A fixed point theorem for non-self multi-maps in metric spaces

B.C. Dhage

Abstract. A fixed point theorem is proved for non-self multi-valued mappings in a metrically convex complete metric space satisfying a slightly stronger contraction condition than in Rhoades [3] and under a weaker boundary condition than in Itoh [2] and Rhoades [3].

Keywords: metrically convex metric space, multi-valued non-self map, fixed point

Classification: 47H10, 54H25

Let (X, d) be a metric space. Then X is said to be metrically convex if for every pair $x, y \in X$, $x \neq y$, there is a point $z \in X$ such that $d(x, y) = d(x, z) + d(z, y)$. We need the following lemma in the sequel.

Lemma 1 ([1]). *Let K be a non-empty and closed subset of a metrically convex metric space X. Then for any $x \in K$ and $y \notin K$, there exists a point $z \in \partial K$ such that $d(x, y) = d(x, z) + d(z, y)$, where ∂K denotes the boundary of K.*

Let $CB(X)$ denote the family of all non-empty, closed and bounded subsets of X. Denote for $A, B \in CB(X)$

$$D(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$$

and

$$\delta(A, B) = \sup\{d(a, b) \mid a \in A, b \in B\}.$$

Note that $D(A, B) \leq H(A, B) \leq \delta(A, B)$, where $H(A, B)$ denotes the Hausdorff distance of A and B.

In [2] Itoh proved a fixed point theorem for the non-self maps $F : K \to CB(X)$ satisfying certain contraction condition in terms of Hausdorff metric H on $CB(X)$ under the boundary condition $F(\partial K) \subset K$. Recently Rhoades [3] generalized this result to a wider class of non-self multi-maps on K. In this paper we prove a fixed point theorem for non-self multi-maps on K satisfying a slightly stronger contraction condition than that in Rhoades [3] and under a weaker boundary condition.

Theorem 1. *Let (X,d) be a metrically convex complete metric space and K a non-empty closed subset of X. Let $F : K \to CB(X)$ be a multi-map satisfying*

$$\delta(Fx,Fy) \le \alpha \max\{d(x,y), D(x,Fx), D(y,Fy)\} + \beta[D(x,Fy) + D(y,Fx)] \tag{1}$$

for all $x,y \in K$, where $\alpha \ge 0$, $\beta \ge 0$ satisfy

$$2\alpha + 3\beta < 1. \tag{2}$$

Further, if $Fx \cap K \ne \emptyset$ for each $x \in \partial K$, then F has a unique fixed point $p \in K$ such that $Fp = \{p\}$ and F is continuous at p in the Hausdorff metric on X.

Proof: Let $x \in K$ be arbitrary and consider a sequence $\{x_n\}$ in K as follows: Let $x_0 = x$ and take a point $x_1 \in Fx_0 \cap K$ if $Fx_0 \cap K \ne \emptyset$. Otherwise choose a point $x_1 \in \partial K$ such that

$$d(x_0, x_1') = d(x_0, x_1) + d(x_1, x_1')$$

for some $x_1' \in Fx_0 \subset X \setminus K$.
Similarly pick $x_2 \in Fx_1 \cap K$ if $Fx_1 \cap K \ne \emptyset$, otherwise choose a point $x_2 \in \partial K$ such that

$$d(x_1, x_2) + d(x_2, x_2') = d(x_1, x_2')$$

for some $x_2' \in Fx_1 \subset X \setminus K$.
Continuing in this way we have

$$x_{n+1} \in Fx_n \cap K \quad \text{if} \quad Fx_n \cap K \ne \emptyset,$$

or $x_{n+1} \in \partial K$ satisfying

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+1}') = d(x_n, x_{n+1}')$$

for some $x_{n+1}' \in Fx_n \subset X \setminus K$.
By the construction of $\{x_n\}$, we can write

$$\{x_n\} = P \cup Q \subset K,$$

where

$$P = \{x_n \in \{x_n\} : x_n \in Fx_{n-1}\}$$

and

$$Q = \{x_n \in \{x_n\} : x_n \in \partial K, x_n \notin Fx_{n-1}\}.$$

Then for any two consecutive terms x_n, x_{n+1} of the sequence $\{x_n\}$, we observe that there are only the following three possibilities:

(i) $x_n, x_{n+1} \in P$,
(ii) $x_n \in P$, $x_{n+1} \in Q$, and
(iii) $x_n \in Q$ and $x_{n+1} \in P$.

First we show that $\{x_n\}$ is a Cauchy sequence in K. Now for any $x_n, x_{n+1} \in \{x_n\}$, we have the following estimates:

Case I. Suppose that $x_n, x_{n+1} \in P$, then we have

$$\begin{aligned} d(x_n, x_{n+1}) &\le \delta(Fx_{n-1}, Fx_n) \\ &\le \alpha \max\{d(x_{n-1}, x_n), D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\} \\ &\quad + \beta[D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})] \\ &\le \alpha \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &\quad + \beta[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ &= \alpha \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + \beta d(x_{n-1}, x_{n+1}) \\ &\le \alpha \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &\quad + \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &= \max\{(\alpha+\beta) d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}), \\ &\qquad (\alpha+\beta) d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n)\} \end{aligned}$$

and hence

$$d(x_n, x_{n+1}) \le k d(x_{n-1}, x_n),$$

where $k = \max\{\frac{\alpha+\beta}{1-\beta}, \frac{\beta}{1-(\alpha+\beta)}\} < 1$, since $2\alpha + 3\beta < 1$.

Case II. Let $x_n \in P$ and $x_{n+1} \in Q$. Then

$$d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1})$$

for some $x'_{n+1} \in Fx_n$. Clearly,

$$\begin{cases} d(x_n, x_{n+1}) \le d(x_n, x'_{n+1}) \\ d(x_n, x'_{n+1}) \le \delta(Fx_{n-1}, Fx_n). \end{cases} \tag{3}$$

Now following arguments similar to those in Case I, we obtain

$$d(x_n, x'_{n+1}) \le k d(x_{n-1}, x_n), \tag{4}$$

where again $k = \max\{\frac{\alpha+\beta}{1-\beta}, \frac{\beta}{1-(\alpha+\beta)}\} < 1$.

From (3) and (4) it follows that

$$d(x_n, x_{n+1}) \le k d(x_{n-1}, x_n). \tag{5}$$

Case III. Suppose that $x_n \in Q$ and $x_{n+1} \in P$. Note that then $x_{n-1} \in P$ and there is a point $x'_n \in Fx_{n-1}$ such that

$$d(x_{n-1}, x_n) + d(x_n, x'_n) = d(x_{n-1}, x'_n). \tag{6}$$

Now,

$$\begin{aligned} d(x_n, x_{n+1}) &\le d(x_n, x'_n) + d(x'_n, x_{n+1}) \\ &\le d(x_n, x'_n) + \delta(Fx_{n-1}, Fx_n) \\ &\le d(x_n, x'_n) + \alpha \max\{d(x_{n-1}, x_n), D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\} \\ &\quad + \beta[D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})] \\ &\le d(x_n, x'_n) + \alpha \max\{d(x_{n-1}, x_n), d(x_{n-1}, x'_n), d(x_n, x_{n+1})\} \\ &\quad + \beta[d(x_{n-1}, x_{n+1}) + d(x_n, x'_n)] \\ &\le d(x_n, x'_n) + \alpha \max\{d(x_{n-1}, x'_n), d(x_n, x_{n+1})\} \\ &\quad + \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, x'_n)] \\ &= d(x_n, x'_n) + \alpha \max\{d(x_{n-1}, x'_n), d(x_n, x_{n+1})\} \\ &\quad + \beta[d(x_{n-1}, x'_n) + d(x_n, x_{n+1})] \\ &\le d(x_{n-1}, x'_n) + \alpha \max\{d(x_{n-1}, x'_n), d(x_n, x_{n+1})\} \\ &\quad + \beta[d(x_{n-1}, x'_n) + d(x_n, x_{n+1})]. \end{aligned}$$

From (4) of Case II applied to $n-1$, we have $d(x_{n-1}, x'_n) \le kd(x_{n-2}, x_{n-1})$ and hence

$$\begin{aligned} d(x_n, x_{n+1}) &\le kd(x_{n-2}, x_{n-1}) + \max\{kd(x_{n-2}, x_{n-1}), d(x_n, x_{n+1})\} \\ &\quad + \beta[kd(x_{n-2}, x_{n+1}) + k(x_n, x_{n+1})] \\ &= \max\{(1+\alpha+\beta)kd(x_{n-2}, x_{n-1}) + \beta d(x_n, x_{n+1}), \\ &\qquad (1+\beta)kd(x_{n-2}, x_{n-1}) + (\alpha+\beta)d(x_n, x_{n+1})\}. \end{aligned}$$

This implies

$$\begin{aligned} d(x_n, x_{n+1}) &\le \max\{(1+\alpha+\beta)k/(1-\beta), (1+\beta)k/[1-(\alpha+\beta)]\}d(x_{n-2}, x_{n-1}) \\ &= qd(x_{n-2}, x_{n-1}), \end{aligned}$$

where

$$\begin{aligned} q &= \max\{(1+\alpha+\beta)k/(1-\beta), (1+\beta)k/[1-(\alpha+\beta)]\} \\ &= k \max\{(1+\alpha+\beta)/(1-\beta), (1+\beta)/[1-(\alpha+\beta)]\} = k(1+\beta)/[1-(\alpha+\beta)] \\ &= (1+\beta)/[1-(\alpha+\beta)] \max\{(\alpha+\beta)/(1-\beta), \beta/[1-(\alpha+\beta)]\} \\ &= \max\{(1+\beta)(\alpha+\beta)/[(1-\beta)(1-(\alpha+\beta))], \beta(1+\beta)/[1-(\alpha+\beta)]^2\} \\ &< 1. \end{aligned}$$

To see this, the inequality (2) yields

$$\begin{aligned} &\alpha+\beta < 1-2\beta-\alpha \\ &\Rightarrow \alpha+\beta+\alpha\beta+\beta^2 < 1-2\beta-\alpha+\alpha\beta+\beta^2 \\ &\Rightarrow (\alpha+\beta+\alpha\beta+\beta^2)/(1-2\beta-\alpha+\alpha\beta+\beta^2) < 1 \\ &\Rightarrow (1+\beta)(\alpha+\beta)/[(1-\beta)(1-\alpha-\beta)]\} < 1. \end{aligned}$$

Similarly again from (2) we have

$$\begin{aligned}
&2\alpha + 3\beta < \alpha^2 + 2\alpha\beta + 1 \\
&\Rightarrow \beta + \beta^2 < 1 - 2\alpha - 2\beta + \alpha^2 + 2\alpha\beta + \beta^2 \\
&\Rightarrow \beta(1+\beta) < 1 - 2(\alpha+\beta) + (\alpha+\beta)^2 \\
&\Rightarrow \beta(1+\beta) < [1-(\alpha+\beta)]^2 \\
&\Rightarrow \beta(1+\beta)/[1-(\alpha+\beta)]^2 < 1.
\end{aligned}$$

Now for any $n \in N$, we have

$$d(x_{2n}, x_{2n+1}) \le q d(x_{2n-2}, x_{2n-1}) \le q^n d(x_0, x_1). \tag{7}$$

Since n is arbitrary, one has

$$d(x_n, x_{n+1}) \le q^n d(x_0, x_1). \tag{8}$$

Then from Cases I–III, it easily follows that $\{x_n\}$ is a Cauchy sequence in K. As K is closed it is complete and hence $\lim_n x_n = p$ exists. We show that p is a fixed point of F. Without loss of generality we may assume that $x_{n+1} \in Fx_n$ for some $n \in N$. Then

$$\begin{aligned}
D(p, Fp) &= \lim_n D(x_{n+1}, Fp) \\
&\le \lim_n \delta(Fx_n, Fp) \\
&\le \lim_n \max\{d(x_n,p), D(x_n, Fx_n), D(p, Fp)\} \\
&\quad + \beta \lim_n [D(x_n, Fp) + D(p, Fx_n)] \\
&= \alpha \lim_n \max\{d(x_n,p), d(x_n, x_{n+1}), D(p, Fp)\} \\
&\quad + \beta \lim_n [D(x_n, Fp) + d(p, x_{n+1})] \\
&= (\alpha+\beta) D(p, Fp)
\end{aligned}$$

which is possible only when $p \in Fp$.

Further, we have

$$\begin{aligned}
\delta(p, Fp) &\le \delta(Fp, Fp) \\
&\le \alpha \max\{d(p,p), D(p,Fp), D(p,Fp)\} + \beta[\delta(p,Fp) + D(p,Fp)] \\
&= \beta\delta(p, Fp)
\end{aligned}$$

and hence $Fp = \{p\}$.

To show the uniqueness of p, let q $(\ne p)$ be another fixed point of F. Then

$$\begin{aligned}
d(p,q) &\le \delta(Fp, Fq) \\
&\le \alpha \max\{d(p,q), D(p,Fp), D(q,Fq)\} + \beta[D(p,Fq) + D(q,Fp)] \\
&= (\alpha + 2\beta) d(p,q).
\end{aligned}$$

This is a contradiction since $\alpha + 2\beta < 1$ and hence $p = q$.

Finally we prove the continuity of F at p. Let $\{z_n\} \subset X$ by any sequence such that $z_n \to p$ as $n \to \infty$. Now

$$\begin{aligned}
\lim_n H(Fz_n, F) \leq\ & \lim_n \delta(Fz_n, Fp) \\
& \leq \alpha \lim_n \max\{d(z_n, p), D(z_n, Fz_n), D(p, Fp)\} \\
& \quad + \beta \lim_n [D(z_n, Fp) + D(p, Fz_n)] \\
& \leq \alpha \lim_n \max\{d(z_n, p), D(z_n, Fz_n)\} \\
& \quad + \beta \lim_n [d(z_n, p) + D(p, Fz_n)] \\
& = (\alpha + \beta) H(Fz_n, Fp)
\end{aligned}$$

where $\alpha + \beta < 1$. Therefore $\lim_n H(Fz_n, Fp) = 0$, showing that F is continuous at p. This completes the proof. □

The following fixed point theorem for non-self multi-maps on a complete convex metric space satisfying a slightly weaker contraction condition and under a stronger boundary condition than ours has been proved by Rhoades [3].

Theorem 2 ([3]). *Let (X, d) be a metrically convex metric space and K a non-empty closed subset of X.*

Let $F : K \to CB(X)$ satisfy

$$H(Fx, Fy) \leq \alpha d(x, y) + \beta \max\{D(x, Fx), D(y, Fy)\} + \gamma[D(x, Fy) + D(y, Fx)] \tag{9}$$

for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ such that

$$\left(\frac{1 + \alpha + \gamma}{1 - \beta - \gamma}\right) \left(\frac{\alpha + \beta + \gamma}{1 - \gamma}\right) < 1. \tag{10}$$

Further if $Fx \subset K$ for each $x \in \partial K$, then there exists a $p \in K$ such that $p \in Fp$ and F is upper semi-continuous at p.

Proof: The existence of such a fixed point $p \in K$ follows from Theorem 1 of Rhoades [3]. We only show the upper semi-continuity of F at p.

Let $\{z_n\} \subset K$ be any sequence such that $z_n \to p$ as $n \to \infty$.

Let $\{y_n\}$ be a sequence in K such that $y_n \in Fx_n$ for each $n \in N$ and $y_n \to q$. To finish, we shall prove that $q \in Fp$. Now

$$\begin{aligned}
d(q, p) =\ & \lim_n d(y_n, p) \leq \lim_n H(Fz_n, Fp) \\
=\ & \lim_n d(z_n, p) + \beta \lim_n \max\{D(z_n, Fz_n), D(p, Fp)\} \\
& + \gamma \lim_n [D(z_n, Fp) + D(p, Fz_n)] \\
=\ & \beta \lim_n \max\{d(z_n, y_n), 0\} + \gamma \lim_n d(p, y_n) \\
=\ & \beta d(p, q) + \gamma d(p, q) = (\beta + \gamma) d(p, q)
\end{aligned}$$

which is possible only when $d(q,p)=0$ as $\beta+\gamma<1$. Hence $q \in Fp$ and the proof ot the theorem is complete. □

Next we prove two fixed point theorems for multi-maps on a metric space satisfying a contractive condition more general than (1) and under certain compactness type conditions.

Theorem 3. *Let (X,d) be a complete metrically convex metric space and K a non-empty compact subset of X. Suppose that $F : K \to CB(X)$ is a continuous multi-map satisfying*

(11) $\delta(Fx,Fy) < \alpha \max\{d(x,y), D(x,Fx), D(y,Fy)\} + \beta[D(x,Fy)+D(y,Fx)]$

for all $x,y \in K$, $x \notin Fx$, $y \notin Fy$, where $\alpha,\beta > 0$ satisfy $2\alpha+3\beta \le 1$. If $Fx \cap K \neq \emptyset$ for each $x \in \partial K$ then the multi-map F has a unique fixed point.

Proof: First we note that if the multi-map F has a fixed point then from condition (11) it follows that the fixed point is unique.

Since K is compact, both sides of the inequality (11) are bounded on K. Now there are two possibilities:

Case I. Suppose that the right hand side of (11) is zero for some $(x,y) \in K \times K$, then we have $x = y \in Fy$. Thus F has a fixed point and so it is unique.

Case II. Suppose that the right hand side of (11) is positive for all $x,y \in K$. Denote for brevity

$$M(x,y) = \alpha \max\{d(x,y), D(x,Fx), D(y,Fy)\} + \beta[D(x,Fy)+D(y,Fx)].$$

Now in the case when $2\alpha+3\beta<1$, the conclusion of Theorem 3 follows from Theorem 1. Therefore we treat only the case when $2\alpha+3\beta=1$.

Define a function $T : K^2 \to \mathbb{R}^+$ by

$$T(x,y) = \frac{\delta(Fx,y)}{M(x,y)}\,. \tag{12}$$

Clearly the function T is well defined since $M(x,y) \neq 0$ for all $x,y \in K$.

Since F, D and δ are continuous, T is continuous and from the compactness of K it follows that there is a point $(u,v) \in K^2$ such that T attains its maximum at this point. Call the value c. From (11) we get $0<c<1$. By the definition of T, we obtain

$$\begin{aligned}\delta(Fx,Fy) &\le cM(x,y)\\ &= \alpha' \max\{d(x,y), D(x,Fx), D(y,Fy)\} + \beta'[D(x,Fy)+D(y,Fx)]\end{aligned}$$

for all $x,y \in K$, where $2\alpha'+3\beta' = c(2\alpha+3\beta) < 1$. As K is compact, it is closed and so the desired conclusion follows by an application of Theorem 1. The proof is complete. □

Theorem 4. *Let (X, d) be a complete metrically convex metric space and K a compact subset of X. Suppose that $F : K \to CB(X)$ is a continuous multi-map satisfying*

$$(13) \quad H(Fx, Fy) < \alpha d(x, y) + \beta \max\{D(x, Fx), D(y, Fy)\} + \gamma[D(x, Fy) + D(y, Fx)]$$

for all $x, y \in X$, $x \notin Fx$, $y \notin Fy$, where $\alpha, \beta, \gamma > 0$ satisfy $\left(\frac{1+\alpha+\gamma}{1-\beta-\gamma}\right)\left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right) \leq 1$. If $Fx \subset K$ for each $x \in \partial K$ then the multi-map F has a fixed point.

Proof: The proof is similar to Theorem 3 and now the desired conclusion follows by an application of Theorem 2. □

Acknowledgment. The author expresses sincere thanks to the learned referee for his careful reading and suggesting some improvement upon the original version of this paper. He is also thankful to Prof. B.E. Rhoades for providing the reprints of his papers.

References

[1] Blumenthal L.M., *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford, 1943.
[2] Itoh S., *Multi-valued generalized contraction and fixed point theorems*, Comment. Math. Univ. Carolinae **18** (1977), 247–248.
[3] Rhoades B.E., *A fixed point theorem for a multi-valued non-self mappings*, Comment. Math. Univ. Carolinae **37** (1996), 401–404.

Mathematics Research Centre, Mahatma Gandhi Mahavidyalaya, Ahmedpur 413 515 (Maharashtra), India

(*Received* June 9, 1997, *revised* September 4, 1998)