Differentially trivial left Noetherian rings

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Abstract. We characterize left Noetherian rings which have only trivial derivations.

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0. Let R be an associative ring with an identity element. A mapping $D : R \longrightarrow R$ is called a derivation of R if

$$D(x + y) = D(x) + D(y)$$

and

$$D(xy) = D(x)y + xD(y)$$

for all elements x and y in R. A ring having no non-zero derivations will be called here differentially trivial ([1]). Every differentially trivial ring is commutative.

Note that the class of differentially trivial rings is contained in the class of ideally differential rings, i.e. rings R in which every ideal is invariant with respect to all derivations of R. The ideally differential rings were studied in [1–3].

In this paper we characterize differentially trivial Noetherian rings.

For convenience of the reader we recall some notation and terminology.

R^+ will always denote the additive group of R, $\mathcal{J}(R)$ the Jacobson radical of R and $\mathcal{F}(R)$ the periodic part of R^+, $Q(R)$ the field of quotients of a commutative domain R, $char(R)$ the characteristic of R, $\mathcal{N}il(R)$ the prime radical of R, $Ann(x)$ the annihilator of x in R, and $\mathcal{D}_R(N) = \{c \in R \mid c + N$ is a regular element of $R/N\}$.

Throughout the paper p is a prime and $\mathbb{Z}_{p^t}$ is the ring of integers modulo a prime power p^t.

Let us recall that a ring R is called local if the factor ring $R/\mathcal{J}(R)$ is a skew field.

We will also use some other terminology from [4].

1. Let R be a commutative Noetherian ring and $\mathcal{A}ss(R)$ be the set of all prime ideals M of R for which there is a nont-zero element x such that

$$M = Ann(x).$$

By Corollary 2 of [5, Chapter II, § 2, $n^\circ 2$]

$$\mathcal{A}ss(R) \leq Supp(R)$$

(see Definition 5 of [5, Chapter IV, § 1, $n^\circ 3$]) and therefore by Corollary 1 of [5, Chapter IV, § 1, $n^\circ 3$] every minimal prime ideal of a commutative Noetherian ring R with zero-divisors is an annihilator. The subring of R generated by the identity element of R is called the prime subring of R. If the field of quotients $Q(R)$ of R is algebraic over its prime subfield we say that R is algebraic over its prime subring.

Proposition 1 (see [6]). *A (commutative) domain R is differentially trivial if and only if at least one of the following two cases takes place:*

(1) *$char(R) = 0$ and R is algebraic over its prime subring;*
(2) *$char(R) = q > 0$ and $R = \{a^q \mid a \in R\}$.*

Lemma 2. *Let Z be a prime subring of a commutative domain R. If R is algebraic over Z then every non-zero prime ideal of R is maximal.*

Proof: Put $q = char(R)$. If $q > 0$ then Z is a finite field and, for every non-zero $u \in R$, the transformation

$$\phi : x \longrightarrow xu, \ x \in S,$$

is an injective endomorphism of the vector space S_Z, $S = Z[u]$. Since R is algebraic over Z, the space S_Z is finite-dimensional, ϕ is an automorphism and u is invertible in R. We have proved that R is a field in this case, and so we may assume that $q = 0$ and $Z = \mathbb{Z}$ (the ring of integers). Now, let P be a non-zero prime ideal of R and I an ideal of R such that $P \subseteq I$ and $P \neq I \neq R$. Since R is algebraic over Z, we have

$$P \cap Z = pZ = I \cap Z$$

for some prime number p of Z. Further, if $u \in I \setminus P$ then there are $n \geq 1$ and $a_0, \dots, a_n \in Z$ such that

$$a_0 + a_1 u + \ldots + a_n u^n = 0,$$

$a_0 \neq 0 \neq a_n$ and the numbers a_i are relatively prime ($i = 0, \dots, n$). Clearly,

$$a_0 \in I \cap Z = pZ,$$

p divides a_0 and

$$a_1 u + \ldots + a_n u^n = u(a_1 + \ldots + a_n u^{n-1}) \in P.$$

Thus

$$a_1 + \ldots + a_n u^{n-1} \in P \subseteq I$$

and, again,

$$a_1 \in I \cap Z = pZ.$$

Proceeding similarly further, we show that p divides all numbers $a_0, \dots, a_n$, a contradiction. This means that P is a maximal ideal in R, as desired. $\square$

Proposition 3. *Let R be a differentially trivial Noetherian domain of characteristic q.*

(i) *If $q = 0$ then every non-zero prime ideal of R is maximal.*
(ii) *If $q > 0$ then R is a field.*

Proof: (i). Just combine Proposition 1 (1) and Lemma 2.

(ii). Let P be a prime ideal of R. From Proposition 1 (2) it follows that $P^q = P$. On the other hand,

$$\bigcap_{n=1}^{\infty} P^n = \{0\}$$

by the Krull Theorem (see [7, Chapter IV, § 7, Theorem 12]). Thus $P = \{0\}$ and we conclude that R is a field. The proposition is proved. □

Remark 4. Let R be a differentially trivial Noetherian domain of characteristic 0. With respect to Proposition 1 (1), we may assume that

$$\mathbb{Z} \subseteq R \subseteq Q(R) \subseteq \mathbb{A},$$

where $\mathbb{A}$ is the field of algebraic complex numbers. Now, it follows from Proposition 3 (i) that the integral closure S of R in $Q(R)$ is a Dedekind domain.

Lemma 5. *Let R be a differentially trivial Noetherian ring such that R is not a domain and let the additive group R^+ be torsion-free. Then R is a subdirect product of finitely many differentially trivial domains of characteristic* 0.

Proof: If $char(R/P) = q > 0$ for some $P \in \mathcal{A}ss(R)$ then there is an $x \in R$ such that $x \neq 0$, $P = (0 : x)$ and

$$qxR = \{0\},$$

and, therefore, $x \in \mathcal{F}(R)$, a contradiction. Thus $char(R/P) = 0$ for every $P \in \mathcal{A}ss(R)$.

If R/P is a field for every $P \in \mathcal{A}ss(R)$ then R is an Artinian ring by the Akizuki Theorem (see [7, Chapter IV, § 2, Theorem 2]). Applying Corollary 2.12 of [6] we obtain that R is the ring direct sum of finitely many differentially trivial fields of characteristic 0.

Suppose that the quotient ring R/P is not a field for some $P \in \mathcal{A}ss(R)$. If $\mathcal{A}ss(R) = \{P\}$ then $P^n = \{0\}$ for some integer $n \geq 1$. Now, let M be a nil ideal of R such that P/M is a minimal ideal of R/M. By Proposition 4.1.3 (iii) of [4]

$$\mathcal{D}_{R/M}(\overline{0}) = \mathcal{D}_{R/M}(P/M),$$

and, therefore,

$$\overline{a} \cdot P/M = P/M$$

for every $\overline{a} \in \mathcal{D}_{R/M}(P/M)$. Then, by Robson's Theorem (see [4, Theorem 4.1.9])

$$R/M = S \oplus A_1 \oplus \ldots \oplus A_m$$

is a ring direct sum of a semiprime ring S and finitely many local Artinian rings $A_1, \dots, A_m$ $(m \in \mathbb{N})$. Since the factor ring R/M is differentially trivial, each A_i is a field $(i = 1, \dots, m)$ by Lemma 2.2 of [6], a contradiction. Consequently, $\mathcal{A}ss(R) = \{P_1, \dots, P_n\}$ for an integer $n \geq 2$.

Assume that $N = \mathcal{N}il(R) \neq \{0\}$ and put $S = \bigcap_{i=1}^{n}(R \setminus P_i)$. If $P_i \leq P_j$, where i and j are distinct integers and $1 \leq i, j \leq n$, and $u \in P_j \setminus P_i$ then there exist $k \geq 1$ and $a_0, \dots, a_k$ in the prime subring of R such that $a_0 \notin P_i$ and

$$a_0 + a_1 u + \cdots + a_k u^k \in P_i$$

(use Proposition 1(1)). Then, however, $a_0 \in P_j$, and this is a contradiction with $char(R/P_j) = 0$. Consequently, all prime ideals $P_1, \dots, P_n$ are pair-wise incomparable. By Proposition 10(ii) of [5, Chapter IV, § 2, $n^\circ 5$] the total ring of quotients $A = S^{-1}R$ is Artinian, and by Theorem 4.1.4 of [4] the factor ring R/N is a Goldie ring.

Let M be a nil ideal of R such that N/M is a minimal ideal of R/M. By Proposition 4.1.3(iii) of [4] we have

$$\mathcal{D}_{R/M}(\overline{0}) = \mathcal{D}_{R/M}(N/M),$$

and, hence,

$$\overline{a} \cdot N/M = N/M$$

for every element $\overline{a} \in \mathcal{D}_{R/M}(\overline{0})$. Using Robson's Theorem again, we conclude that the factor ring R/M is a ring direct sum of a semiprime ring X and finitely many local Artinian rings $B_1, \dots, B_l$ $(l \in \mathbb{N})$.

Thus to complete the proof we show that a differentially trivial local Artinian ring $A = B_i$ of characteristic 0 is a field. Let $\pi : A \to A/\mathcal{J}(A)$ be a canonical epimorphism and $K = A/\mathcal{J}(A)$. By P we denote the prime subring of A. Since $char(A) = 0$, P is a field. The family Γ of all subfields of A ordered by inclusion has a maximal element M by Zorn's Lemma. If $\beta \in K$ is transcendental over $\pi(M)$ then every non-zero element of the polynomial ring $M[\beta]$ is not contained in $\mathcal{J}(A)$. Therefore $M[\beta]$ is a field, a contradiction. Hence K is an algebraic extension of $\pi(M)$.

Let $\overline{f}(Y) = Y^n + \alpha_1 Y^{n-1} + \ldots + \alpha_n \in \pi(M)[Y]$ be a minimal polynomial of $\eta \in K$ over $\pi(M)$. By a_i we denote the inverse image of α_i in M $(i = 1, \dots, n)$. Since $\overline{f}(Y)$ have no multiple roots, by Hensel's Lemma (see e.g. [8, Chapter 10, Exercises 9 and 10]) the polynomial

$$f(Y) = Y^n + a_1 Y^{n-1} + \ldots + a_n \in M[Y]$$

has a unique root z such that

$$\pi(z) = \eta.$$

This means that the ring $M[z]$ is isomorphic to the ring $\pi(M)[\eta]$ which is a field. The maximality of M yields that $\eta \in \pi(M)$. Hence $\pi(M) = K$ and

$$A = \mathcal{J}(A) + M.$$

Thus for every element a of A there are unique elements $j \in \mathcal{J}(A)$ and $m \in M$ such that

$$a = j + m. \tag{1}$$

It is well known that $\mathcal{J}(A)$ is a nilpotent ideal with index of nilpotency, say, $t \geq 2$. Furthermore, $Ann(\mathcal{J}(A)) \neq \mathcal{J}(A)^{t-2}$ if $t > 2$. The map $\sigma : A \to A$ given by

$$\sigma(a) = bj,$$

where b is a fixed element of $\mathcal{J}(A)^{t-2} \setminus (Ann\mathcal{J}(A))$ if $t > 2$, and

$$\sigma(a) = j$$

if $t = 2$ with j as in (1), determines a non-zero derivation σ of A, a contradiction. Hence $\mathcal{J}(B_i) = \mathcal{J}(A) = \{\overline{0}\}$ $\quad(i = 1, \dots, l)$, a contradiction. This means that

$$\bigcap_{s=1}^{n} P_s = \mathcal{N}il(R) = \{0\}.$$

By Proposition 10 of [9, § 2.1], R is a subdirect product of differentially trivial rings R/P_s $\quad(s = 1, \dots, n)$. The lemma is proved. □

Lemma 6. *Let R be a differentially trivial Noetherian ring such that R is not a domain and let the additive group R^+ be torsion. Then*

$$R \cong \sum_{i=1}^{n}{}^{\oplus} \mathbb{Z}_{{p_i}^{k_i}} \quad (k_i \in \mathbb{N}).$$

Proof: By Proposition 3 (ii) every non-zero prime ideal of R is maximal. Consequently, R is an Artinian ring (see e.g. [7, Chapter IV, § 2]) and the result follows from Corollary 2.12 of [6]. □

Lemma 7. *If R is a differentially trivial semiprime Noetherian ring with the mixed additive group R^+ then*

$$R = A \oplus B$$

is the ring direct sum of a differentially trivial ring A of characteristic 0 and a differentially trivial ring B of finite characteristic.

Proof: Let $\mathcal{A}ss(R) = \{P_1, \dots, P_n\}$. From $\mathcal{N}il(R) = \{0\}$ it follows that $n \geq 2$. Moreover, there are ideals $P, Q \in \mathcal{A}ss(R)$ such that $char(R/P) = p$ for some prime p and $char(R/Q) = 0$. Let π be the set of all primes p such that there is an ideal $P \in \mathcal{A}ss(R)$ with $char(R/P) = p$.

We will show that $\mathcal{F}(R)^+$ is a π-group. For doing this, suppose by contrary that $\mathcal{F}(R)^+$ contains some non-zero element a of order q and $q \notin \pi$. Then

$$a \cdot qR = \{0\}$$

and, consequently, $qR \leq \bigcup_{i=1}^{n} P_i$, a contradiction. Since R is a Noetherian ring, the set π is finite and, further, $\mathcal{F}(R)^+$ is a group of exponent $p_0 = \prod_{p\in\pi} p$. If F_p is the Sylow p-subgroup of $\mathcal{F}(R)^+$ $(p \in \pi)$ then

$$\mathcal{F}(R)^+ = F_p \oplus (\mathcal{F}(R) \cap pR)$$

is a group direct sum, where $\mathcal{F}(R) \cap pR$ is a p'-subgroup. Since the factor ring R/pR is differentially trivial and $(F_p + pR)/pR \cong F_p$, in view of Proposition 3 (ii) and Lemma 6 the ideal $\mathcal{F}(R)$ is a differentially trivial ring with the identity element e. Thus $eR \leq \mathcal{F}(R)$ and

$$R = eR \oplus (1 - e)R$$

is a ring direct sum. If $eR \neq \mathcal{F}(R)$ and $f \in \mathcal{F}(R) \setminus eR$ then

$$f = eu + (1 - e)v$$

for some elements $u, v \in R$, and thus

$$f = e \cdot f = eu,$$

a contradiction. Hence $eR = \mathcal{F}(R)$ and $(1 - e)R$ is a differentially trivial ring of characteristic 0. The lemma is proved. □

Theorem 8. *Let R be a Noetherian ring. Then R is differentially trivial if and only if it is of one of the following types:*

1) *R is a differentially trivial Noetherian domain (i.e. R is algebraic over its prime subring if $char(R) = 0$, and $R = \{a^p \mid a \in R\}$ if $char(R) = p$);*
2) *R is a subdirect product of finitely many differentially trivial Noetherian domains of characteristic 0;*
3) $R \cong \sum_{i=1}^{n}{}^{\oplus} \mathbb{Z}_{p_i^{k_i}}$;
4) *$R = F \oplus S$ is a ring direct sum, where S is a ring of type 2) and F is a ring of type 3).*

PROOF: ($\Rightarrow$). Let R be a differentially trivial Noetherian ring. If R is a domain then by Proposition 1 it is a ring of type 1).

Suppose that R is not a domain. If the additive group R^+ is torsion-free (periodic, respectively) then R is a ring of type 2) by Lemma 5 (of type 3) by Lemma 6, respectively).

Assume that the additive group R^+ is mixed. As a consequence of Lemma 5

$$\mathcal{N}il(R) \leq \mathcal{F}(R).$$

If $\mathcal{A}ss(R) = \{P\}$ then $char(R/P) = 0$ and

$$P = \mathcal{N}il(R) = \mathcal{F}(R).$$

Let a be a non-zero element of $\mathcal{F}(R)^+$ of finite order p. Then

$$a \cdot pR = \{0\}$$

and in view of Corollary 2 of [5, Chapter IV, § 1, $n^\circ 3$] $pR \leq P$, a contradiction. Hence $\mathcal{A}ss(R) = \{P_1, \dots, P_n\}$ for an integer $n \geq 2$. Since the group R^+ is nonperiodic, $char(R/Q) = 0$ for some ideal $Q \in \mathcal{A}ss(R)$. If $char(R/P_i) = 0$ for all $i \quad (i = 1, \dots, n)$ then

$$\mathcal{F}(R) = \mathcal{N}il(R)$$

and for every non-zero element a of $\mathcal{F}(R)$ of order p we have

$$a \cdot pR = \{0\}.$$

By Corollary 2 of [5, Chapter IV, § 1, $n^\circ 3$]

$$pR \leq P_s$$

for some integer $s \quad (1 \leq s \leq n)$, a contradiction. Thus R has an ideal $M \in \mathcal{A}ss(R)$ such that $char(R/M) = p$.

Now it is clear that

$$c \cdot \mathcal{N}il(R) = \mathcal{N}il(R)$$

for every element $c \in \mathcal{D}_R(0)$. From Theorem 2.2.15 of [4] it follows that $R/\mathcal{N}il(R)$ is a Goldie ring. Then by Proposition 4.1.3 (ii) of [4] we obtain

$$\mathcal{D}_R(0) = \mathcal{D}_R(\mathcal{N}il(R)).$$

Using Robson's Theorem (see [4, Theorem 4.1.9]), one sees that R is the ring direct sum of a semiprime ring S and finitely many local Artinian rings $A_1, \dots, A_k$ $(k \in \mathbb{N})$. In view of Corollary 2.12 of [6], A_i is either a differentially trivial field or isomorphic to some $\mathbb{Z}_{p^k}$. Finally, we can apply Lemma 7.

($\Leftarrow$). It is obvious that R of type 1), 2) or 4) is differentially trivial. We will show that R of type 3) is differentially trivial. Since R is a subdirect product of finitely many differentially trivial domains $R_1, \dots, R_n$ of characteristic 0, Proposition 10 of [9, § 2.1] yields that there are the ideals $P_1, \dots, P_n$ of R such that

$$\bigcap_{i=1}^{n} P_i = \{0\} \text{ and } R_i = R/P_i \quad (i = 1, \dots, n).$$

By Theorem 2.2.15 of [4] R is a Goldie ring. Then by Theorem 2.3.6 of [4] $S = \mathcal{D}_R(0)$ is an Ore set and $S^{-1}R$ is an Artinian ring. $\mathcal{N}il(R) = \{0\}$ yields that $\mathcal{J}(S^{-1}R) = \{0\}$ and, consequently,

$$S^{-1}R = B_1 \oplus \ldots \oplus B_n$$

is the ring direct sum of fields $B_1, \ldots, B_n$ such that $Q(R/P_i) \cong B_i \ (i = 1, \ldots, n)$. Clearly, every derivation of R can be extended to a derivation of $S^{-1}R$. Since the ring $S^{-1}R$ is differentially trivial we conclude that R is as desired. $\square$

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