite cardinal κ.*

It is a consequence of Theorem 1.3 that X^{κ} is not extraresolvable for every $\kappa \geq nw(X)$, for every space X, and compact topological groups cannot be extraresolvable, since $|G| = 2^{w(G)}$ for every compact topological group G (see [Co]). It is then natural to ask whether, for every space X and for every infinite cardinal κ, the power space X^{κ} has a dense extraresolvable subspace. It is shown in [CG3] that every totally bounded group has a dense (strongly) extraresolvable subgroup; hence, every compact topological group has a dense extraresolvable subgroup. The authors of [CG2] pointed out that every product of infinitely many metric separable spaces admits a dense extraresolvable subspace (this result will follow directly from Lemma 3.15 below). Next, we shall give some partial affirmative answers to the question suggested above.

The next lemma plays a very important rule in the construction of some extraresolvable examples.

Lemma 3.9. *Let α and κ be cardinal numbers with $\omega \leq \alpha \leq \kappa$ and let $\{X_\xi : \xi < \kappa\}$ be a set of infinite spaces. Suppose that*

1) *D_ξ is a dense subset of X_ξ for each $\xi < \kappa$;*
2) *$\gamma = \sup\{|D_\xi| : \xi < \kappa\}$;*
3) *$d = (d_\xi)_{\xi<\kappa} \in \prod_{\xi<\kappa} D_\xi$, and*
4) *there are cardinals δ and λ such that $((\gamma \cdot \kappa)^{<\alpha})^{\delta} < \lambda^{\kappa}$ and D_ξ is λ-resolvable for every $\xi < \kappa$.*

If
$$\Sigma = \{x \in \prod_{\xi<\kappa} D_\xi : |\{\xi < \kappa : x_\xi \neq d_\xi\}| < \alpha\},$$
then Σ^{δ} is a dense extraresolvable subspace of $(\prod_{\xi<\kappa} X_\xi)^{\delta}$.

Proof: We have that $\Delta(\Sigma^{\delta}) = |\Sigma|^{\delta} \leq ((\gamma \cdot \kappa)^{<\alpha})^{\delta} < \lambda^{\kappa}$. For each $\xi < \kappa$, we choose a family $\{D^{\xi}_{\zeta} : \zeta < \lambda\}$ of pairwise disjoint dense subsets of D_ξ. Consider the set $\mathcal{F} = \{\sigma : E \to [\kappa]^{<\omega} : E \in [\delta]^{<\omega}\}$. If $f \in \lambda^{\kappa}$, then we define
$$E_f = \bigcup_{\sigma\in\mathcal{F}} [[\prod_{\nu\in dom(\sigma)} (\prod_{\xi\in\sigma(\nu)} D^{\xi}_{f(\xi)} \times \prod_{\xi\in\kappa-\sigma(\nu)} \{d_\xi\})] \times \{d^{\delta-dom(\sigma)}\}].$$

It is evident that E_f is dense in $\prod_{\xi<\kappa} X^{\delta}_{\xi}$, for each $f \in \lambda^{\kappa}$. Let $f, g \in \lambda^{\kappa}$ and let $\zeta < \kappa$ be such that $f(\zeta) \neq g(\zeta)$. Fix $\nu < \delta$ and let $\pi_\nu : (\prod_{\xi<\kappa} D_\xi)^{\delta} \to \prod_{\xi<\kappa} D_\xi$

be the projection map on the νth-coordinate. Then, we have that

$$\pi_\nu[E_f \cap E_g] \subseteq \{d\} \cup [(\bigcup_{F\in[\kappa]^{<\omega}} (\prod_{\xi\in F} D^\xi_{f(\xi)} \times \prod_{\xi\in\kappa-F} \{d_\xi\})) \cap (\bigcup_{F\in[\kappa]^{<\omega}} (\prod_{\xi\in F} D^\xi_{g(\xi)} \times \prod_{\xi\in\kappa-F} \{d_\xi\}))].$$

Now, if $\pi_\zeta : \prod_{\xi<\kappa} D_\xi \to D_\zeta$ is the projection map, then

$$\pi_\zeta[\pi_\nu[E_f \cap E_g]] \subseteq \{d_\zeta\} \cup (D^\zeta_{f(\zeta)} \cap D^\zeta_{g(\zeta)}) = \{d_\zeta\}.$$

This shows that $E_f \cap E_g$ is nowhere dense in Σ^δ. Therefore, Σ^δ is extraresolvable. □

We remark that if, in Lemma 3.9, D_ξ is a dense subgroup of a topological group X_ξ, for every $\xi < \kappa$, then $\Sigma = \{x \in \prod_{\xi<\kappa} D_\xi : |\{\xi < \kappa : x_\xi \neq d_\xi\}| < \alpha\}$ is a dense subgroup of $\prod_{\xi<\kappa} X_\xi$. If D_ξ is a dense ω-bounded subspace of X_ξ, for every $\xi < \kappa$, and $\omega < cf(\alpha)$, then Σ is a dense ω-bounded subspace of $\prod_{\xi<\kappa} X_\xi$.

Lemma 3.10. *Let α and κ be infinite cardinals such that $\omega < \alpha \leq \kappa$. If $(\kappa^{<\alpha})^\lambda < 2^\kappa$ for some cardinal λ, then $G(\kappa,\alpha)^\lambda$ is extraresolvable.*

Proof: Before applying Lemma 3.9, we shall verify that $((2^\omega \cdot \kappa)^{<\alpha})^\lambda < 2^\kappa$. In fact, if $\kappa \leq 2^\omega$, then $\kappa^{<\alpha} = 2^{<\alpha} = (2^\omega \cdot \kappa)^{<\alpha} < 2^\kappa$, and if $2^\omega < \kappa$, then $(2^\omega \cdot \kappa)^{<\alpha} = \kappa^{<\alpha} < 2^\kappa$. Let $\{A_\xi : \xi < \kappa\} \subseteq [\kappa]^\omega$ be a partition of κ. Note that $\{0,1\}^{A_\xi} \cong \{0,1\}^\omega$ for every $\xi < \kappa$. Enumerate A_ξ as $\{\theta(\xi,n) : n < \omega\}$ for every $\xi < \kappa$. It is not hard to see that the function $h : \{0,1\}^\kappa \to \prod_{\xi<\kappa}\{0,1\}^{A_\xi}$ defined by $h(x) = ((x_{\theta(\xi,n)})_{n<\omega})_{\xi<\kappa}$, for every $x \in 2^\kappa$, is a homeomorphism and isomorphism, and $h[G(\kappa,\alpha)] = \Sigma$, where

$$\Sigma = \{x \in \prod_{\xi<\kappa}\{0,1\}^{A_\xi} : |\{\xi < \kappa : (x_{\theta(\xi,n)})_{n<\omega} \neq 0^{A_\xi}\}| < \alpha\}.$$

Hence, by Lemma 3.9, Σ^λ is extraresolvable. Therefore, $G(\kappa,\alpha)^\lambda$ is extraresolvable. □

A straightforward application of Lemma 3.10 is the following.

Corollary 3.11. *Let α and κ be infinite cardinals such that $\omega < \alpha \leq \kappa$, and $\kappa^{<\alpha} < 2^\kappa$. Then the compact group $\{0,1\}^\kappa$ contains a dense $< cf(\alpha)$-bounded extraresolvable subgroup.*

Next, we have another application of Lemma 3.9.

Corollary 3.12. *Let X be a resolvable space. If either $\pi w(X) < 2^\kappa$ or $|X| < 2^\kappa$, then X^κ contains a dense extraresolvable subspace.*

PROOF: We only prove the case when $\pi w(X) < 2^\kappa$. Choose a dense resolvable subspace D of X of size $\pi w(X)$. We replace in Lemma 3.9 D_ξ by D and X_ξ by X for every $\xi < \kappa$, α by ω, γ by $\pi w(X)$, δ by 1, and λ by 2. Since $(\pi w(X) \cdot \kappa)^{<\omega} = \pi w(X) \cdot \kappa < 2^\kappa$, by Lemma 3.9, X^κ has a dense extraresolvable subspace □

It is worthy to mention that if X is resolvable and $\log_2(\pi w(X)) \leq \kappa$, then X^κ has a dense extraresolvable subspace. Hence, by Theorem 1.3, the power X^κ is ω-resolvable for every $\kappa \geq \log_2(\pi w(X))$. Hence, $\mathbb{R}^\kappa$ contains a dense extraresolvable subset for every infinite cardinal κ.

Theorem 1.2 and Corollary 3.12 imply the following.

Corollary 3.13. *If X is a space such that either $\pi w(X) < 2^\omega$ and $\pi w(X) \leq \Delta(X)$ or X is a resolvable space with $|X| < 2^\omega$, then X^κ is an ω-resolvable space that contains a dense extraresolvable subspace, for every infinite cardinal κ. In particular, if X is a countable space with $\pi w(X) = \omega$, then X^κ is ω-resolvable and contains a dense extraresolvable subspace, for every infinite cardinal κ.*

In the category of resolvable spaces, the following remains unsolved.

Question 3.14. *If X is a resolvable space, must X^κ contain a dense extraresolvable subspace, for every infinite cardinal κ?*

Corollaries 3.12 and 3.13 contain responses to Question 3.14 in the positive fashion. Another partial affirmative answers to Question 3.14 are stated in the next three corollaries.

Lemma 3.15. *Let $\{X_\xi : \xi < \kappa\}$ be a set of spaces such that X_ξ has a dense extraresolvable subset D_ξ with $|D_\xi| = \Delta(D_\xi)$, for each $\xi < \kappa$. Then, $\prod_{\xi<\kappa} X_\xi$ has a dense extraresolvable subset.*

PROOF: In virtue of Theorem 1.4, we have that finite products of extraresolvable spaces are extraresolvable. Hence, we may assume that $\omega \leq \kappa$. Fix $x \in \prod_{\xi<\kappa} X_\xi$. Define

$$D = \bigcup_{F \in [\kappa]^{<\omega}} (\prod_{\xi \in F} D_\xi \times \{(x_\xi)_{\xi \in \kappa - F}\}).$$

It is evident that D is a dense subset of $\prod_{\xi<\kappa} X_\xi$ and $\Delta(D) = \sum_{\xi<\kappa} |D_\xi|$. For each $\xi < \kappa$, choose a family $\mathcal{D}_\xi$ of dense subsets of D_ξ witnessing its extraresolvability. Since $|\mathcal{D}_\xi| = \Delta(D_\xi)$ for each $\xi < \kappa$, by König's Lemma,

$$\prod_{\xi<\kappa} |\mathcal{D}_\xi| > \sum_{\xi<\kappa} \Delta(D_\xi) = \sum_{\xi<\kappa} |D_\xi| = \Delta(D).$$

For $f \in \prod_{\xi<\kappa} \mathcal{D}_\xi$, we define

$$E_f = \bigcup_{F \in [\kappa]^{<\omega}} (\prod_{\xi \in F} f(\xi) \times \{(x_\xi)_{\xi \in \kappa - F}\}).$$

Notice that E_f is a dense subset of D for each $f \in \prod_{\xi<\kappa} \mathcal{D}_\xi$. If $f, g \in \prod_{\xi<\kappa} \mathcal{D}_\xi$, $f(\zeta) \neq g(\zeta)$ for some $\zeta < \kappa$, and $\pi_\zeta : \prod_{\xi<\kappa} X_\xi \to X_\zeta$ is the projection map, then $\pi_\zeta[E_f \cap E_g] \subseteq \{x_\zeta\} \cup (f(\zeta) \cap g(\zeta))$, and this implies that $E_f \cap E_g$ is nowhere dense. This shows that D is extraresolvable. $\square$

The following three results are immediate consequences of Lemma 3.15.

Corollary 3.16. *If X has a dense extraresolvable subset D with $|D| = \Delta(D)$, then X^κ has a dense extraresolvable subset for every cardinal κ. Hence, if X is a space and X^ω is extraresolvable, then every infinite power of X has a dense extraresolvable subset.*

Thus, if $\mathcal{D}$ witnesses the extraresolvability of a space X and $|X|^\omega < |\mathcal{D}|$, then X^ω is extraresolvable and hence every power of X has a dense extraresolvable subset. Next, we shall give an example of an extraresolvable space X with $\Delta(D) < |D|$ for every dense subset D of X.

Example 3.17. *Let κ be an uncountable cardinal and let $\Gamma_\kappa = \mathbb{Q} \oplus \kappa^\omega$ be the topological sum of $\mathbb{Q}$ and κ^ω. Then*

1. *$\Delta(\Gamma_\kappa) = \omega$ and $w(\Gamma_\kappa) = \kappa$;*
2. *Γ_κ is extraresolvable;*
3. *$\Delta(D) = \omega < \kappa \leq |D|$ for every dense subset D of Γ_κ; and*
4. *if $\mathcal{D}$ is a family of dense subsets that witnesses the extraresolvability of Γ_κ, then $|\mathcal{D}| \leq 2^\omega \leq \kappa^\omega$.*

We claim that every subset of $(\Gamma_\kappa)^\omega$ of size $< \kappa$ is nowhere dense. Indeed, fix $A \in [(\Gamma_\kappa)^\omega]^{<\kappa}$ and assume that $cl_{(\Gamma_\kappa)^\omega} A$ has nonempty interior. Then, there is $m < \omega$ such that $cl_{\Gamma_\kappa} \pi_m[A] = \Gamma_\kappa$, where $\pi_m : (\Gamma_\kappa)^\omega \to \Gamma_\kappa$ is the projection map. That is, $\pi_m[A]$ is dense in Γ_κ and, by clause (3), we must have that $|A| \geq |\pi_m[A]| \geq \kappa$, which is impossible. This shows our claim. This observation and Lemma 2.5 imply that $(\Gamma_\kappa)^\omega$ is extraresolvable whenever $\kappa = \kappa^\omega$, because of $w((\Gamma_\kappa)^\omega) = \kappa = \Delta((\Gamma_\kappa)^\omega) = \kappa^\omega$. Hence, by Corollary 3.16, all powers of Γ_κ have a dense extraresolvable subspace for every uncountable cardinal κ with $\kappa = \kappa^\omega$. Now, let κ be an uncountable cardinal with $cf(\kappa) = \omega$ and choose a dense subset D of $(\Gamma_\kappa)^\omega$ with $|D| = \Delta(D) = \kappa$ (this is possible since $(\Gamma_\kappa)^\omega$ is maximally resolvable and $w((\Gamma_\kappa)^\omega) = \kappa \leq \kappa^\omega = \Delta((\Gamma_\kappa)^\omega)$). We have that $w(D) \leq \kappa = \kappa^{<\omega} = \Delta(D)$. By Lemma 2.1, D is an extraresolvable dense subspace of $(\Gamma_\kappa)^\omega$ and $|D| = \Delta(D)$. Thus, by Corollary 3.16, all powers of Γ_κ have a dense extraresolvable subspace, for every uncountable cardinal κ with $cf(\kappa) = \omega$. We have shown that GCH implies that all powers of Γ_κ have a dense extraresolvable subspace for every uncountable cardinal κ. Unfortunately, we do not know whether, in ZFC, all powers of Γ_κ have a dense extraresolvable subspace, for every uncountable cardinal κ.

Corollary 3.18. *If X is extraresolvable and $|X| = \Delta(X)$, then all powers of X have a dense extraresolvable subset.*

Corollary 3.19. *If X^λ is extraresolvable for some infinite cardinal λ, then X^κ has a dense extraresolvable subset for every cardinal $\kappa > \lambda$.*

We saw in Theorem 2.4 that $(\kappa^+)^\kappa$ is extraresolvable iff $2^\kappa < 2^{\kappa^+}$, for every infinite cardinal κ. Hence, GCH implies that $(\kappa^+)^\kappa$ is extraresolvable for every infinite cardinal κ, and in a model M of ZFC in which $M \models 2^\omega = 2^{\omega_1}$, we have that $M \models (\omega_1)^\omega$ is not extraresolvable. However, all these spaces have, in any model of ZFC, a dense extraresolvable subspace as is shown in the next theorem.

Theorem 3.20. *Let λ and κ be cardinal numbers such that $\lambda \geq 2$ and $\kappa \geq \omega$. Then λ^κ has a dense extraresolvable subspace.*

Proof: If $2 \leq \lambda < \omega$, then λ^κ is homeomorphic to a compact topological group and the conclusion follows from Theorem 1 of [CG3]. Suppose that $\lambda \geq \omega$. According to Corollary 3.16, it suffices to show that λ^ω has a dense extraresolvable subspace D with $|D| = \Delta(D)$. Let us consider the following subspace of λ^ω:

$$D = \{x \in \lambda^\omega : |\{n < \omega : x_n \neq 0\}| < \omega\}.$$

It is evident that D is dense in λ^ω and $\lambda = w(D) = \Delta(D) = |D|$. We shall verify that D is extraresolvable. In virtue of Lemma 2.5, we only need to show that every subset D of size $< \lambda$ is nowhere dense. In fact, let $A \in [D]^{<\lambda}$ and suppose that $cl_D A$ has nonempty interior. Then there is $V = \bigcup_{j \leq k} \pi_{n_j}^{-1}(\{\theta_j\})$, where $\pi_{n_j} : \lambda^\omega \to \lambda$ is the projection map, $n_j < \omega$ and $\theta_j < \lambda$ for every $j \leq k$, such that $V \cap D \subseteq cl_D A$. Since $|\{\theta < \lambda : \theta = a_n$ for some $a \in A$ and $n < \omega\}| < \lambda$, we may choose $\nu \in \lambda - \{\theta < \lambda : \theta = a_n$ for some $a \in A$ and $n < \omega\}$. Fix $m \in \omega - \{n_0, \ldots, n_k\}$ and define $W = V \cap \pi_m^{-1}(\{\nu\})$. Notice that $\emptyset \neq W \cap D \subseteq cl_D A$. If $a \in A \cap W$, then $\pi_m(a) = a_m = \nu$, which is a contradiction. Therefore, D is an extraresolvable dense subspace of λ^ω with $|D| = \Delta(D)$. □

In the context of connected spaces, let us consider the following topological groups:

For $\omega \leq \alpha \leq \kappa$ and the unit circle $\mathbb{T}$, we define

$$T(\kappa, \alpha) = \{x \in \mathbb{T}^\kappa : |\{\xi < \kappa : x_\xi \neq 1\}| < \alpha\}$$

satisfies that:

(1) $T(\kappa, \alpha)$ is a connected, dense topological subgroup of $\mathbb{T}^\kappa$;

(2) $T(\kappa, \alpha)$ is $< cf(\alpha)$-bounded;

(3) $|T(\kappa, \omega)| = \kappa \cdot 2^\omega$ and if $\alpha > \omega$, then $|T(\kappa, \alpha)| = \kappa^{<\alpha}$;

(4) $\Delta(T(\kappa, \omega)) = \kappa \cdot 2^\omega$ and if $\alpha > \omega$, then $T(\kappa, \alpha) = \kappa^{<\alpha}$;

(5) $w(T(\kappa, \alpha)) = \kappa$;

(6) If $\alpha < \kappa$, then every subset of $T(\kappa, \alpha)$ of size $< \kappa$ is nowhere dense; and

(7) If $\alpha = \kappa$, then every subset of $T(\kappa, \kappa)$ of size $< cf(\kappa)$ is nowhere dense.

Now, we again apply Lemma 3.9 to get the following result.

Corollary 3.21. *Let κ be an infinite cardinal.*

1. *If $2^\kappa > \mathfrak{c}$, then $T(\kappa,\omega)$ is a totally bounded, connected, extraresolvable, topological Abelian group of size and dispersion character equal to κ.*
2. *If $\kappa = \kappa^\omega$, then $T(\kappa,\omega_1)$ is an ω-bounded, normal, connected, extraresolvable, topological Abelian group of size and dispersion character equal to κ.*

References

[Ce] Ceder J.G., *On maximally resolvable spaces*, Fund. Math. **55** (1964), 87–93..

[Co] Comfort W.W., *Topological Groups*, in Handbook of Set-Theoretic Topology, K. Kunen and J.E. Vaughan, Eds., North-Holland, Amsterdam, 1984, pp. 1143–1263.

[CG1] Comfort W.W., García-Ferreira S., *Resolvability: a selective survey and some new results*, Topology Appl. **74** (1996), 149–167.

[CG2] Comfort W.W., García-Ferreira S., *Strongly extraresolvable groups and spaces*, manuscript submitted for publication, 1998.

[CG3] Comfort W.W., García-Ferreira S., *Dense subsets of maximally almost periodic groups*, to appear in Proc. Amer. Math. Soc.

[CN] Comfort W.W., Negrepontis S., *The Theory of Ultrafilters*, Grudlehren der Mathematischen Wissenschaften vol. 211, Springer-Verlag, Berlin, 1974.

[El] El'kin A.G., *On the maximal resolvability of products of topological spaces*, Soviet Math. Dokl. **10** (1969), 659–662.

[GMT] Garcia-Ferreira S., Malykhin V.I., Tomita A.H., *Extraresolvable spaces*, to appear in Topology Appl.

[He] Hewitt E., *A problem of set-theoretic topology*, Duke Math. J. **10** (1943), 309–333.

[Ma] Malykhin V.I., *Irresolvability is not descriptively good*, manuscript submitted for publication.

[Pa] Pavlov O., *On resolvability of topological spaces*, manuscript submitted for publication.

[Py] Pytkeev E.G., *On maximally resolvable spaces*, Proc. Steklov Institute of Mathematics **154** (1984), 225–230.

Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, CEP 05315-970, São Paulo, Brasil

E-mail: alas@ime.usp.br

Instituto de Matemáticas, Ciudad Universitaria (UNAM), D.F. 04510, México

E-mail: sgarcia@zeus.ccu.umich.mx; garcia@servidor.unam.mx

Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, CEP 05315-970, São Paulo, Brasil

E-mail: tomita@ime.usp.br

(*Received* May 21, 1998, *revised* February 15, 1999)