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On the solvability of commutative loops and their multiplication groups

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Abstract. We investigate the situation when the inner mapping group of a commutative loop is of order $2p$, where $p = 4t + 3$ is a prime number, and we show that then the loop is solvable.

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Introduction

A groupoid Q is called a loop if Q has a unique division and a neutral element (thus loops are nonassociative versions of groups). The mappings $L_a(x) = ax$ and $R_a(x) = xa$ are permutations on Q for every $a \in Q$ and the permutation group $M(Q) = \langle L_a, R_a : a \in Q\rangle$ is called the multiplication group of Q. The stabilizer of the neutral element is denoted by $I(Q)$ and we say that $I(Q)$ is the inner mapping group of Q. This link between loop theory and group theory was established by Bruck [1] and he was the first to investigate the properties of loops by using the corresponding multiplication groups. One of those properties that we are interested in, is the solvability of loops. Recall that a loop Q is solvable if it has a series $Q = Q_0 \geq \cdots \geq Q_n = 1$, where Q_i is normal in Q_{i-1} and Q_{i-1}/Q_i is an abelian group.

What then is the relation between a solvable loop Q and its multiplication group $M(Q)$? In 1996 Vesanen [11] showed that the solvability of $M(Q)$ implies the solvability of Q if Q is a finite loop. After this it is quite natural to be interested in those properties of $I(Q)$ which imply the solvability of $M(Q)$. We have been able to show for a finite loop Q that if $I(Q)$ is abelian or if $I(Q)$ is a dihedral 2-group, then $M(Q)$ is solvable (see [9] and [6]). In [5] we managed to show that $M(Q)$ is solvable if $|I(Q)| = 6$ (thus we covered the smallest nonabelian case). The more general problem, where $|I(Q)| = pq$ (here p and q are prime numbers), was investigated in [7]. By using the classification of finite simple groups we were able to show that $M(Q)$ is solvable provided that $q = 2$ and $p \leq 61$, $q = 3$ and $p \leq 31$, $q = 5$ and $p \leq 11$.

In this short note we consider the case when Q is a finite commutative loop and $|I(Q)| = 2p$, where $p = 4t + 3$ is an odd prime. It follows that $M(Q)$ is a solvable group. Our method — in fact, quite elementary permutation group theory —

cannot be generalized to the noncommutative case or to the case $p = 4t + 1$. We assume that a more complicated machinery is needed here.

Many properties of loops can be reduced to the properties of connected transversals in the multiplication group and thus our problem can be posed entirely in group theoretic terms. We give basic information about connected transversals together with other preliminary lemmas in Section 1. Section 2 contains our main theorem and the corresponding solvability criteria for finite commutative loops.

1. H-connected transversals

Let G be a group, $H \leq G$ and A and B be two left transversals to H in G. We say that the two transversals A and B are H-connected if the commutator subgroup $[A, B]$ is contained in H. If $A = B$, then we say that A is an H-selfconnected transversal in G. By $L_G(H)$ we denote the core of H in G. If Q is a loop, then it is easy to see that $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$ are $I(Q)$-connected transversals in $M(Q)$. Clearly, the core of $I(Q)$ in $M(Q)$ is trivial. The connection between multiplication groups of commutative loops and connected transversals is given by

Lemma 1.1. *A group G is isomorphic to the multiplication group of a commutative loop if and only if there exist a subgroup H satisfying $L_G(H) = 1$ and an H-selfconnected transversal A satisfying $G = \langle A \rangle$.*

For the proof, see [8, Corollary 4.2].

In the following lemmas we assume that G is a group, $H \leq G$ and A is an H-selfconnected transversal in G.

Lemma 1.2. *Now A is a left and right transversal to H in G. If $L_G(H) = 1$, then $1 \in A$.*

Lemma 1.3. *If $L_G(H) = 1$, then $N_G(H) = H \times Z(G)$.*

Lemma 1.4. *If $C \subseteq A$ and $K = \langle H, C \rangle$, then $C \subseteq L_G(K)$.*

Lemma 1.5. *If H is cyclic, then G is solvable.*

For the proofs, see [8, p. 113]; [8, Proposition 2.7]; [8, Lemma 2.5 and 6, Lemma 2.6].

We still need three general group theoretic results for the proof of our main theorem.

Lemma 1.6. *Let G be a finite group and M an abelian subgroup of G. If M is a maximal subgroup of G, then G is solvable.*

For the proof, see [3, Theorem 1].

Lemma 1.7. *Let G be a permutation group on a set X. Furthermore, let $fix_X(g) = \{i \in X \mid g(i) = i\}$, where $g \in G$. Then the number of orbits of G on X is $\frac{1}{|G|} \sum_{g \in G} |fix_X(g)|$.*

For the proof, see [4, Theorem 9.1].

Lemma 1.8. *Let G be a finite group and let Q be an abelian Sylow subgroup contained in the center of its normalizer, or $Q \leq Z(N_G(Q))$. Then Q has a normal complement K.*

For the proof, see [2, Theorem 7.4.3].

2. Main theorem

Now we shall investigate the following situation: G is a group, $H \leq G$ and A is an H-selfconnected transversal in G. We also assume that $|H| = 2p$, where p is an odd prime number. We shall next introduce our main theorem.

Theorem 2.1. *If $H \leq G$ and $|H| = 2p$, where $p = 4t + 3$ is a prime number, then G is solvable.*

Proof: We first assume that G is finite. By Lemma 1.5, we may assume that H is not abelian. Assume now that G is a minimal counterexample.

If $L_G(H) > 1$, then the group $H/L_G(H)$ is cyclic. It is easy to see that the transversal $AL_G(H)/L_G(H)$ is $H/L_G(H)$-selfconnected in $G/L_G(H)$ and hence $G/L_G(H)$ is solvable by Lemma 1.5. Now H is solvable and so $L_G(H)$ is solvable. Hence G is also solvable and we must assume that $L_G(H) = 1$. Furthermore, $1 \in A$ by Lemma 1.2.

We may continue as in the proof of Theorem 3.1 of [7] and we conclude that H is a maximal subgroup in G and G is a simple group. We denote by P the Sylow p-subgroup of H and by Q a Sylow 2-subgroup of H.

Now P is a Sylow p-subgroup of G and $N_G(P) = H$, so $[G : H] = 1 + kp$. If $1 + kp$ is an odd number, then Q is a Sylow 2-subgroup of G and by Lemma 1.8 Q has a normal complement. Because the group G is simple we assume that k is an odd number. We can consider G as a permutation group acting on the set with $1 + kp$ points, and H is a one point stabilizer. Now H stabilizes one point and in the action of H on the remaining kp points the orbits have length p or $2p$. Assume that all orbits have length p. If we consider one orbit it is clear that H acts transitively on the p points. By Lemma 1.7 every involution of H must fix one and only one point in the orbit. Thus, if $y \in H$ is an involution, y is a product of $k(p-1)/2 = k(2t+1)$ distinct transpositions. But then y is an odd permutation, which is a contradiction, since G contains only even permutations. Thus we can assume that there exists at least one orbit of length $2p$. Therefore we have $1 \neq d \in A$ such that $H \cap H^d = 1$. Since A is both left and right transversal we get that also A^{-1} is both left and right transversal and so we have $a \in A$ such that $H \cap H^{a^{-1}} = 1$.

If $a, b \in A$ and $H \cap H^{a^{-1}} = 1$, then we have a unique $g(a,b) \in A$ such that $abH = g(a,b)H$. If we write $h(a,b) = g(a,b)^{-1}ab$, then $h(a,b) \in H$. In addition, since A is H-selfconnected in G, we conclude that $h(a,b)aH = g(a,b)^{-1}abaH = g(a,b)^{-1}aabH = g(a,b)^{-1}ag(a,b)H = aH$. Thus $h(a,b)aH = aH$ and $h(a,b) \in H^{a^{-1}}$, hence $h(a,b) \in H \cap H^{a^{-1}} = 1$ and $g(a,b) = ab \in A$. This means that $ab \in A$ whenever $b \in A$. Thus $a^2 \in A$, hence $[a^2, b] = a^{-2}b^{-1}a^2b = a^{-1}[a,b]b^{-1}ab \in H$.

Then $H = a^{-1}[a,b]b^{-1}abH = a^{-1}[a,b]b^{-1}baH = a^{-1}[a,b]aH$ and so $[a,b] \in H^{a^{-1}}$. From this it follows that $[a,b] \in H \cap H^{a^{-1}} = 1$ or $ab = ba$ for each $b \in A$.

If $G = \langle A \rangle$, then $a \in Z(G)$, a contradiction because G is simple. With the same argument the case $[G : \langle A \rangle] = 2$ is not possible. If $[G : \langle A \rangle] = p$, then we can consider G as a permutation group on the set with p points. It is clear that involutions from H are products of $(p-1)/2$ transpositions, which is not possible. Thus we are left with the case that $[G : \langle A \rangle] = 2p$. But then $\langle A \rangle = A$ and since $[A, A] \le A \cap H = 1$, it follows that A is an abelian group. Thus $G = AH$, where A is an abelian group. Let M be a maximal subgroup of G which contains A. As before, $[G : M] = 2$ and $[G : M] = p$ are not possible. If $[G : M] = 2p$, then $M = A$ and G has an abelian maximal subgroup, hence G is solvable by Lemma 1.6, a contradiction. This means that G is solvable in the case that G is a finite group.

Next we prove that our theorem also holds when G is infinite. If $L_G(H) > 1$, then $H/L_G(H)$ is cyclic and thus $G/L_G(H)$ is solvable by Lemma 1.5 and hence G is also solvable. From now on we assume that $L_G(H) = 1$.

First assume that $G = \langle A \rangle$. Let a be a fixed element of A and h a fixed element of H and write $F(a, h) = \{b \in A : a^{-1}b^{-1}ab = h\}$. If b and c are elements of $F(a, h)$, then $bc^{-1} \in C_G(a)$ and $b \in C_G(a)c$. Thus $F(a, h) \subseteq C_G(a)b_h$, where b_h is a fixed element from $F(a, h)$, and $A = \bigcup F(a, h)$, where h goes through all the elements of H. Now $G = AH \subseteq C_G(a)\{b_h : h \in H\}H$ and thus $[G : C_G(a)] \le |H|^2$. It follows that $[G : C_G(H)]$ is finite, whence $[G : N_G(H)]$ is also finite. By Lemma 1.3 $N_G(H) = H \times Z(G)$ and thus $[G : Z(G)]$ is finite. Clearly $HZ(G)/Z(G)$ is of order $2p$ and the set $AZ(G)/Z(G)$ contains a subset which is an $HZ(G)/Z(G)$-selfconnected transversal in $G/Z(G)$. Hence we conclude by the first part of our proof that $G/Z(G)$ and G are solvable.

Then let $K = \langle A \rangle$ be a proper subgroup of G. Hence A is $K \cap H$-selfconnected in K and thus K is a solvable group. Now $[G : K]$ is finite and we have a normal subgroup $L_G(K) \le K$ such that $[G : L_G(K)]$ is finite. Since $HL_G(K)/L_G(K)$ is cyclic or of order $2p$, it follows that $G/L_G(K)$ is solvable. Now $L_G(K)$ is solvable and therefore G is solvable. The proof is complete. □

From Lemma 1.1 and Theorem 2.1 we conclude that if Q is a commutative loop and $|I(Q)| = 2p$, where $p = 4t+3$ is a prime number, then $M(Q)$ is solvable. The following result of Vesanen [11] is of fundamental importance.

Theorem 2.2. *Let Q be a finite loop. If $M(Q)$ is solvable, then Q is solvable.*

By combining all these results we get

Theorem 2.3. *If Q is a finite commutative loop such that $|I(Q)| = 2p$, where $p = 4t + 3$ is a prime number, then Q is a solvable loop.*

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