Two remarks on weaker connected topologies

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Abstract. It is shown that no generalized Luzin space condenses onto the unit interval and that the discrete sum of $\aleph_1$ copies of the Cantor set consistently does not condense onto a connected compact space. This answers two questions from [2].

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We say that a topological space $\langle X, \mathcal{T}_0\rangle$ *condenses onto a space* $\langle Y, \mathcal{T}_1\rangle$ if there exists a topology $\mathcal{T}_2$ on X such that $\mathcal{T}_2 \subseteq \mathcal{T}_0$, and the spaces $\langle X, \mathcal{T}_2\rangle$ and $\langle Y, \mathcal{T}_1\rangle$ are homeomorphic. In [2], Tkačenko, Tkachuk, Uspenskij, and Wilson showed that hereditarily separable Luzin spaces do not condense onto the unit interval, and asked whether there are any Luzin spaces that do (Problem 3.4). They also asked whether it is true in ZFC that the discrete sum of $\aleph_1$ copies of the Cantor set condenses onto a connected compact space (Problem 3.10). We answer both questions in the negative. We thank M. Tkačenko for commenting on an earlier version of this note and suggesting an improvement.

Theorem 1. *Assume that the real line cannot be covered by $\aleph_1$ nowhere dense sets. Then no discrete sum of $\aleph_1$ zero-dimensional compact Hausdorff spaces of which at least two are nonempty condenses onto a connected compact Hausdorff space.*

Proof: Let $X = \bigoplus_{\xi<\omega_1} C_\xi$, where each C_ξ is a zero-dimensional compact Hausdorff space. Let $\mathcal{T}_0$ denote the topology of the discrete sum on X, and suppose $\mathcal{T}_1$ is a topology on X such that $\mathcal{T}_1 \subset \mathcal{T}_0$, and $\langle X, \mathcal{T}_1\rangle$ is a connected compact Hausdorff space.

Assuming that the real line cannot be covered by $\aleph_1$ nowhere dense sets, one can show that no compact Hausdorff space is a union of $\aleph_1$ pairwise disjoint nowhere dense closed subsets (see [1]). From that we will derive a contradiction.

Lemma 2. *Each C_ξ is closed in $\mathcal{T}_1$. More generally, each closed* (*in $\mathcal{T}_0$*) *subset of C_ξ remains closed in $\mathcal{T}_1$.*

Proof: Since $\mathcal{T}_1 \subseteq \mathcal{T}_0$, the identity mapping $id : \langle X, \mathcal{T}_0\rangle \to \langle X, \mathcal{T}_1\rangle$ is continuous. Each closed (in $\mathcal{T}_0$) subset of a C_ξ is compact, hence its continuous image is compact in the Hausdorff space $\langle X, \mathcal{T}_1\rangle$, and thus is in particular closed. $\square$

Lemma 3. *Each C_ξ is nowhere dense in $\langle X, \mathcal{T}_1\rangle$.*

Proof: Suppose towards a contradiction that for some $\xi_0 \in \omega_1$ and $U \in \mathcal{T}_1 \backslash \{\emptyset\}$, the set $C_{\xi_0} \cap U$ is dense in U. Since C_{ξ_0} is closed, this implies $U \subset C_{\xi_0}$. Let $x \in U$. Since C_{ξ_0} is zero-dimensional, there exists $V \in \mathcal{T}_0$ with $x \in V = cl_{\mathcal{T}_0}(V) \subset U$. Because the restrictions of $\mathcal{T}_0$ and $\mathcal{T}_1$ to C_{ξ_0} coincide, Lemma 2 implies that V is nonempty, clopen in $\mathcal{T}_1$, and different from the whole space X. This contradicts connectedness of $\mathcal{T}_1$. □

Thus $\langle X, \mathcal{T}_1\rangle$ is the union of the pairwise disjoint nowhere dense closed subsets $\{C_\xi : \xi < \omega_1\}$, and we get a contradiction. □

Definition 4. *An uncountable space X without isolated points is Luzin if all its nowhere dense sets are countable. A space X is a generalized Luzin space if $|X| = 2^{\aleph_0}$, X has no isolated points, and every nowhere dense subset of X has cardinality $< 2^{\aleph_0}$.*

Theorem 5. *No generalized Luzin space can be condensed onto the unit segment.*

Proof: Let X be a generalized Luzin space. Recall that a space X satisfies the κ-c.c. if each family of pairwise disjoint nonempty open subsets of X has cardinality $< \kappa$.

Lemma 6. *The space X satisfies the $2^{\aleph_0}$-c.c.*

Proof: Let $\mathcal{U}$ be a family of pairwise disjoint nonempty open subsets of X. For each $U \in \mathcal{U}$, pick $x_U \in U$, and let $A = \{x_U : U \in \mathcal{U}\}$. Then A is a nowhere dense subset of X, and hence has cardinality $< 2^{\aleph_0}$. On the other hand, $|A| = |\mathcal{U}|$, which gives the desired result. □

Lemma 7. *If $f : X \to [0,1]$, f is 1-1, onto, and continuous, $Y \subset [0,1]$ with $|Y| = 2^{\aleph_0}$, and Y is closed in $[0,1]$, then $f^{-1}(Y)$ has nonempty interior.*

Proof: Follows immediately from the definition of a generalized Luzin space. □

Fact 8. *There exists a family $\mathcal{K} = \{K_\xi : \xi < 2^{\aleph_0}\}$ of pairwise disjoint closed subspaces of $[0,1]$ such that $|K_\xi| = 2^{\aleph_0}$ for each ξ.*

Proof: This follows immediately from the existence of a continuous mapping of $[0,1]$ onto $[0,1]^2$. □

Suppose $f : X \to [0,1]$ is such that f is continuous and bijective. By Fact 8, there exists $\mathcal{K} = \{K_\xi : \xi < 2^{\aleph_0}\}$, a collection of pairwise disjoint closed subspaces of $[0,1]$, such that $|K_\xi| = |2^{\aleph_0}|$ for each $\xi < 2^{\aleph_0}$. By Lemma 7, for every $K_\xi \in \mathcal{K}$, the inverse image $f^{-1}(K_\xi)$ has nonempty interior. Thus $\mathcal{U} = \{int(f^{-1}(K_\xi)) : \xi < 2^{\aleph_0}\}$ is a collection of pairwise disjoint nonempty open sets in X. But this implies that X does not satisfy the $2^{\aleph_0}$-c.c., contradicting Lemma 6. □

References

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