An independency result in connectification theory

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Abstract. A space is called connectifiable if it can be densely embedded in a connected Hausdorff space.

Let ψ be the following statement: "a perfect T_3-space X with no more than $2^{\mathfrak{c}}$ clopen subsets is connectifiable if and only if no proper nonempty clopen subset of X is feebly compact".

In this note we show that neither ψ nor $\neg\psi$ is provable in ZFC.

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The problem of finding those spaces which can be densely embedded in a connected Hausdorff space has been extensively studied in the last years and many results have been obtained (see, e.g., [1], [2], [6], [10] and [13]).

Despite all the efforts, a characterization of connectifiable spaces is still unknown.

In this note we present a characterization of connectifiable perfect T_3-spaces with no more than $2^{\mathfrak{c}}$ clopen subsets, which can be neither proved nor disproved in ZFC.

We recall that a space X is called:

(i) perfect if every closed subset of X is a G_δ-set;

(ii) H-closed if every open cover of X has a finite subfamily whose union is dense, or equivalently, X is a closed subspace of every Hausdorff space in which it is contained;

(iii) feebly compact if every countable open cover of X has a finite subfamily whose union is dense.

As usual, p will stand for the smallest cardinality of a maximal subfamily of $[\omega]^\omega$ with the strong finite intersection property (see, e.g., [4] and [12]).

Regarding connectifiability observe that

(1) A connectifiable space contains no proper nonempty open H-closed subset ([13]).

(2) Let X be a Hausdorff space with no more than $2^{\mathfrak{c}}$ clopen subsets. If every proper nonempty clopen subsets of X is not feebly compact, then X is connectifiable ([10]).

(3) There exists, in ZFC, a nonconnectifiable Hausdorff space of cardinality $\mathfrak{c}$ with no proper nonempty H-closed subspace ([10]).

(4) It is consistent with ZFC that there is a nonconnectifiable normal Hausdorff space of cardinality $\mathfrak{c}$ which has no proper nonempty H-closed subspace ([10]).

In our result we will make use of the following set-theoretic statements (which are consistent with ZFC):

(a) $p > \omega_1$;

(b) (Jensen's Combinatorial Principle $\diamondsuit$) There are sets $A_\alpha \subset \alpha$ for $\alpha < \omega_1$ such that for every $A \subset \omega_1$ the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary.

We refer the reader to [5] for topological terminology. For set-theoretic terminology see [8] and [4].

Theorem. *The following statement:*

> *"a perfect T_3-space X with no more than $2^{\mathfrak{c}}$ clopen subsets X is connectifiable if and only if no proper nonempty clopen subset of X is feebly compact"*

is independent of ZFC.

PROOF: First let us show that, under $p > \omega_1$, the above statement is true.

If X is not connectifiable then, by one of the above mentioned result, there is a proper nonempty feebly compact clopen subset A of X.

Now let us suppose that there is a proper nonempty feebly compact clopen subset A of X. Since X is T_3 and perfect, it follows that A is a countably compact perfect T_3-space. So by a theorem of Weiss (here we use $p > \omega_1$) A is compact ([14], see also [12]).

Hence X is not connectifiable. Therefore the statement is consistent with ZFC.

Now let us prove the independency by showing that, under the Jensen's principle $\diamondsuit$, there exists a connectifiable T_6-space with exactly two proper nonempty clopen subsets, each of which is feebly compact.

Let S be the Ostaszewskii's space, this space is, under $\diamondsuit$, an example of a noncompact countably compact perfectly normal space ([9]).

Let Z be the cone over S, i.e., let Z be the quotient of $S \times I$ obtained by identifying $S \times \{1\}$ with a point.

Now Z is noncompact ($S \times \{0\}$ is a noncompact space homeomorphic to a closed subspace of Z), countably compact and perfectly normal (Z is the continuous closed image of the countably compact perfectly normal space $S \times I$ under the natural mapping).

Now let $X = Z \oplus Z$, X is a perfectly normal space. The only proper nonempty clopen subsets of X are the two copies of Z, which are countably compact (= feebly compact) but not compact.

Since a Hausdorff space with open components is connectifiable if and only if it has no proper nonempty open H-closed subspace ([7]), it follows that X is connectifiable. $\square$

Example. It is worth noting that there is a ZFC example of a connectifiable perfect Hausdorff space, with no more than $2^{\mathfrak{c}}$ clopen subsets, which has proper nonempty feebly compact clopen subsets.

In fact let $\mathcal{F}$ be the set of all free ultrafilters on ω and let Y be $\omega \cup \mathcal{F}$ endowed with the topology generated by the points of ω and all sets of the form $G \cup \{p\}$ where $G \in p \in \mathcal{F}$. Now fix $p \in \mathcal{F}$ and let X be the subspace $Y \setminus \{p\}$ of Y.

X is a Hausdorff space which is not H-closed (it is not closed in Y).

Now let us show that X is feebly compact. By a result in [3] it is enough to show that every locally finite system of pairwise disjoint nonempty open subsets of X is finite.

Suppose that $\mathcal{A} = \{A_n : n \in \omega\}$ is an infinite locally finite family of pairwise disjoint nonempty open subsets of X. Without loss of generality we may assume that, for every n, $A_n = \{\kappa_n\}$ for some $\kappa_n \in \omega$.

Let q be a free open ultrafilter on ω such that $q \neq p$ and $\{\kappa_n : n \in \omega\} \in q$. Since every neighbourhood of q in X meets infinitely many members of $\mathcal{A}$, we reach a contradiction. Therefore X is feebly compact.

Moreover X is perfect. In fact, every open subset A of X, is the union of the F_σ-set $\omega \cap A$ and the closed set $A \setminus \omega$ ($A \setminus \omega$ is a subset of the closed discrete subspace $X \setminus \omega$ of X).

Now let C be the cone over X and set $Z = C \oplus C$.

Z is a perfect Hausdorff space, and the only two proper nonempty clopen subsets of Z (namely the copies of X) are feebly compact.

Nonetheless Z has open components and no proper nonempty H-closed subspaces, therefore Z is connectifiable.

Remarks. (i) If L is the long line, then $X = L \oplus L$ is a ZFC example of a connectifiable hereditarily normal space of cardinality $\mathfrak{c}$ which has proper nonempty feebly compact clopen subsets.

(ii) In [10] it is shown that, under $MA+\neg CH$, a disconnected perfectly normal space with no more than $2^{\mathfrak{c}}$ clopen subsets is connectifiable if and only if no nonempty clopen subset is relatively pseudocompact.

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