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Representing lattices by homotopy groups of graphs

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Abstract. In this paper we represent every lattice by subgroups of free groups using the concept of the homotopy group of a graph.

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In this paper we present a method how to represent a given lattice L as a sublattice of the subgroup lattice of a free group. The method is based on the idea of the homotopy group of a graph. Our construction is such that if the lattice L is finite then the free group and all its subgroups representing the elements of L are finitely generated. The first part of the proof is formulated in a more general way to enable further possible modifications of the proof replacing the free group by a finite group $\mathbf{G}$.

The first proof that every lattice can be embedded into the subgroup lattice of a group was given in [Wh]. The proof was later simplified by the author of the present note in [Tů2] using the solution of the word problem for HNN-extensions.

Let Ω be a set. By a *twist* on Ω we mean a bijection $t : A \to B$ between two subsets A, B of Ω. The set A is called the *domain* of t and denoted by $\mathrm{Dom}(t)$, while B is called the *range* of t and denoted by $\mathrm{Rng}(t)$. If $t : A \to B$ is a twist on Ω, then the inverse mapping $t^{-1} : B \to A$ is also a twist on Ω and called the *inverse* of t. The value of a twist t at a point $a \in \mathrm{Dom}(t)$ will be written as at.

By a *twisting structure* on Ω we mean a set $\mathcal{T} = \{t_i : i \in I\}$ of twists on Ω such that with every $t \in \mathcal{T}$ the inverse t^{-1} of t is also contained in $\mathcal{T}$.

The Cayley graph $\mathbf{G}(\mathcal{T})$ of a twisting structure $\mathcal{T}$ is defined as follows. The vertex set of $\mathbf{G}(\mathcal{T})$ is Ω. The edge set of $\mathbf{G}(\mathcal{T})$ is the set $E = \{(a,t) : a \in \mathrm{Dom}(t), t \in \mathcal{T}\}$. If $e = (a,t) \in E$, then a is the initial vertex $\alpha(e)$ of e and at is the terminal vertex $\omega(e)$ of e. Since $\mathcal{T}$ contains with every twist t also the inverse t^{-1} of t, with every edge $e = (a,t) \in E$ there is also the edge $(at, t^{-1}) \in E$. The edge (at, t^{-1}) is called the *inverse* of e and denoted by e^{-1}. It is obvious that $\alpha(e^{-1}) = \omega(e)$ and $\omega(e^{-1}) = \alpha(e)$, thus the graph $\mathbf{G}(\mathcal{T})$ is a symmetric graph possibly with loops and parallel edges. We further define the value $\nu(e)$ of an edge $e = (a,t)$ as the twist t. Thus the Cayley graph $\mathbf{G}(\mathcal{T}) = (\Omega, E, \alpha, \omega, \nu)$ of a twisting structure $\mathcal{T}$ is a symmetric graph with edges valued by elements of $\mathcal{T}$. It

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is the union of the graphs of all partial bijections $t \in \mathcal{T}$. Note also that for every $t \in \mathcal{T}$ and $a \in \Omega$ there is at most one edge of $\mathbf{G}(\mathcal{T})$ with initial vertex a and value $\nu(e) = t$.

By a *congruence* of a twisting structure $\mathcal{T}$ on Ω we mean an equivalence relation π on Ω satisfying the following condition:

$$(*) \quad \text{whenever} \quad (a,b) \in \pi,\ t \in \mathcal{T} \quad \text{and} \quad a,b \in \mathrm{Dom}(t),\ \text{then also} \quad (at,bt) \in \pi.$$

Thus a congruence of $\mathcal{T}$ is a congruence of the partial algebra $(\Omega, \mathcal{T})$, where each $t \in \mathcal{T}$ is considered to be a partial unary operation on Ω. Obviously the set $\mathbf{C}(\mathcal{T})$ of all congruences of the twisting structure $\mathcal{T}$ on Ω is closed under arbitrary intersections and contains the least and the greatest equivalence relations on Ω. Thus $\mathbf{C}(\mathcal{T})$ when ordered by inclusion is a complete lattice. It is called the *congruence lattice* of $\mathcal{T}$. The meet $\pi \wedge \rho$ of two congruences π, ρ of $\mathcal{T}$ is their set-theoretical intersection $\pi \cap \rho$, while their join $\pi \vee \rho$ in $\mathbf{C}(\mathcal{T})$ is the least equivalence relation on Ω containing the set-theoretical union $\pi \cup \rho$ and satisfying the condition $(*)$.

The following simple representation result was proved in [Tů1].

Theorem 1. *Every lattice L can be represented as a sublattice of $\mathbf{C}(\mathcal{T})$ for some twisting structure $\mathcal{T}$.* □

For the sake of completeness we present the construction. Given a lattice L, we may assume that it has a least element 0. For any two non-zero elements $a < b$ of L we define a twist $t_{a,b}$ on L with $\mathrm{Dom}(t_{ab}) = \mathrm{Rng}(t_{ab}) = \{0, a, b\}$ and such that

$$bt_{ab} = b,\ at_{ab} = 0,\ 0t_{ab} = a.$$

Moreover, if $a, b \in L$ are two non-comparable and non-zero elements, then we define a twist t_{ab} on L such that $\mathrm{Dom}(t_{ab}) = \mathrm{Rng}(t_{ab}) = \{a, b, a \vee b\}$ and

$$at_{ab} = a,\ bt_{ab} = a \vee b,\ (a \vee b)t_{ab} = b.$$

Let the twisting structure $\mathcal{T}$ on L consist of all twists of the form t_{ab}, where $0 \neq a, b \in L$. Then the mapping assigning to every $x \in L$ the partition of L with one block the interval $[0, x]$ and the other blocks singletons is an embedding of L into $\mathbf{C}(\mathcal{T})$.

The main purpose of this note is to investigate a canonical mapping Φ from $\mathbf{C}(\mathcal{T})$ into the subgroup lattice of the free group $\mathbf{F}(\mathcal{T})$. Here $\mathbf{F}(\mathcal{T})$ denotes the free group generated by the set $\mathcal{T}$ of free generators. Under sufficiently general conditions on $\mathcal{T}$ we can prove that the canonical mapping Φ is an embedding of any member from a large class of sublattices of $\mathbf{C}(\mathcal{T})$. To this end we recall some basic ideas from combinatorial group theory related to homotopy groups of graphs. By a graph we mean a quadruple (V, E, α, ω), where V, E are non-empty sets and $\alpha, \omega : E \to V$ are two incidence functions, $\alpha(e)$ is called the initial vertex of an edge $e \in E$ and $\omega(e)$ is the terminal vertex of e. By a path in the graph (V, E, α, ω) we mean a sequence $p = e_1 e_2 \cdots e_k$ of edges of E such that $\omega(e_i) = \alpha(e_{i+1})$ for every $i = 1, 2, \dots, k-1$. A path $p = e_1 e_2 \cdots e_k$ is called a loop if $\omega(e_k) = \alpha(e_1)$, and a loop $p = e_1 e_2 \cdots e_k$ is called a loop at a vertex $v \in V$ if $\alpha(e_1) = v$.

We will generalize these concepts to the Cayley graph of a twisting structure $\mathcal{T}$. Let π be a congruence of $\mathcal{T}$. A sequence $p = e_1 e_2 \cdots e_k$ of edges of the Cayley graph $\mathbf{C}(\mathcal{T})$ of $\mathcal{T}$ is called a π-path if $(\omega(e_i), \alpha(e_{i+1})) \in \pi$ for every $i = 1, 2, \dots k-1$. A path $p = e_1 e_2 \cdots e_k$ is called a π-loop if $(\alpha(e_1), \omega(e_k)) \in \pi$ and it is called a π-loop at a vertex $v \in \Omega$ if moreover $(v, \alpha(e_1)) \in \pi$. If $(\omega(e_i), \alpha(e_{i+1})) \in \tau$ for some relation τ on Ω, then we say that $(\omega(e_i), \alpha(e_{i+1}))$ is a τ-jump.

Suppose moreover that $\mathcal{T} \subset \mathbf{G}$, where $\mathbf{G}$ is a group. Then we can assign to every π-path $p = e_1 e_2 \cdots e_k$ its $\mathbf{G}$-value

$$\nu(p) = \nu(e_1)\nu(e_2) \cdots \nu(e_k) \in \mathbf{G}.$$

The set of $\mathbf{G}$-values of all π-loops at a vertex $v \in \Omega$ in the Cayley graph $\mathbf{C}(\mathcal{T})$ of a twisting structure $\mathcal{T}$ is obviously a subgroup of $\mathbf{G}$. Indeed, $1 \in \mathbf{G}$ is the value of the empty π-path, if $g = \nu(p)$ for a π-loop $p = e_1 e_2 \cdots e_k$ at v, then g^{-1} is the value of the inverse path $p^{-1} = e_k^{-1} \cdots e_1^{-1}$. And if $g = \nu(p)$, $h = \nu(q)$, then $gh = \nu(pq)$. Thus we can define a mapping

$$\Phi_{\mathbf{G}} : \mathbf{C}(\mathcal{T}) \to \mathrm{Sub}(\mathbf{G})$$

from the congruence lattice of $\mathcal{T}$ into the subgroup lattice of $\mathbf{G}$ by

$$\Phi_{\mathbf{G}}(\pi) = \{\nu(p) : p \text{ is a } \pi\text{-loop at } v\}.$$

If $\pi \subseteq \rho$ are two congruences of $\mathcal{T}$, then obviously any π-loop at v is also a ρ-loop at v, thus we get the following simple lemma.

Lemma 2. *The mapping $\Phi_{\mathbf{G}}$ is order-preserving.* $\square$

We say that a twisting structure $\mathcal{T}$ is connected if its Cayley graph $\mathbf{G}(\mathcal{T})$ is connected. For connected twisting structures we have the following result.

Theorem 3. *If $\mathcal{T}$ is a connected twisting structure, then the mapping $\Phi_{\mathbf{G}}$ is join-preserving.*

Proof: First of all we describe the join $\pi \vee \rho$ of two congruences $\pi, \rho \in \mathbf{C}(\mathcal{T})$. Set $\sigma_0 = \pi \cup \rho$. If σ_{2i} is already defined for a natural number i, we define

$$\sigma_{2i+1} = \sigma_{2i} \cup \{(at, bt) : t \in \mathcal{T}, a, b \in \mathrm{Dom}(t), (a, b) \in \sigma_{2i}\},$$

and

$$\sigma_{2i+2} \text{ is the transitive closure of } \sigma_{2i+1}.$$

Obviously, any congruence σ of $\mathcal{T}$ containing both π and ρ must contain also σ_n for any natural number n. On the other hand,

$$\sigma = \bigcup_n \sigma_n$$

is an equivalence relation satisfying the condition $(*)$, hence a congruence of $\mathcal{T}$. Thus $\sigma = \pi \vee \rho$ in $\mathbf{C}(\mathcal{T})$.

Since $\Phi_{\mathbf{G}}$ is order-preserving, we get that

$$\Phi_{\mathbf{G}}(\pi) \vee \Phi_{\mathbf{G}}(\rho) \subseteq \Phi_{\mathbf{G}}(\pi \vee \rho)$$

for any two congruences π, ρ of $\mathcal{T}$. To prove the opposite inclusion we have to show that the value $\nu(p)$ of any $(\pi \vee \rho)$-path p at v is contained in the subgroup of $\mathbf{G}$ generated by $\Phi_{\mathbf{G}}(\pi) \cup \Phi_{\mathbf{G}}(\rho)$.

So let $p = e_1 e_2 \cdots e_k$ be an arbitrary $(\pi \vee \rho)$-loop at v. Thus $(\omega(e_i), \alpha(e_{i+1})) \in \pi \vee \rho$ for any $i = 1, 2, \ldots, k-1$ as well as $(v, \alpha(e_1)), (\omega(e_k), v) \in \pi \vee \rho$. Since $\pi \vee \rho = \bigcup \sigma_n$, there exists a natural number m such that

$$(v, \alpha(e_1)), (\omega(e_k), v) \in \sigma_m, (\omega(e_i), \alpha(e_{i+1})) \in \sigma_m$$

for every $i = 1, 2, \ldots, k-1$. Let us call such a $(\pi \vee \rho)$-loop $p = e_1 e_2 \cdots e_k$ at v a σ_m-loop. We are going to prove that the value $\nu(p)$ of any σ_m-loop at v, $m \geq 1$, belongs to the subgroup of $\mathbf{G}$ generated by the values of σ_{m-1}-loops at v.

If m is odd, let $i \leq k$ be such that $(\omega(e_i), \alpha(e_{i+1})) \in \sigma_m \setminus \sigma_{m-1}$. Thus there exist a twist $t \in \mathcal{T}$ and $a, b \in \sigma_{m-1}$ such that $at = \omega(e_i)$ and $bt = \alpha(e_{i+1})$. Consider the loop $p' = e_1 e_2 \cdots e_i (at, t^{-1})(b, t) e_{i+1} \cdots e_k$. Since $\omega(e_i) = \alpha(at, t^{-1})$, $(\omega(at, t^{-1}), \alpha(b, t)) = (a, b) \in \sigma_{m-1}$ and $\omega(b, t) = bt = \alpha(e_{i+1})$, p' is also a σ_m-loop at v and the number of $\sigma_m \setminus \sigma_{m-1}$-jumps in p' is one less than the number of $\sigma_m \setminus \sigma_{m-1}$-jumps in p. Similarly, if $(v, \alpha(e_1)) \in \sigma_m \setminus \sigma_{m-1}$, then again there are a twist $t \in \mathcal{T}$ and $(a, b) \in \sigma_m \setminus \sigma_{m-1}$ such that $at = v$ and $bt = \alpha(e_1)$. Again the loop $p' = (at, t^{-1})(b, t)p$ is a σ_m-loop at v (since $(v, t^{-1}) = (at, t^{-1})$) and the number of $\sigma_m \setminus \sigma_{m-1}$-jumps in p' is one less than the number of $\sigma_m \setminus \sigma_{m-1}$-jumps in p. The case $(\omega(e_k), v) \in \sigma_m \setminus \sigma_{m-1}$ is treated in exactly the same way. In all cases, $\nu(p') = \nu(p)$ and the number of $(\sigma_m \setminus \sigma_{m-1})$-jumps in the path p' is one less than the number of $(\sigma_m \setminus \sigma_{m-1})$-jumps in p. By a simple induction on the number of $(\sigma_m \setminus \sigma_{m-1})$-jumps in p we prove that for every σ_m-loop p at v there exists a σ_{m-1}-loop p'' at v such that $\nu(p'') = \nu(p)$. Thus if m is odd, then the value $\nu(p)$ of any σ_m-loop p at v is equal to the value of a σ_{m-1}-loop at v.

If $m > 0$ is even and $p = e_1 e_2 \cdots e_k$ a σ_m-loop at v that is not a σ_{m-1}-loop, then either there exists some $i = 1, 2, \ldots, k-1$ such that $(\omega(e_i), \alpha(e_{i+1})) \in \sigma_m \setminus \sigma_{m-1}$ or $(v, \alpha(e_1)) \in \sigma_m \setminus \sigma_{m-1}$ or $(\omega(e_k), v) \in \sigma_m \setminus \sigma_{m-1}$. Let the first of the three possibilities occur. Since σ_m is the transitive closure of σ_{m-1}, there are elements $\omega(e_i) = a_1, a_2, \ldots, a_l = \alpha(e_{i+1})$ such that $(a_j, a_{j+1}) \in \sigma_{m-1}$. Since the Cayley graph $\mathbf{G}(\mathcal{T})$ of $\mathcal{T}$ is connected, there are paths q_i in $\mathbf{G}(\mathcal{T})$ of $\mathcal{T}$ from v to a_i. Then

$p' = e_1e_2\cdots e_i q_1^{-1}q_1q_2^{-1}\cdots q_{l-1}q_l^{-1}q_le_{i+1}\cdots e_k$ is again a σ_m-loop at v in which the number of $\sigma_m \setminus \sigma_{m-1}$-jumps is one less than the number of $\sigma_m \setminus \sigma_{m-1}$-jumps in p. Moreover, $\nu(p') = \nu(p)$. The other two cases are treated in exactly the same way. Thus also in this case the value of any σ_m-loop at v coincides with the value of a σ_{m-1}-loop at v.

Hence the value $\nu(p)$ of any $(\pi \vee \rho)$-loop at v equals the value $\nu(p')$ of a σ_0-loop p' at v. Recall that $\sigma_0 = \pi \cup \rho$. Let $p' = f_1f_2\cdots f_l$. For every $i = 1, 2, \ldots, l$ let q_i be a path in $\mathbf{G}(\mathcal{T})$ from v to $\omega(f_i)$. Then

$$p'' = f_1q_1^{-1}q_1f_2q_2^{-1}q_2f_3\cdots q_{l-1}^{-1}q_{l-1}f_l$$

is also a σ_0-loop at v. Obviously, $\nu(p'') = \nu(p)$. Finally, observe that each $q_{i-1}f_iq_i^{-1}$, $i = 2, \ldots, l-1$ is either a π-loop or a ρ-loop, since $(\omega(f_{i-1}), \alpha(f_i)) \in \sigma_0 = \pi \cup \rho$. Thus $\nu(q_{i-1}f_iq_i^{-1}) \in \Phi_{\mathbf{G}}(\pi) \cup \Phi_{\mathbf{G}}(\rho)$.

Similarly, we prove that also $\nu(f_1)q_1^{-1} \in \Phi_{\mathbf{G}}(\pi) \cup \Phi_{\mathbf{G}}(\rho)$ and $\nu(q_{l-1}f_l) \in \Phi_{\mathbf{G}}(\pi) \cup \Phi_{\mathbf{G}}(\rho)$. Thus $\nu(p'') \in \Phi_{\mathbf{G}}(\pi) \vee \Phi_{\mathbf{G}}(\rho)$. □

In the rest of the paper we restrict ourselves to the case that $\mathbf{G}$ is the free group $\mathbf{F}$ freely generated by $\mathcal{T}$. Let L be any non-empty collection of partitions on the set Ω closed under finite meets. We say that a twisting structure $\mathcal{T}$ on Ω is *balanced* with respect to L if for every twist $t \in T$, an element $a \in \Omega$ and any two $x, y \in \mathrm{Dom}(t)$, whenever $(a, x) \in \pi \in L$ and $(a, y) \in \rho \in L$, then there exists some $z \in \mathrm{Dom}(t)$ such that $(a, z) \in \pi \wedge \rho \in L$. The following lemma from [Tů1] gives us a way to construct balanced sets.

Lemma 4. *Let X be a set and $\Omega = \mathbf{S}_X$, the group of all permutations of X of a finite type (i.e. generated by transpositions). For a set $Y \subset X$ let S_Y be the subgroup of $\mathbf{S}_X$ consisting of all permutations p such that $p(x) = x$ for every $x \in X \setminus Y$. Let L be the set of partitions of Ω into left cosets of subgroups S_Y, $Y \subseteq X$. Then every left coset of every S_Y, $Y \subseteq X$, is balanced with respect to L.* □

By modifying Example 2.5. and Proposition 2.8. of [Tů1] we get the following lemma.

Lemma 5. *For every lattice $\mathbf{L}$ there exist a set Ω, a twisting structure $\mathcal{T}$ on Ω with finite domains and a lattice embedding $\phi : \mathbf{L} \to \mathbf{C}(\mathcal{T})$ such that the twisting structure $\mathcal{T}$ is balanced with respect to the lattice $L = Im(\phi)$.* □

No we are ready to prove the following counterpart to Theorem 3.

Theorem 6. *Let $\mathcal{T}$ be a twisting structure on a set Ω and $L \subset \mathbf{C}(\mathcal{T})$ a sublattice of $\mathbf{C}(\mathcal{T})$. Suppose moreover that the domains of the elements of $\mathcal{T}$ are finite and that $\mathcal{T}$ is balanced with respect to L. Then the restriction of the canonical mapping $\Phi_{\mathbf{F}}$ to the lattice L is meet-preserving.*

Proof: Let $\pi \in \mathbf{C}(\mathcal{T})$. First of all we prove that for any π-loop $p = e_1e_2\cdots e_k$ in $\mathbf{G}(\mathcal{T})$ at v such that $\nu(e_1)\nu(e_2)\cdots\nu(e_k)$ is not a reduced word in $\mathbf{F}$ there exists a subpath $p' = e_{i_1} \cdots e_{i_l}$ of p that is also a π-loop at v and the word $\nu(e_{i_1}) \cdots \nu(e_{i_l})$ is reduced. Indeed, if $\nu(e_1)\nu(e_2) \cdots \nu(e_k)$ is not reduced, then there is some $i = 1, 2, \dots, k-1$ such that $\nu(e_i) = t = \nu(e_{i+1})^{-1}$. Thus there are some $a, b \in \mathrm{Dom}(t)$ such that $e_i = (a, t)$ and $e_{i+1} = (bt, t^{-1})$. Moreover, since p is a π-path, we have $(\omega(e_i), \alpha(e_{i+1})) = (at, bt) \in \pi$. Since $\pi \in \mathbf{C}(\mathcal{T})$, we have also $(att^{-1}, btt^{-1}) = (a, b) \in \pi$. But we have also $(\omega(e_{i-1}), \alpha(e_i)) = (\omega(e_{i-1}), a) \in \pi$ and $(\omega(e_{i+1}), \alpha(e_{i+2})) = (b, \alpha(e_{i+2})) \in \pi$, we get $(\omega(e_{i-1}), \alpha(e_{i+2})) \in \pi$. Thus we can delete from p the edges e_i, e_{i+1} and the remaining path p' is still a π-loop at v. In this way we can subsequently delete from p pairs of subsequent edges with mutually inverse values to get a π-loop p' with required properties. Let us call such a π-path a *reduced* π-path.

Take now arbitrary congruences $\pi, \rho \in L$. Since $\Phi_{\mathbf{F}}$ is order-preserving, we have

$$\Phi_{\mathbf{F}}(\pi) \cap \Phi_{\mathbf{F}}(\rho) \supseteq \Phi_{\mathbf{F}}(\pi \cap \rho).$$

To prove the opposite inclusion take any reduced word $w = t_1 \cdots t_k \in \Phi_{\mathbf{F}}(\pi) \cap \Phi_{\mathbf{F}}(\rho)$. Then by the previous paragraph there is a reduced π-loop $p = e_1 e_2 \cdots e_k$ at v such that $\nu(p) = w$. Hence $\nu(e_i) = t_i$ for every $i = 1, 2, \dots, k$. Similarly, there is a reduced ρ-loop $q = f_1 f_2 \cdots f_l$ at v such that $\nu(f_i) = t_i$ for every $i = 1, 2, \dots, k$. Thus in particular, $(v, \alpha(e_1)) \in \pi$ and $(v, \alpha(f_1)) \in \rho$. Let us denote $\alpha(e_1) = a$ and $\alpha(f_1) = b$. Thus $a, b \in \mathrm{Dom}(t_1)$. Since $\mathrm{Dom}(t_1)$ is balanced with respect to L, there exists some $c \in \mathrm{Dom}(t_1)$ such that $(v, c) \in \pi \cap \rho$. Hence also $(a, c) \in \pi$ and $(b, c) \in \rho$. Since both π and ρ are congruences of $\mathcal{T}$, we get also $(at_1, ct_1) \in \pi$ and $(bt_1, ct_1) \in \rho$. Moreover, $(at_1, \alpha(e_2)) = (\omega(e_1), \alpha(e_2)) \in \pi$ and $(bt_1, \alpha(f_2)) = (\omega(f_1), \alpha(f_2)) \in \rho$, we get $(ct_1, \alpha(e_2)) \in \pi$ and $(ct, \alpha(f_2)) \in \rho$. Denote by g_1 the edge (c, ct_1). Thus $g_1 e_2 \cdots e_k$ is another π-loop at v and $g_1 f_2 \cdots f_k$ is another ρ-loop at v. Moreover, $(v, \alpha(g_1)) = (v, c) \in \pi \cap \rho$, and $\nu(g_1) = t_1$.

By repeating the same procedure with ct_1, e_2 and f_2 in place of v, e_1 and f_1, we get another edge g_2 that can replace e_2 in p and f_2 in q and satisfies $\nu(g_2) = \nu(e_2) = \nu(f_2)$ and $(\omega(g_1), \alpha(g_2)) \in \pi \cap \rho$. After k steps we construct a $(\pi \cap \rho)$-loop $r = g_1 g_2 \cdots g_k$ at v with $\nu(r) = w$. Hence $w \in \Phi_{\mathbf{F}}(\pi \cap \rho)$. $\square$

Finally, connectedness of $\mathcal{T}$ also implies injectivity of $\Phi_{\mathbf{F}}$. Since the method of the proof will be also used in the proof of Theorem 8, we introduce some definitions here. If π is a congruence of $\mathcal{T}$, we define the quotient $\mathcal{T}/\pi$ of $\mathcal{T}$ as follows. The twisting structure $\mathcal{T}/\pi$ will be defined on the set Ω/π of blocks of π. For any twist $t \in \mathcal{T}$ we define another twist t_π on Ω/π. The domain $\mathrm{Dom}(t_\pi)$ consists of all blocks of π intersecting the domain $\mathrm{Dom}(t)$. If $a \in \mathrm{Dom}(t)$, then we define $[a]t_\pi = [at]$, where $[x]$ denotes the block of π containing x. The definition of t_π is correct since π is a congruence of $\mathcal{T}$. Hence t_π is also a bijection between two subsets of Ω/π and $\{t_\pi : t \in \mathcal{T}\}$ is a twisting structure on Ω/π. Thus $\mathcal{T}/\pi$ is simply the quotient of the partial unary algebra $(\Omega, \mathcal{T})$ by the congruence π.

It is also useful to mention that the graph of $\mathcal{T}/\pi$ is naturally isomorphic to a quotient of the graph of $\mathcal{T}$. The vertices of $\mathbf{G}(\mathcal{T}_\pi)$ are blocks of the partition π on Ω. Whenever $a, b \in \mathrm{Dom}(t)$ are such that $(a, b) \in \pi$, then the two edges (a, t)

and (b,t) of $\mathbf{G}(\mathcal{T})$ are identified into a single edge $([a],[a]t_\pi)$ of $\mathbf{G}(\mathcal{T}/\pi)$. If we assign to each edge $([a],[a]t_\pi)$ of $\mathbf{G}(\mathcal{T}/\pi)$ the value $t \in \mathbf{F}$, then we see that the values of π-loops at v in the graph $\mathbf{G}(\mathcal{T})$ are in one-to-one correspondence with the values of ordinary loops at $[v]$ in the graph of $\mathbf{G}(\mathcal{T}/\pi)$.

Lemma 7. *If $\mathcal{T}$ is a connected twisting structure on Ω and $\pi < \rho$ two congruences of $\mathcal{T}$, then $\Phi_{\mathbf{F}}(\pi) \neq \Phi_{\mathbf{F}}(\rho)$.*

Proof: Let $(a,b) \in \rho \setminus \pi$. Since $\mathcal{T}$ is connected, there exist a reduced path $p = e_1 e_2 \cdots e_k$ in $\mathbf{G}(\mathcal{T})$ from v to a and a reduced path $q = f_1 f_2 \cdots f_l$ in $\mathbf{G}(\mathcal{T})$ from v to b. Then pq^{-1} is a ρ-loop in $\mathbf{G}(\mathcal{T})$ at v but it is not a π-loop at v. If pq^{-1} is not reduced, then there exists a twist $t \in \mathcal{T}$ such that $e_k = (at^{-1}, t)$ and $f_l = (bt^{-1}, t)$. Then also $(at^{-1}, bt^{-1}) \in \rho \setminus \pi$. So we can replace p by $p' = e_1 e_2 \cdots e_{k-1}$ and q by $q' = f_1 f_2 \cdots f_{l-1}$ to get a shorter ρ-loop $p'q'^{-1}$ at v that is not a π-loop at v. Hence we may assume that the ρ-loop $r = pq^{-1}$ at v is already reduced and it is not a π-loop. But then $\nu(r) \in \Phi_{\mathbf{F}}(\rho) \setminus \Phi_{\mathbf{F}}(\pi)$. □

Putting together previous results we get the following theorem.

Theorem 8. *Every lattice L can be embedded into the subgroup lattice of a free group $\mathbf{F}$. If the lattice L is finite, then the group $\mathbf{F}$ and all the subgroups of $\mathbf{F}$ representing elements of L can be taken finitely generated.*

Proof: It remains to prove the second assertion. However, if the lattice L is finite, then the twisting structure $\mathcal{T}$ of Lemma 4 can be taken finite by [Tů1]. But then the group $\Phi_{\mathbf{F}}(\pi)$ is isomorphic to the homotopy group of the graph $\mathbf{G}(\mathcal{T}/\pi)$, by the remarks preceding Lemma 7. Since the graph $\mathbf{G}(\mathcal{T}/\pi)$ is finite, its homotopy group is a finitely free group. □

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