Kuratowski convergence on compacta and Hausdorff metric convergence on compacta

P. Brandi, R. Ceppitelli, L'. Holá

Abstract. This paper completes and improves results of [10]. Let (X, d_X), (Y, d_Y) be two metric spaces and G be the space of all Y-valued continuous functions whose domain is a closed subset of X. If X is a locally compact metric space, then the Kuratowski convergence τ_K and the Kuratowski convergence on compacta τ_K^c coincide on G. Thus if X and Y are boundedly compact metric spaces we have the equivalence of the convergence in the Attouch-Wets topology τ_{AW} (generated by the box metric of d_X and d_Y) and τ_K^c convergence on G, which improves the main result of [10]. In the second part of paper we extend the definition of Hausdorff metric convergence on compacta for general metric spaces X and Y and we show that if X is locally compact metric space, then also τ-convergence and Hausdorff metric convergence on compacta coincide in G.

Keywords: Kuratowski convergence, Attouch-Wets convergence, τ-convergence, Kuratowski convergence on compacta and Hausdorff metric convergence on compacta

Classification: 54B20, 54C35

1. Introduction

Topologies and convergences of graph spaces (spaces of functions identified with their graphs or epigraphs) has been applied to different fields of mathematics, including differential equations, convex analysis, optimization, mathematical economics, programming models, calculus of variation, etc.

The problem of continuous dependence on the data for the solutions of functional differential equations led to the problem of defining a suitable notion of convergence in the space of continuous functions with moving domains; in [4] so called τ-convergence and Hausdorff metric convergence on compacta $\tau^\star$ were introduced and studied for $Y = \mathbb{R}^m$ and X a closed connected subset of $\mathbb{R}$; in [5], [6], [7] τ-convergence was extended for general metric spaces X and Y; in [10], [11] the Kuratowski convergence on compacta τ_K^c was considered.

In [10] the authors proved that if (X, d_X), (Y, d_Y) are locally connected boundedly compact metric spaces, then the convergence in the Attouch-Wets topology τ_{AW} (generated by the box metric of d_X and d_Y) and τ_K^c convergence in G coincide. In our paper we show that the assumption of locally connectedness is useless. We use here the result of our Theorem 3.1 which claims that if $C \in CL(X)$ then for every $B \in CL(X)$ and every $\epsilon > 0$ there is $L \in CL(X)$ such that $B \subset L$, the Hausdorff distance between B and L is smaller than ϵ and $L \cap C = \overline{L^\circ \cap C}$. Theorem 3.1 improves Proposition 2.4 from [10].

Moreover we introduce Hausdorff metric convergence on compacta τ_H^c for general metric spaces X and Y and as a further application of mentioned Theorem 3.1 we prove the coincidence of τ_H^c and τ-convergence in G for X locally compact.

2. Notations and definitions

Let (X, d) be a metric space. For basic notions and definitions the reader is referred to the recent Beer's monograph [1]. Given two subsets A, B of X, the excess or Hausdorff semi-distance of A over B is denoted by $e_d(A, B) = \sup_{a\in A} \inf_{b\in B} d(a, b)$ with the convention $e_d(A, \emptyset) = +\infty$ if $A \neq \emptyset$ and $e_d(\emptyset, B) = 0$.
It is well known that $H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}$ defines the Hausdorff distance between A and B.

The gap $D_d(A, B)$ between nonempty subsets A and B is given by $D_d(A, B) = \inf\{d(a, B) : a \in A\}$, where by $d(a, B)$ we mean $\inf\{d(a, b) : b \in B\}$.

Denote by $CL(X)$ the family of all non-empty closed subsets of X and by $\mathcal{K}(X)$ the family of all compact sets in $CL(X)$.

The open (resp. closed) ball with center x and radius $r > 0$ will be denoted by $S(x, r)$ (resp. $B(x, r)$). The open (resp. closed) r-enlargement of A is the set $S(A, r) = \{x \in X : d(x, A) < r\}$ $(B(A, r) = \{x \in X : d(x, A) \leq r\})$.

Recall that a net $\{C_\sigma : \sigma \in \Sigma\}$ in $CL(X)$ is Kuratowski convergent to $C \in CL(X)$ if $LiC_\sigma = LsC_\sigma = C$, where $LiC_\sigma = \{x \in X$: each nbd of x intersects C_σ for all σ in some residual subset of $\Sigma\}$ and $LsC_\sigma = \{x \in X$: each nbd of x intersects C_σ for all σ in some cofinal subset of $\Sigma\}$.

Now let (X, d_X), (Y, d_Y) be two metric spaces. Denote by $\mathcal{D}$ the box metric of d_X and d_Y and consider $X \times Y$ equipped with this metric $\mathcal{D}$.

For every $\Omega \in CL(X)$, $C(\Omega, Y)$ denotes, as usual, the space of all continuous functions $f : \Omega \to Y$. If $f \in C(\Omega, Y)$ we denote by $\Gamma(f, \Omega) = \{(\omega, f(\omega)) : \omega \in \Omega\}$ the graph of f. Let $G = \{\Gamma(f, \Omega) : \Omega \in CL(X), f \in C(\Omega, Y)\}$ denote the set of all graphs.

In [10] the following definition was introduced:

Definition 2.1. *A net $\{\Gamma(f_\sigma, \Omega_\sigma) : \sigma \in \Sigma\}$ in G is said to be Kuratowski convergent on compacta to $\Gamma(f, \Omega)$ (τ_K^c-convergent to $\Gamma(f, \Omega)$) if $\{\Gamma(f_\sigma, \Omega_\sigma \cap K) : \sigma \in \Sigma\}$ Kuratowski converges to $\Gamma(f, \Omega \cap K)$ for every $K \in \mathcal{K}(X)$ such that*

$$(\star) \qquad \overline{K^\circ \cap \Omega} = K \cap \Omega.$$

By A° we mean the interior of A, by $\overline{A}$ the closure of A and by ∂A the boundary of A.

3. Main result

In the first part of our paper we are interested in the property $(\star)$. We greatly improve Proposition 2.4 in [10] by proving that if $C \in CL(X)$ is fixed then for every $B \in CL(X)$ we can find arbitrarily close to B (with respect to the Hausdorff metric) a set $L \in CL(X)$ such that $L \supset B$ and L has the property $(\star)$ with respect to C; i.e. $\overline{L^\circ \cap C} = L \cap C$.

Theorem 3.1. *Let (X,d) be a metric space and $C \in CL(X)$. For every $B \in CL(X)$ and every $\epsilon > 0$ there is $L \in CL(X)$ such that $B \subset L^\circ$, $L \cap C = \overline{L^\circ \cap C}$ and $H_d(B,L) < \epsilon$.*

Proof: Let $B \in CL(X)$ and $\epsilon > 0$. If $B \cap C = \emptyset$, then the normality of X implies that there is an open set V such that $B \subset V \subset \overline{V} \subset X \setminus C$. Then the set $L = \overline{V \cap S(B, \epsilon/2)}$ does the job.

Now suppose that $B \cap C \neq \emptyset$. Put $V_0 = S(B, \epsilon/4)$ and $B_0 = \overline{V_0}$. Then $H_d(B_0, B) \leq \epsilon/4$.
Now put $V_1 = \bigcup\{S(x, \epsilon/2^{2+1}) : x \in (\overline{V_0} \setminus V_0) \cap C\}$ and put $B_1 = B_0 \cup \overline{V_1}$. Then $B_0 \cap C \subset B_1^\circ$, $H_d(B_1, B_0) = e_d(B_1, B_0) = e_d(V_1, B_0) = \sup_{x \in (\overline{V_0} \setminus V_0) \cap C} e_d(S(x, \epsilon/2^{2+1}), B_0) \leq \epsilon/2^{2+1}$ and $H_d(B_1, B) \leq H_d(B_1, B_0) + H_d(B_0, B) \leq \epsilon/2^{2+1} + \epsilon/2^2$.
Let $V_2 = \bigcup\{S(x, \epsilon/2^{2+2}) : x \in (\overline{V_1} \setminus V_1) \cap C\}$ and put $B_2 = B_1 \cup \overline{V_2}$.
Then $B_1 \cap C \subset B_2^\circ$, $H_d(B_2, B_1) = e_d(B_2, B_1) = e_d(V_2, B_1) \leq \epsilon/2^{2+2}$ and $H_d(B_2, B) \leq H_d(B_2, B_1) + H_d(B_1, B) \leq \epsilon/2^{2+2} + \epsilon/2^{2+1} + \epsilon/2^2$.

Suppose now, we defined open sets $V_1, \dots, V_n$ and closed sets $B_1, \dots, B_n$ such that $V_i = \bigcup\{S(x, \epsilon/2^{2+i}) : x \in (\overline{V_{i-1}} \setminus V_{i-1}) \cap C\}$ and $B_i = B_{i-1} \cup \overline{V_i}$, $B_{i-1} \cap C \subset B_i^\circ$, $H_d(B_i, B) \leq \sum_{j=0}^{i} \epsilon/2^{2+j}$, for every $i = 1, \dots n$.
We will define now V_{n+1} and B_{n+1}.

Put $V_{n+1} = \bigcup\{S(x, \epsilon/2^{2+n+1}) : x \in (\overline{V_n} \setminus V_n) \cap C\}$ and $B_{n+1} = B_n \cup \overline{V_{n+1}}$. Then $B_n \cap C \subset B_{n+1}^\circ$ as $B_n \cap C = (B_{n-1} \cup \overline{V_n}) \cap C = (B_{n-1} \cap C) \cup (\overline{V_n} \setminus V_n) \cap C \cup (V_n \cap C) \subset B_n^\circ \cup V_{n+1} \cup V_n \subset B_{n+1}^\circ$.
Moreover, since $H_d(B_{n+1}, B_n) = e_d(B_{n+1}, B_n) = e_d(V_{n+1}, B_n) \leq \epsilon/2^{2+n+1}$, thus $H_d(B_{n+1}, B) \leq H_d(B_{n+1}, B_n) + H_d(B_n, B) \leq \epsilon/2^{2+n+1} + \sum_{i=0}^{n} \epsilon/2^{2+i} = \sum_{i=0}^{n+1} \epsilon/2^{2+i}$.

Put $L = \overline{\cup\{B_n : n = 0, 1, 2, \dots\}}$. Then $B \subset L^\circ$ and $H_d(L, B) = e_d(\bigcup B_n, B) = \sup_n e_d(B_n, B) \leq \sup_n \sum_{i=0}^{n} \epsilon/2^{2+i} = \sum_{i=0}^{\infty} \epsilon/2^{2+i} = \epsilon/4(\sum_{i=0}^{\infty} 1/2^i) = 2\epsilon/4 = \epsilon/2$.

Now we show that $L \cap C = \overline{L^\circ \cap C}$. Let $z \in L \cap C$ and suppose there is $n \in Z^+$ such that $z \in B_n$. By above $B_n \cap C \subset B_{n+1}^\circ \subset L^\circ$ thus $z \in L^\circ \cap C$.

Suppose now there is no $n \in Z^+$ with $z \in B_n$. Let $\eta > 0$. We must show that $S(z, \eta) \cap (L^\circ \cap C) \neq \emptyset$. Let $k \in Z^+$ be such that $\epsilon/2^{2+k} < \eta/4$. Let H_z be an open nbd of z such that $H_z \subset (X \setminus B_k) \cap S(z, \eta/4)$. Thus there must exist $m > k$ with $z_m \in H_z \cap B_m$. Without loss of generality we can suppose that m is such that $z_m \in B_m \setminus B_{m-1}$. Thus by the assumption $z_m \in \overline{V_m}$, where $V_m = \bigcup\{S(x, \epsilon/2^{2+m}) : x \in (\overline{V_{m-1}} \setminus V_{m-1}) \cap C\}$. Thus there is $v_m \in H_z \cap V_m$ with $v_m \in S(v, \epsilon/2^{2+m})$, where $v \in C$. Clearly $S(v, \epsilon/2^{2+m}) \subset S(z, \eta)$ and $v \in V_m \cap C \subset B_m^\circ \cap C \subset L^\circ \cap C$. $\square$

Remark 3.2. Let (X,d) be a locally compact metric space. Then for every compact set K and every $\epsilon > 0$ there is a compact set L such that $K \subset L^\circ$, $L \cap C = \overline{L^\circ \cap C}$ and $H_d(K,L) < \epsilon$.

Remark 3.3. Let (X,d) be a metric space and $C \in CL(X)$. If $L_1, L_2, \ldots, L_n$ is a finite family of closed subsets satisfying property $(\star)$ with respect to C, then also $\bigcup_{i=1}^n L_i$ has property $(\star)$ with respect to C (Lemma 2.2 in [10]).

The following example shows that if $L_1, L_2, \ldots, L_n, \ldots$ is an infinite family of closed sets satisfying property $(\star)$ with respect to C then $\overline{\cup_{n=1}^\infty L_n}$ can fail to have property $(\star)$.

Example 3.4. Let X be the set of reals with the usual metric and Q^+ be the set of positive rationals. Let $\{q_1, q_2, \ldots, q_n, \ldots\}$ be an enumeration of Q^+. Put $C = (-\infty, 0]$ and $L_n = \{q_n\}$ for every $n \in Z^+$. It is easy to verify that every L_n has property $(\star)$ with respect to C, since $L_n \cap C = \emptyset$ for every $n \in Z^+$. But $\overline{\cup_{n=1}^\infty L_n}$ fails to have property $(\star)$ with respect to C.

4. Applications to Kuratowski convergence

In this part we apply Theorem 3.1 to prove a coincidence of Kuratowski and Kuratowski convergence on compacta τ_K^c in G.

By using of the following proposition we give a shorter and better proof of the main result of [10].

Proposition 4.1. *Let (X, d_X) and (Y, d_Y) be metric spaces. The following are equivalent:*

(1) *X is locally compact;*

(2) *the Kuratowski convergence and the τ_K^c-convergence in G coincide.*

Proof: (2) $\Rightarrow$ (1). Suppose X is not locally compact. There is a point $x \in X$ which has no compact nbd. Denote by $\mathcal{U}(x)$ the family of all open nbds of x. For every $U \in \mathcal{U}(x)$ and every $K \in \mathcal{K}(X)$ there is a point $x_{U,K} \in U \setminus K$.

Consider the following directions on $\mathcal{U}(x)$ and $\mathcal{K}(X)$: if $U, V \in \mathcal{U}(x)$ then $U \geq V \Leftrightarrow U \subset V$ and if $B, C \in \mathcal{K}(X)$ then $B \geq C \Leftrightarrow B \supset C$. Let $\mathcal{U}(x) \times \mathcal{K}(X)$ be equipped with the natural direction induced by the above ones.

For every $(U,K) \in \mathcal{U}(x) \times \mathcal{K}(X)$ put $\Omega_{U,K} = \{x_{U,K}, z\}$, where z is a point in X different from x (such a point z exists, otherwise x has a compact nbd). Define $f_{U,K}$ on $\Omega_{U,K}$ as $f_{U,K}(z) = f_{U,K}(x_{U,K}) = 0$. Let further $\Omega = \{z\}$ and f is defined on Ω as $f(z) = 0$.

Then $\{\Gamma(f_{U,K}, \Omega_{U,K}) : (U,K) \in \mathcal{U}(x) \times \mathcal{K}(X)\}$ fails to Kuratowski converge to $\Gamma(f, \Omega)$, since $(x, 0) \in Li\Gamma(f_{U,K}, \Omega_{U,K}) \setminus \Gamma(f, \Omega)$.

Now we show that $\{\Gamma(f_{U,K}, \Omega_{U,K}) : (U,K) \in \mathcal{U}(x) \times \mathcal{K}(X)\}$ τ_K^c-converges to $\Gamma(f, \Omega)$.

Let $C \in \mathcal{K}(X)$ be such that $C \cap \Omega = \overline{C^\circ \cap \Omega}$.

If $C \cap \Omega = \emptyset$, then for every $K \supset C$ and every $U \in \mathcal{U}(x)$ we have also $C \cap \Omega_{U,K} = \emptyset$, so we are done.

If $C \cap \Omega \neq \emptyset$, then $\{z\} = C \cap \Omega$ and for every $K \supset C$ and every $U \in \mathcal{U}(x)$ we have $C \cap \Omega_{U,K} = \{z\}$, since $x_{U,K} \in K^c \subset C^c$. Thus $\{\Gamma(f_{U,K}, \Omega_{U,K} \cap C) : (U, K) \in \mathcal{U}(x) \times \mathcal{K}(X)\}$ Kuratowski converges to $\Gamma(f, \Omega \cap C)$.

(1) $\Rightarrow$ (2). Suppose first that $\{\Gamma(f_\sigma, \Omega_\sigma) : \sigma \in \Sigma\}$ Kuratowski converges to $\Gamma(f, \Omega)$. Let $K \in \mathcal{K}(X)$ be such that $K \cap \Omega = \overline{K^\circ \cap \Omega}$.
We must show that $\{\Gamma(f_\sigma, \Omega_\sigma \cap K) : \sigma \in \Sigma\}$ Kuratowski converges to $\Gamma(f, \Omega \cap K)$.
First, we show that $\Gamma(f, \Omega \cap K) \subset Li\Gamma(f_\sigma, \Omega_\sigma \cap K)$.

Let $(x, f(x)) \in \Gamma(f, \Omega \cap K)$ and $O_x \times O_{f(x)}$ be a nbd of $(x, f(x))$. We must find $\sigma_0 \in \Sigma$ with $\Gamma(f_\sigma, \Omega_\sigma \cap K) \cap O_x \times O_{f(x)} \neq \emptyset$ for every $\sigma \geq \sigma_0$.

The continuity of f at x implies there is an open set U such that $U \subset O_x$, $x \in U$ and $f(U \cap K) \subset O_{f(x)}$.
Moreover, the point x belongs to $\overline{K^\circ \cap \Omega}$, i.e. there is $z \in K^\circ \cap \Omega \cap U$.

Let V be an open set such that $z \in V \subset U \cap K^\circ$. Since $(z, f(z)) \in Li\Gamma(f_\sigma, \Omega_\sigma)$ and $V \times O_{f(x)}$ is a nbd of $(z, f(z))$ there is $\sigma_0 \in \Sigma$ such that $\Gamma(f_\sigma, \Omega_\sigma) \cap V \times O_{f(x)} \neq \emptyset$ for every $\sigma \geq \sigma_0$.
So for every $\sigma \geq \sigma_0$ there is $(z_\sigma, f_\sigma(z_\sigma)) \in V \times O_{f(x)} \subset K^\circ \times O_{f(x)}$. Thus for every $\sigma \geq \sigma_0$, $\Gamma(f_\sigma, \Omega_\sigma \cap K) \cap O_x \times O_{f(x)} \neq \emptyset$.

Now we show that $Ls\Gamma(f_\sigma, \Omega_\sigma) \subseteq \Gamma(f, \Omega \cap K)$. Clearly $Ls\Gamma(f_\sigma, \Omega_\sigma \cap K) \subseteq Ls\Gamma(f_\sigma, \Omega_\sigma) \subseteq \Gamma(f, \Omega)$.
Let $(x, y) \in Ls\Gamma(f_\sigma, \Omega_\sigma \cap K)$. So $(x, y) \in \Gamma(f, \Omega)$ and it is easy to verify that $x \in K$. So $x \in \Omega \cap K$ and thus $(x, y) \in \Gamma(f, \Omega \cap K)$, we are done.

Suppose now that a net $\{\Gamma(f_\sigma, \Omega_\sigma) : \sigma \in \Sigma\}$ τ^c_K-converges to $\Gamma(f, \Omega)$. We prove $Ls\Gamma(f_\sigma, \Omega_\sigma) \subset \Gamma(f, \Omega) \subset Li\Gamma(f_\sigma, \Omega_\sigma)$.

Concerning the first inclusion, let $(x, y) \in Ls\Gamma(f_\sigma, \Omega_\sigma)$ and suppose $(x, y) \notin \Gamma(f, \Omega)$. Then $x \in \Omega$, otherwise $x \notin \Omega$ implies that there is a compact ball $B(x, \delta)$ such that $B(x, \delta) \cap \Omega = \emptyset = \overline{B(x, \delta)^\circ \cap \Omega}$. Then $(x, y) \in Ls\Gamma(f_\sigma, \Omega_\sigma \cap B(x, \delta)) \subset \Gamma(f, \Omega \cap B(x, \delta)) = \emptyset$, a contradiction. Thus $x \in \Omega$.
By Remark 3.2 there is a compact set C such that $x \in C^\circ$ and $C \cap \Omega = \overline{C^\circ \cap \Omega}$. It is easy to verify that $(x, y) \in Ls\Gamma(f_\sigma, \Omega_\sigma \cap C) \subset \Gamma(f, \Omega \cap C)$ by the assumption. Thus $(x, y) \in \Gamma(f, \Omega)$.

To prove $\Gamma(f, \Omega) \subset Li\Gamma(f_\sigma, \Omega_\sigma)$, let $(x, f(x)) \in \Gamma(f, \Omega)$. Let C be a compact nbd of x such that $C \cap \Omega = \overline{C^\circ \cap \Omega}$ (Remark 3.2). Then $(x, f(x)) \in \Gamma(f, \Omega \cap C) \subset Li\Gamma(f_\sigma, \Omega_\sigma \cap C) \subset Li\Gamma(f_\sigma, \Omega_\sigma)$. □

Corollary 4.2. *From the proof* (1) $\Rightarrow$ (2) *in the previous proposition we can see that the Kuratowski convergence always implies* τ^c_K*-convergence and also that the Kuratowski convergence of* $\{\Gamma(f_\sigma, \Omega_\sigma) : \sigma \in \Sigma\}$ *to* $\Gamma(f, \Omega)$ *implies Kuratowski convergence of* $\{\Gamma(f_\sigma, \Omega_\sigma \cap B) : \sigma \in \Sigma\}$ *to* $\Gamma(f, \Omega \cap B)$ *for every* $B \in CL(X)$ *such that* $B \cap \Omega = \overline{B^\circ \cap \Omega}$ (*without a locally compactness assumption*) (*see also* [11]).

Corollary 4.3. *Let* X *be a locally compact metric space. Let* $\{C_\sigma : \sigma \in \Sigma\}$ *and* $C \in CL(X)$. *Then* $\{C_\sigma : \sigma \in \Sigma\}$ *Kuratowski converges to* C *if and only if for every* $K \in \mathcal{K}(X)$ *with* $K \cap C = \overline{K^\circ \cap C}$, $\{C_\sigma \cap K : \sigma \in \Sigma\}$ *Kuratowski converges to* $C \cap K$.

The following corollary improves the main result of [10].

Corollary 4.4. *Let* (X, d_X) *and* (Y, d_Y) *be boundedly compact metric spaces,* $\mathcal{D}$ *be the box metric of* d_X *and* d_Y*. Then* τ_K^c*-convergence and the Attouch-Wets convergence generated by* $\mathcal{D}$ *in* G *coincide.*

Proof: By the previous proposition τ_K^c-convergence and Kuratowski convergence coincide in G. It is a well-known result that the Kuratowski convergence and the Attouch-Wets convergence coincide in boundedly compact metric spaces ([1]). □

For X and Y locally connected boundedly compact spaces, Corollary 4.4 was proved in [10].

Notice here that the Attouch-Wets topology on graphs and epigraphs of functions with the same domain has been studied in literature (see [1], [9]).

5. Applications to τ-convergence

The definition of a new graph topology τ was motivated by concrete problems in the theory of hereditary differential equations.

In [2], [3], [8], the authors studied a Cauchy problem (P) for ordinary differential equations with delay. By virtue of the generality of the hereditary structure, the solutions of problem (P) are elements of the graph set G (where $X = E$ is a closed interval of $\mathbb{R}$ and $Y = \mathbb{R}^m$). To study problem (P), the authors introduced the topology $\tau^\star$ in G ([4]); it arose as a localization on compact sets of the Hausdorff metric topology; the connection between $\tau^\star$ and Hausdorff metric topology is the same as that between the compact-open topology τ_{CO} and the uniform convergence topology in $C(E, \mathbb{R}^m)$.

Definition ([4, Definition 2]). *A sequence* $(\Gamma(x_n, \Omega_n))_n$, $\Gamma(x_n, \Omega_n) \in G$ *is said to be convergent to* $\Gamma(f_0, \Omega_0) \in G$ *according to the Hausdorff metric on the compact subsets (or more simply,* $\tau^\star$*-convergent to* $\Gamma(f_0, \Omega_0)$*) if for every* $K^\star \in \mathcal{K}_{\Omega_0}$ *the sequence* $\Gamma(x_n, \Omega_n \cap K^\star)$ $H_{\mathcal{D}}$*-converges to* $\Gamma(x_0, \Omega_0 \cap K^\star)$*,*
where K_Ω *is the family of compact intervals with the properties*

(i) $cl(]a, b[\cap\Omega) = [a, b] \cap \Omega \neq \emptyset$*, for every* $[a, b] \in \mathcal{K}_\Omega$*;*
(ii) *for every* $K \in \mathcal{K}(X)$ *there exists an interval* $[a, b] \in \mathcal{K}_\Omega$ *such that* $K \subset [a, b]$.

The peculiar property of $\tau^\star$, which makes it useful in applications to hereditary differential equations, is the homeomorphism between the topological space $(G, \tau^\star)$ and the quotient space $[(CL(E), \tau_F) \times (C(E, \mathbb{R}^m), \tau_{CO})]/\mathcal{R}$ with respect to a suitable equivalence relation, where τ_F is the Fell topology. In force of this homeomorphism, the theory of hereditary differential equations in G has been reduced to the classical theory in $C(E, \mathbb{R}^m)$.

In [4] the proof of the homeomorphic property was constructive, and the choice of compact intervals $K^\star \subset \mathbb{R}$ was an important tool.

This definition is not operative to put it in its proper place in the literature. To this aim, the authors introduced in [4] an equivalent definition of $\tau^\star$, called τ-convergence, they proved that this convergence is topological and found relations between τ and other known topologies.

In [5], [6] the authors extended the τ-topology over the graphs of functions defined on subsets of a metric space X, preserving its main properties. The aim is to introduce the same general hereditary structure in the theory of partial differential equations. The existence of the homeomorphism was proved in locally compact separable metric spaces by using the Dugundji's continuous extension and Michael's continuous selection theorem.

Given two elements $\Gamma(f,\Omega)$, $\Gamma(g,\Delta)$ in G and a set $K \in \mathcal{K}(X)$, we define

$$\rho_K(\Gamma(f,\Omega),\Gamma(g,\Delta)) = \\ = \max\{e_{\mathcal{D}}(\Gamma(f,\Omega\cap K),\Gamma(g,\Delta)), e_{\mathcal{D}}(\Gamma(g,\Delta\cap K),\Gamma(f,\Omega))\}.$$

Remark 5.1. Note that ρ_K is non decreasing with respect to K, i.e. if $K_1 \subset K_2$ then $\rho_{K_1}(\cdot,\cdot) \le \rho_{K_2}(\cdot,\cdot)$. We also have that $\rho_K(\cdot,\cdot) \le H_{\mathcal{D}}(\cdot,\cdot)$.
Moreover, we have

$$e_{\mathcal{D}}(\Gamma(f,\Omega\cap K),\Gamma(g,\Delta)) = e_{\mathcal{D}}(\Gamma(f,\Omega\cap K),\Gamma(g,\Delta\cap B(K,r)))$$

for every number $r > e_{\mathcal{D}}(\Gamma(f,\Omega\cap K),\Gamma(g,\Delta))$.

Definition 5.2. *Let (X,d_X) and (Y,d_Y) be metric spaces. A net $\{\Gamma(f_\sigma,\Omega_\sigma) : \sigma \in \Sigma\}$ in G is said to be τ-convergent to $\Gamma(f,\Omega)$ if for every $K \in \mathcal{K}(X)$ the numerical net $\{\rho_K(\Gamma(f,\Omega),\Gamma(f_\sigma,\Omega_\sigma))\colon \sigma \in \Sigma\}$ converges to zero.*

It is natural to ask whether also in general metric spaces we can describe τ convergence as Hausdorff metric convergence on compacta (in similar way as in [4]). In this section we use the result of our Theorem 3.1 to show that in locally compact metric spaces it is possible.

Definition 5.3. *A net $\{\Gamma(f_\sigma,\Omega_\sigma) : \sigma \in \Sigma\}$ in G is said to be τ_H^c-convergent to $\Gamma(f,\Omega)$ in G if $\{\Gamma(f_\sigma,\Omega_\sigma\cap K) : \sigma \in \Sigma\}$ converges in the Hausdorff metric $H_{\mathcal{D}}$ to $\Gamma(f,\Omega\cap K)$ for every $K \in \mathcal{K}(X)$ such that $\overline{K^\circ \cap \Omega} = K \cap \Omega$.*

Of course, a more "natural" generalization of $\tau^\star$-convergence would be that one where compact connected sets with the nonempty intersection with the domain of the limit function are taken instead of compact sets (compare [4, Definition 2]); but to guarantee such a description of τ-convergence we need to work in spaces with an additional structure; more precisely in locally connected locally compact spaces in which every compact set can be covered by a compact connected set.
However, the homeomorphism result holds in locally compact separable metric spaces [6], so we found a reasonable generalization of $\tau^\star$-convergence for this class of spaces.
The question of nonempty intersection with the domain of the limit function is solved in the following remark.

Remark 5.4. *Let (X, d_X) be a locally compact metric space, (Y, d_Y) be a metric space, $\Gamma(f,\Omega) \in G$ and $\{\Gamma(f_\sigma, \Omega_\sigma) : \sigma \in \Sigma\}$ be a net in G.*
If $\{H_{\mathcal{D}}(\Gamma(f_\sigma, \Omega_\sigma \cap K), \Gamma(f, \Omega \cap K)) : \sigma \in \Sigma\} \to 0$ for every $K \in \mathcal{K}(X)$ with $K \cap \Omega = \overline{K^\circ \cap \Omega} \neq \emptyset$ then $\{\Gamma(f_\sigma, \Omega_\sigma) : \sigma \in \Sigma\}$ τ^c_H-converges to $\Gamma(f, \Omega)$.

Proof: It is sufficient to show that if $K \cap \Omega = \emptyset$ then also $K \cap \Omega_\sigma = \emptyset$ eventually.

Suppose this is not true, thus $K \cap \Omega_\sigma \neq \emptyset$ frequently. There is $\epsilon > 0$ such that $S(K, \epsilon) \cap S(\Omega, \epsilon) = \emptyset$. Take $x \in \Omega$. By Remark 3.2 there is a compact set L such that $\{x\} \cup K \subset L^\circ$, $H_{d_X}(\{x\} \cup K, L) < \epsilon$ and $L \cap \Omega = \overline{L^\circ \cap \Omega} \neq \emptyset$. By the assumption there is $\sigma_0 \in \Sigma$ such that $H_{\mathcal{D}}(\Gamma(f_\sigma, \Omega_\sigma \cap L), \Gamma(f, \Omega \cap L)) < \epsilon$ for every $\sigma \leq \sigma_0$. Let $\sigma > \sigma_0$ be such that $K \cap \Omega_\sigma \neq \emptyset$. Let $z_\sigma \in K \cap \Omega_\sigma$. Since $\mathcal{D}((z_\sigma, f_\sigma(z_\sigma)), \Gamma(f, \Omega \cap L)) \leq H_{\mathcal{D}}(\Gamma(f_\sigma, \Omega_\sigma \cap L), \Gamma(f, \Omega \cap L)) < \epsilon$ there must exist $z \in \Omega \cap L$ with $d_X(z_\sigma, z) < \epsilon$, a contradiction. □

Notice that if a metric space (X, d_X) is not locally compact, then the above property always fails as the following argument shows.

Let $x \in X$ be a point which has no compact nbd. Put $\Omega = \{x\}$ and for every $C \in \mathcal{K}(X)$ put $\Omega_C = C$. Consider the natural direction on $\mathcal{K}(X)$. Put further $f(x) = 0$ and $f_C(z) = 0$ for every $z \in C$.
Then $\rho_K(\Gamma(f, \Omega), \Gamma(f_C, \Omega_C)) \to 0$ for every $K \in \mathcal{K}(X)$ with $K \cap \Omega = \overline{K^\circ \cap \Omega} \neq \emptyset$ since there is no such $K \in \mathcal{K}(X)$ (otherwise $x \in K^\circ$).

Now let $K \in \mathcal{K}(X)$ be such that $K \cap \Omega = \emptyset$. For every $C \geq K$ we have $H_{\mathcal{D}}(\Gamma(f, \Omega \cap K), \Gamma(f_C, \Omega_C \cap K)) = \infty$, thus $\{\Gamma(f_C, \Omega_C) : C \in \mathcal{K}(X)\}$ fails to τ^c_H-converge to $\Gamma(f, \Omega)$.

Lemma 5.5. *Let (X, d_X) be a locally compact metric space and (Y, d_Y) be a metric space. Let $\Omega \in CL(X)$ and $K \in \mathcal{K}(X)$ be such that $K \cap \Omega = \overline{K^\circ \cap \Omega} \neq \emptyset$. For every $\epsilon > 0$ there is $0 < \delta < \epsilon$ such that whenever $\eta < \delta$ and $(\Gamma(f, \Omega), \Gamma(g, \Delta)) \in G \times G$ with $\rho_K(\Gamma(f, \Omega), \Gamma(g, \Delta)) < \eta$, then $H_{\mathcal{D}}(\Gamma(f, \Omega \cap K), \Gamma(g, \Delta \cap K)) < \eta + \max\{\epsilon, \omega(\epsilon)\}$, where ω is the modulus of continuity of f in the compact set $\Omega \cap B(K, \eta)$.*

Proof: Let $\epsilon > 0$. Let $0 < \gamma < \epsilon$ be such that $B(K, \gamma)$ is compact. We divide the proof in two steps.

1. First we show there is $0 < \delta < \epsilon$ such that if $\eta < \delta$ and $\rho_K(\Gamma(f, \Omega), \Gamma(g, \Delta)) < \eta$, then

$$e_{\mathcal{D}}(\Gamma(f, \Omega \cap K), \Gamma(g, \Delta \cap K)) < \eta + \max\{\epsilon, \omega(\epsilon)\}.$$

If $\partial K \cap \Omega = \emptyset$, then $K \cap \Omega \subset K^\circ$. There is $\alpha < \gamma$ with $S(K \cap \Omega, \alpha) \subset K^\circ$. With $\delta = \alpha$ we are done.

If $\partial K \cap \Omega \neq \emptyset$, for $\epsilon/3$ there are $w_1, w_2, \dots, w_n \in \partial K \cap \Omega$ such that $\partial K \cap \Omega \subset \cup_{i=1}^n S(w_i, \epsilon/3)$. For every $i = 1, 2, \dots, n$ let $z_i \in S(w_i, \epsilon/3) \cap K^\circ \cap \Omega$.
Put $\pi = \mathcal{D}_{d_X}(\{z_1, \dots, z_n\}, \partial K \cap \Omega)$ and $L = K \setminus S(\partial K \cap \Omega, \pi/2)$. Then $L \in CL(X)$, $z_i \in L$ for every $i = 1, 2, \dots, n$ and $\Omega \cap L \subset K^\circ$.
There exists $\mu > 0$ such that $\mu < \gamma$ and $S(\Omega \cap L, \mu) \subset K^\circ$. Put $\delta = \mu$. If $\eta < \delta$ and $\rho_K(\Gamma(f, \Omega), \Gamma(g, \Delta)) < \eta$, then $\Delta \cap K \neq \emptyset$ and $e_{\mathcal{D}}(\Gamma(f, \Omega \cap L), \Gamma(g, \Delta \cap K)) < \eta$.

To estimate $e_{\mathcal{D}}(\Gamma(f,\Omega\cap K),\Gamma(f,\Omega\cap L))$ we show that for every $z\in(\Omega\cap K)\setminus L$, there is $v\in\Omega\cap L$ such that $d_X(z,v)<\epsilon$.
In fact $(\Omega\cap K)\setminus L\subset\Omega\cap K\cap S(\partial K\cap\Omega,\pi/2)$, thus there is $u\in S(\partial K\cap\Omega,\pi/2)$ with $d_X(u,z)<\pi/2<\epsilon/3$. There is $w_i\in\partial K\cap\Omega$ with $d_X(u,w_i)<\epsilon/3$. Now we have $d_X(z,z_i)\le d_X(z,u)+d_X(u,w_i)+d_X(w_i,z_i)<\epsilon$ and $z_i\in L\cap\Omega$.

Thus

$$\begin{aligned}e_{\mathcal{D}}(\Gamma(f,\Omega\cap K),\Gamma(g,\Delta\cap K))&\le e_{\mathcal{D}}(\Gamma(f,\Omega\cap K),\Gamma(f,\Omega\cap L))+\\&+e_{\mathcal{D}}(\Gamma(f,\Omega\cap L),\Gamma(g,\Delta\cap K))<\max\{\epsilon,\omega(\epsilon)\}+\eta.\end{aligned}$$

2. Second we show that there is $0<\delta<\epsilon$ such that if $\eta<\delta$ and $\rho_K(\Gamma(f,\Omega),\Gamma(g,\Delta))<\eta$, then $e_{\mathcal{D}}(\Gamma(g,\Delta\cap K),\Gamma(f,\Omega\cap K))<\eta+\max\{\epsilon,\omega(\epsilon)\}$.

If $\Omega\subset\Omega\cap K$, we are done.

Otherwise there is $0<\alpha<\gamma$ with $\Omega\setminus S(K\cap\Omega,\alpha/2)\neq\emptyset$. Put $\delta_0=\mathcal{D}_{d_X}(K,\Omega\setminus S(K\cap\Omega,\alpha/2))$ and put $\delta=\min\{\delta_0/2,\epsilon/2,\gamma/2\}$.

Let $\eta<\delta$ and $\rho_K(\Gamma(f,\Omega),\Gamma(g,\Delta))<\eta$. Then

$$\begin{aligned}e_{\mathcal{D}}(\Gamma(g,\Delta\cap K),\Gamma(f,\Omega\cap K))&\le e_{\mathcal{D}}(\Gamma(g,\Delta\cap K),\Gamma(f,\Omega\cap B(K,\eta)))+\\&+e_{\mathcal{D}}(\Gamma(f,\Omega\cap B(K,\eta)),\Gamma(f,\Omega\cap K))<\eta+\max\{\epsilon,\omega(\epsilon)\},\end{aligned}$$

since every $z\in\Omega\cap B(K,\eta)$ belongs to $S(\Omega\cap K,\alpha/2)$.
(If $z\notin S(\Omega\cap K,\alpha/2)$, $d_X(z,K)\ge\delta_0>\delta>\eta$ a contradiction). □

Theorem 5.6. *Let (X,d_X) and (Y,d_Y) be metric spaces. The following are equivalent:*

(1) *X is locally compact;*
(2) *τ-convergence and τ_H^c-convergence in G coincide.*

Proof: (2) ⇒ (1). Suppose X is not locally compact. There is a point $x\in X$ which has no compact nbd. Denote by $\mathcal{U}(x)$ the family of all open nbds of x. For every $U\in\mathcal{U}(x)$ and every $K\in\mathcal{K}(X)$ there is a point $x_{U,K}\in U\setminus K$. Consider the same directions on $\mathcal{U}(x)$ and $\mathcal{K}(X)$ as above and equip $\mathcal{U}(x)\times\mathcal{K}(X)$ with the natural product direction. For every $(U,K)\in\mathcal{U}(x)\times\mathcal{K}(X)$ put $\Omega_{U,K}=\{x_{U,K}\}$, and define $f_{U,K}$ on $\Omega_{U,K}$ by $f_{U,K}(x_{U,K})=1$. Let further $\Omega=\{x\}$ and f be the function defined on Ω by $f(x)=0$. It is easy to verify that the net $\{\Gamma(f_{U,K},\Omega_{U,K}):(U,K)\in\mathcal{U}(x)\times\mathcal{K}(X)\}$ fails to τ-converge to $\Gamma(f,\Omega)$, since $\rho_{\{x\}}(\Gamma(f_{U,K},\Omega_{U,K}),\Gamma(f,\Omega))=1$ eventually.

Now we show that $\{\Gamma(f_{U,K},\Omega_{U,K}):(U,K)\in\mathcal{U}(x)\times\mathcal{K}(X)\}$ τ_H^c-converges to $\Gamma(f,\Omega)$. Let $K\in\mathcal{K}(X)$ be such that $K\cap\Omega=\overline{K^\circ\cap\Omega}$. Then $K\cap\Omega=\emptyset$, otherwise $x\in K^\circ$. For every $C\ge K$ and $U\in\mathcal{U}(x)$ we have $K\cap\Omega_{U,C}=\emptyset$, thus $H_{\mathcal{D}}(\Gamma(f,\Omega\cap K),\Gamma(f_{U,C},\Omega_{U,C}\cap K))=0$ for every $C\ge K$ and $U\in\mathcal{U}(x)$.

(1) ⇒ (2). Suppose first that $\{\Gamma(f_\sigma,\Omega_\sigma):\sigma\in\Sigma\}$ τ_H^c-converges to $\Gamma(f,\Omega)$. To prove τ-convergence of $\{\Gamma(f_\sigma,\Omega_\sigma):\sigma\in\Sigma\}$ to $\Gamma(f,\Omega)$, let $K\in\mathcal{K}(X)$.

By Remark 3.2 there is $A \in \mathcal{K}(X)$ such that $A \supset K$ and $A \cap \Omega = \overline{A^\circ \cap \Omega} \neq \emptyset$. Then we have $\rho_K(\Gamma(f_\sigma, \Omega_\sigma), \Gamma(f, \Omega)) \le \rho_A(\Gamma(f_\sigma, \Omega_\sigma), \Gamma(f, \Omega)) \le H_{\mathcal{D}}(\Gamma(f_\sigma, \Omega_\sigma \cap A), \Gamma(f, \Omega \cap A))$.

Thus τ_H^c-convergence implies τ-convergence.

The equivalence of two convergences is an immediate consequence of Lemma 5.5. $\square$

If X is a closed connected subset of $\mathbb{R}$ and $Y = \mathbb{R}^m$, then (1) implies (2) from the above theorem was proved in [4] for a special subclass of $\mathcal{K}(X)$.

References

[1] Beer G., *Topologies on Closed and Closed Convex Sets*, Kluwer, 1993.

[2] Brandi P., Ceppitelli R., *Esistenza, unicitá e dipendenza continua per equazioni differenziali in una struttura ereditaria*, Atti Sem. Mat. Fis. Univ. Modena **35** (1987), 357–363.

[3] Brandi P., Ceppitelli R., *Existence, uniqueness and continuous dependence for hereditary differential equations*, J. Diff. Equations **81** (1989), 317–339.

[4] Brandi P., Ceppitelli R., *A new graph topology. Connections with compact open topology*, Appl. Analysis **53** (1994), 185–196.

[5] Brandi P., Ceppitelli R., *A new graph topology intended for functional differential equations*, Atti Sem. Mat. Univ. Modena **54** (1996), 43–52.

[6] Brandi P., Ceppitelli R., *A hypertopology intended for functional differential equations*, Appl. Analysis **67** (1997), 73–88.

[7] Brandi P., Ceppitelli R., Holá L'., *Topological properties of a new graph topology*, J. Convex Anal. **5** (1998), no. 2, 1–12.

[8] Ceppitelli R., Faina L., *Differential equations with hereditary structure induced by a Volterra type property*, preprint.

[9] Holá L'., *The Attouch-Wets topology and a characterization of normable linear spaces*, Bull. Austral. Math. Soc. **44** (1991), 11–18.

[10] Piccione P., Sampalmieri R., *Attouch-Wets convergence and Kuratowski convergence on compact sets*, Comment. Math. Univ. Carolinae **36** (1995), 551–562.

[11] Sampalmieri R., *Kuratowski convergence on compact sets*, Atti Sem. Mat. Fis. Univ. Modena **39** (1992), 381–390.

Primo Brandi, Rita Ceppitelli:
Department of Mathematics, University of Perugia, Via L. Vanvitelli, 1-06100 Perugia, Italy

L'ubica Holá:
Academy of Sciences, Institute of Mathematics, Štefánikova 49, 81 473 Bratislava, Slovakia

(*Received* October 27, 1997, *revised* June 5, 1998)