Comment.Math.Univ.Carolin. 40,3 (1999)531–536

On AP and WAP spaces

A. Bella, I.V. Yaschenko

Abstract. Several remarks on the properties of approximation by points (AP) and weak approximation by points (WAP) are presented. We look in particular at their behavior in product and at their relationships with radiality, pseudoradiality and related concepts. For instance, relevant facts are:
(a) There is in ZFC a product of a countable WAP space with a convergent sequence which fails to be WAP.
(b) C_p over σ-compact space is AP. Therefore AP does not imply even pseudoradiality in function spaces, while it implies Fréchet-Urysohn property in compact spaces.
(c) WAP and AP do not coincide in C_p.

Keywords: AP space, WAP space, pseudoradial space, radial space, product, compact space, submaximal space, function space

Classification: 54A25, 54D55

All spaces under consideration are assumed to be Tychonoff. For all undefined notions see [En].

The following definitions were originated in Categorical Topology.

Definition 0.1 ([PT]). A space X is said to have the property of Approximation by Points (Weak Approximation by Points) if for every non-closed set A and every (some) point $x \in \mathrm{Cl}_X A \setminus A$ there is a subset $B \subset A$, such that $\mathrm{Cl}_X B \setminus A = \{x\}$.

Simon [Si] was first to study these properties from a General Topological point of view. In particular, he observed that every GO space is WAP; constructed examples of two AP spaces such that their product is not WAP and (under CH) of two countable AP spaces, such that their product is not AP.

Bella [B1], see also [B2] discovered strong connections of these properties with higher convergence properties — radiality, semiradiality and pseudoradiality. E.g. every semiradial space is WAP, every compact WAP space is pseudoradial, a product of compact WAP and compact semiradial space is WAP.

Definition 0.2. A space X is radial (pseudoradial) if for every non-closed set A and every (some) point $x \in \mathrm{Cl}_X A \setminus A$ there is a transfinite sequence $S \subset A$ which converges to x.

A space X is semiradial if for every non-κ-closed set A there is a transfinite sequence $S \subset A$ which converges to a point outside A and satisfies $|S| \leq \kappa$.

Radial $\rightarrow$ Semiradial $\rightarrow$ Pseudoradial.

This work was done while the second author was visiting Catania University. He is grateful to Italian colleagues for generous hospitality and to CNR for financial support.

General remarks on AP spaces

In [PT], it was observed that $\omega_1 + 1$ is not AP. The first author showed in [B1] that every scattered compact AP space is Fréchet-Urysohn. Here we go further by the following

Theorem 1.1. *Every compact AP space is Fréchet-Urysohn.*

PROOF: If X has not countable tightness then X contains an uncountable free sequence ξ ([Ar3]). Since our space is compact, ξ has at least one complete accumulation point (indeed, outside ξ), but on the other hand no subset of ξ can have that point as a unique cluster point. So the tightness of X is countable and X, being a compact AP space, is Fréchet-Urysohn. □

Theorem 1.2. *Every pseudoradial AP space is radial.*

PROOF: Let $A \subset X$ and $x \in \mathrm{Cl}_X A \setminus A$. Since X is AP there is a subset $B \subset A$, so that x is the only point in $\mathrm{Cl}_X B \setminus A$. Since X is pseudoradial, there is a transfinite sequence in $\mathrm{Cl}_X B \setminus \{x\}$ converging to x. Clearly, this sequence lies in A and we are done. □

Recall that a space is submaximal if every dense subset is open or, equivalently, every subset with empty interior is closed and discrete (see [AC] for a recent survey on submaximal spaces).

Proposition 1.3. *Every submaximal space is AP.*

PROOF: Let $x \in \mathrm{Cl}_X A \setminus A$. $A \setminus \mathrm{Int}\, A$ has empty interior in X and hence it is closed and discrete in X. Therefore, $x \in \mathrm{Cl}_X \mathrm{Int}\, A$. Since $\mathrm{Cl}_X \mathrm{Int}\, A \setminus A$ is closed and discrete, there is an open neighborhood W of x, so that

$$\mathrm{Cl}_X W \cap (\mathrm{Cl}_X \mathrm{Int}\, A \setminus A) = \{x\}.$$

It is easy to see, that $B = W \cap \mathrm{Int}\, A$ satisfies $\mathrm{Cl}_X B \setminus A = \{x\}$. □

Countable AP and WAP spaces

In [Si] Simon used CH to construct two countable AP spaces whose product is not AP, leaving it open, whether the same can be done without CH.

We present here two ZFC examples (Theorem 2.1 and Example 2.5), with some stronger properties — the first is the product of a convergent sequence with an AP space which is even non-WAP, and the second is a non-AP product of a Fréchet-Urysohn space and a compact metric space.

Theorem 2.1. *There is a countable AP space C such that the product of C with a convergent sequence is not WAP.*

From Theorem 2.1 we can get the following

Corollary 2.2. *There is a countable non WAP space.*

Theorem 2.1 is an immediate corollary of Proposition 1.3 and Theorem 2.3. In fact, one may take for C an arbitrary countable submaximal space without isolated points. At least one such a space exists due to van Douwen [vD].

Theorem 2.3. *The product of a countable submaximal dense-in-itself space with a convergent sequence is not WAP.*

PROOF: Let X be the product of a countable submaximal space C with the usual convergent sequence $\omega+1$. Fix an arbitrary bijection $f : C \to \omega$. The set $A = \{(x, f(x)) : x \in C\}$ is not closed because $\mathrm{Cl}_X A = A \cup C \times \{\omega\}$.

Suppose that there is $B' \subset C$ so that for $B = \{(x, f(x)) : x \in B'\}$ we have $|\mathrm{Cl}_X B \setminus A| = 1$. Evidently, $\mathrm{Cl}_X B \setminus A = \{(y, \omega)\}$, for some $y \in C$. Since $|B \cap \{y\} \times \omega| \le 1$, it follows that $y \in \mathrm{Cl}_C(B' \setminus \{y\})$. So, B' has non-empty interior in C. Taking any arbitrary point $z \in \mathrm{Int}\, B' \setminus \{y\}$, we see that $(z, \omega) \in \mathrm{Cl}_X B \setminus A$ — a contradiction. □

Observe that the space C above is far from being Fréchet-Urysohn or sequential. Indeed, there is no chance to have a sequential example, because of the following

Proposition 2.4. *The product of a sequential space X with a countably compact WAP space Y is WAP.*

PROOF: Let $A \subset X \times Y$ be a non-closed subset. Take $(x, y) \in \mathrm{Cl}_{X\times Y} A \setminus A$. If $A \cap \{x\} \times Y$ were non-closed, we could immediately apply the WAP property of Y to find a set $B \subset A$ satisfying $\mathrm{Cl}_{X\times Y} B \setminus A = (x, y)$. So, consider the case that $A \cap \{x\} \times Y$ is closed. Passing to a suitable subset, we can assume that $A \cap \{x\} \times Y = \emptyset$. Since $x \in \mathrm{Cl}_X \pi_X(A)$, it follows that $\pi_X(A)$ is not closed in X. So, there is a convergent sequence $\{x_n : n \in \omega\} \subset \pi_X(A)$ with limit outside $\pi_X(A)$. For each $n \in \omega$ take a point $y_n \in Y$ in such a way that $(x_n, y_n) \in A$. Since countably compact WAP spaces are sequentially compact ([B1]), $\{y_n : n \in \omega\}$ contains a convergent subsequence $\{y_{n_k} : k \in \omega\}$. Clearly, $\{(x_{n_k}, y_{n_k}) : k \in \omega\}$ converges to a point outside A and we are done. □

We now show that property AP may disappear even in a product of a Fréchet-Urysohn space with a compact metric space.

Example 2.5. *There are a countable Fréchet-Urysohn space and a countable compact metric space, whose product is not AP.*

PROOF: Let $S_\omega = \{*\} \cup \{(n, m) : n, m \in \omega\}$ be the familiar countable Fréchet-Urysohn fan having $*$ as the unique non-isolated point (here n stands for the number of a sequence converging to $*$ and m for the number of a point in the corresponding sequence). We claim that the product $X = S_\omega \times (\omega+1)^2$ of the Fréchet-Urysohn space S_ω with the compact metric space $(\omega+1)^2$ is not AP. Indeed, let $A = \{((n, m), (n, m)) : n, m \in \omega\}$ be a "diagonal" over sets of isolated points in both factors. Clearly, $(*, (\omega, \omega)) \in \mathrm{Cl}_X A$. If the product were AP then there should exist $B \subset A$, with $\mathrm{Cl}_X B \setminus A = \{(*, (\omega, \omega))\}$. Since, $* \in \mathrm{Cl}_{S_\omega} \pi_{S_\omega}(B)$, there exists $n \in \omega$ so that $\pi_{S_\omega}(B)$ intersects the n-th sequence in an infinite set. Now, it is easy to realize that $(*, (n, \omega)) \in \mathrm{Cl}_X B \setminus A$ and therefore X cannot be an AP space. $\square$

AP and WAP properties in function spaces

We start with a bit technical result. We shall use the following

Definition 3.1 ([Ar2])**.** A space X has countable fan-tightness provided that for every point $x \in X$ and every countable family G_n of subsets of X if $x \in \mathrm{Cl}_X G_n$ for any $n \in \omega$ then there are finite $F_n \subset G_n$, so that $x \in \mathrm{Cl}_X \cup\{F_n : n \in \omega\}$.

Definition 3.2. A space X is ω-ψ-monolithic if the closure of every countable subset has countable pseudocharacter (in itself).

Theorem 3.3. *Every ω-ψ-monolithic space with countable fan-tightness is AP.*

Proof: Let $x \in \mathrm{Cl}_X A \setminus A$. Since X has countable tightness, we can assume that A is countable. Since X is ω-ψ-monolithic, $\psi(x, \mathrm{Cl}_X A) \leq \omega$. So, there is a countable decreasing family $W_n : n \in \omega$ of neighborhoods of x in $\mathrm{Cl}_X A$, so that $\cap\{\mathrm{Cl}_X W_n : n \in \omega\} = \{x\}$. Evidently, $x \in \mathrm{Cl}_X(W_n \cap A)$. Since X has countable fan-tightness, there are finite $F_n \subset W_n \cap A$, such that $x \in \mathrm{Cl}_X(\cup\{F_n : n \in \omega\})$. Let $F = \cup\{F_n : n \in \omega\}$. Since for each $n \in \omega$ $F \setminus W_n$ is finite, F cannot have cluster points other than x. So, $|\mathrm{Cl}_X F \setminus A| = \{x\}$. $\square$

From Theorem 3.3 we can immediately get the following two propositions.

Corollary 3.4. *Every space with countable pseudocharacter and countable fan-tightness is AP.*

Corollary 3.5. *Every ω-monolithic space with countable fan-tightness is AP.*

A well known result concerning function spaces with the topology of pointwise convergence says that $C_p(X)$ is Fréchet-Urysohn if and only if it is sequential if and only if it is a k-space ([GN]). This result was further improved in [GNS], where it was shown that $C_p(X)$ is radial if and only if it is Fréchet-Urysohn.

A natural question is now whether property AP could be added to the previous list of equivalences. It turns out that this is not the case, even if we assume that $C_p(X)$ is countably AP (i.e. countably tight and AP). A simple example witnessing this is $C_p(I)$, where I is the unit segment.

More generally, we have the following

Theorem 3.6. *If X is a σ-compact space then $C_p(X)$ is countably AP.*

Proof: Every σ-compact space is a Hurewicz space. Hence $C_p(X)$ has countable fan-tightness ([Ar2]). Also, a σ-compact space is ω-stable and hence $C_p(X)$ is ω-monolithic ([Ar1]). Now apply Corollary 3.5. $\square$

Notice that the last result cannot be strengthened to Lindelöf-Σ-spaces.

Example 3.7. *C_p over a separable metric space can fail to be WAP.*

PROOF: By Corollary 2.2 there is a countable non WAP space X. Then $Y = C_p(X)$ is separable and metric. Furthermore, X can be embedded into $C_p(Y)$ as a closed subspace by the evaluation mapping. Therefore, $C_p(Y)$ is not WAP. □

The next natural question concerns whether AP and WAP coincide in function spaces.

Theorem 3.8. *Let $k > \omega$ be a regular ω-inaccessible cardinal, i.e. $\lambda^\omega < \kappa$ for any $\lambda < \kappa$. Then $C_p(k)$ is WAP but not AP.*

PROOF: It is known ([GNS]) that $C_p(k)$ is pseudoradial, but not radial. So, $C_p(k)$ is not AP by Theorem 1.2. To reach our target, we improve the above mentioned result of [GNS] by showing that such a $C_p(\kappa)$ is actually semiradial, and hence WAP because of a result in [B1]. In [DT] it was observed that every convergent transfinite sequence in $C_p(\kappa)$ has length either κ or ω. Thus, to check that $C_p(\kappa)$ is semiradial it suffices to look at the non-λ-closed subsets for $\lambda < \kappa$. Clearly, our result will be achieved if we prove that the closure of every set $B \subset C_p(\kappa)$ of cardinality less than κ has still cardinality less than κ. Let B' be the sequential closure of B. It is well known that we have $|B'| \leq |B|^\omega$ and consequently, as κ is ω-inaccessible, we have also $|B'| < \kappa$. We claim that B' gives the full closure of B in $C_p(\kappa)$. In fact on the contrary, since $C_p(\kappa)$ is pseudoradial, there would exist a transfinite sequence in B' converging to some point outside B'. But, thanks to the observation in [DT], such a transfinite sequence must be countable and so it does not converge to any point outside B'. □

Notice that the argument in the proof of the above theorem shows that $C_p(\kappa)$ is indeed R-monolithic. A space X is R-monolithic if for every non-closed set $A \subset \mathrm{Cl}_X B \subset X$ there is a transfinite sequence $S \subset A$ converging to some point outside A and satisfying $|S| \leq |B|$. Thus, Theorem 3.8 answers Question 7 in [B2].

Questions

These are two key questions in the study of compact WAP spaces:

Question 4.1 ([B2])**.** *Does there exist a compact pseudoradial non-WAP space?*

Question 4.2 ([B2])**.** *Is the product of two (countably many) compact WAP spaces still WAP?*

In view of our study of AP and WAP properties in function spaces the following questions could be interesting:

Question 4.3. *When is $C_p(X)$ AP or WAP?*

Question 4.4. *Suppose $C_p(X)$ is AP. Is then $C_p(X)$ a space with countable tightness?*

References

[AC] Arhangel'skij A.V., Collins P.J., *On submaximal spaces*, Topology Appl. **64** (1995), 219–241.

[Ar1] Arhangel'skij A.V., *Factorization theorems and function spaces: stability and monolithicity*, Soviet Math. Dokl. **26** (1982), 177–181.

[Ar2] Arhangel'skij A.V., *Hurewicz spaces, analytic sets and fan-tightness of spaces of functions*, Soviet Math. Dokl. **33** (1986), 396–399.

[Ar3] Arhangel'skij A.V., *On bicompacta hereditarily satisfying Suslin's condition. Tightness and free sequences*, Soviet Math. Dokl. **12** (1971), 1253–1257.

[B1] Bella A., *On spaces with the property of weak approximation by points*, Comment. Math. Univ. Carolinae **35** (1994), 357–360.

[B2] Bella A., *Few remarks and questions on pseudoradial and related spaces*, Topology Appl. **70** (1996), 113–123.

[vD] van Douwen E.K., *Applications of maximal topologies*, Topology Appl. **51** (1993), 125–140.

[DT] Dimov G., Tironi G., *Some remarks on almost radiality in function spaces*, Acta Univ. Carolinae **28** (1987), 49–58.

[GN] Gerlits J., Nagy Z.S., *Some properties of $C(X)$ I*, Topology Appl. **14** (1982), 151–161.

[GNS] Gerlits J., Nagy Z., Szentmiklossy Z., *Some convergent properties in function space*, General Topology and its Relations to Modern Analysis and Algebra VI (Z. Frolík, ed.), Heldermann Verlag, Berlin, 1988, pp. 211–222.

[PT] Pultr A., Tozzi A., *Equationally closed subframes and representation of quotient spaces*, Cahiers de Topologie et Geometrie Differentielle Categoriques **34** (1993), 167–183.

[Si] Simon P., *On accumulation points*, Cahiers de Topologie et Geometrie Differentielle Categoriques **35** (1994), 321–327.

Dipartimento di Matematica, Cittá Universitaria, Viale A. Doria 6, 95125 Catania, Italy

E-mail: BELLA@dipmat.unict.it

Moscow Center for Continuous Mathematical Education, B. Vlas'evskij 11, 121002 Moscow, Russia

E-mail: ivan@mccme.ru

(*Received* September 8, 1997)