On blow-up and asymptotic behavior of solutions to a nonlinear parabolic equation of second order with nonlinear boundary conditions

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Abstract. We obtain some sufficient conditions under which solutions to a nonlinear parabolic equation of second order with nonlinear boundary conditions tend to zero or blow up in a finite time. We also give the asymptotic behavior of solutions which tend to zero as $t \to \infty$. Finally, we obtain the asymptotic behavior near the blow-up time of certain blow-up solutions and describe their blow-up set.

Keywords: blow-up, global existence, asymptotic behavior, maximum principle

Classification: 35K55, 35K60, 35B40

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the following boundary value problem:

$$\frac{\partial \varphi(u)}{\partial t} = Lu - a(x,t)f(u) \quad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

$$\frac{\partial u}{\partial N} = b(x,t)g(u) \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_o(x) \quad \text{in} \quad \Omega, \tag{1.3}$$

where

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u + d(x,t),$$

$$\frac{\partial u}{\partial N} = \sum_{i,j=1}^{n} \cos(\nu, x_i) a_{ij}(x,t)\frac{\partial u}{\partial x_j},$$

ν is the exterior normal unit vector on $\partial\Omega$. The coefficients $a_{ij}(x,t)$, $a_i(x,t)$, $c(x,t)$ and $d(x,t)$ are defined in $\Omega \times (0,T)$. Moreover, a_{ij} satisfy the following inequality

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \alpha|\xi|^2$$

for $\xi \in \mathbb{R}^n$ with positive constant α, $a(x,t)$ is a nonnegative function in $\Omega \times (0,T)$, $b(x,t)$ is a nonnegative function on $\partial\Omega \times (0,T)$. Here $u_o(x) \in C^1(\Omega)$ is a positive function in Ω which satisfies the compatibility condition $\frac{\partial u_o}{\partial N} = b(x,0)g(u_o)$ on $\partial\Omega$. For positive values of s, $\varphi(s)$, $f(s)$, $g(s)$ are positive and increasing functions. We want to determine when the solutions of the problem (1.1)–(1.3) are global, i.e. defined for every $t \in (0,\infty)$.

Definition 1.1. *We say that a solution u of the problem* (1.1)–(1.3) *blows up in a finite time if there exists a finite time T_o such that*

$$\lim_{t\to T_o} \|u(x,t)\|_{L^\infty(\Omega)} = \infty.$$

The time T_o is the blow-up time of the solution u. A point $x \in \overline{\Omega}$ is a blow-up point of the solution u if there exists a sequence (x_n, t_n) such that $x_n \to x$, $t_n \uparrow T_o$ and $\lim_{n\to\infty} u(x_n,t_n) = \infty$. The set

$$E_B = \{x \in \overline{\Omega} \quad \textit{such that } x \textit{ is a blow-up point of the solution } u\}$$

is the blow-up set of the solution u.

The problem of blow-up of solutions to parabolic equations of second order with nonlinear boundary conditions has been the subject of investigation of many authors (see, for instance [1], [2], [3], [6] and others). In [3], Egorov and Kondratiev have considered the problem (1.1)–(1.3). They have given some conditions under which the solutions of (1.1)–(1.3) exist globally, tend to zero as $t \to \infty$ or blow up in a finite time. In [1], we have described the asymptotic behavior of some solutions of (1.1)–(1.3) which tend to zero as $t \to \infty$ in the case where $\varphi(u) = u$, $f(u) = g(u)$, $a(x,t) = a(x)$ and $b(x,t) = b(x)$. An interesting question of the problem (1.1)–(1.3) is the localization of the blow-up set. This problem has been studied in [2] by Fila, Chipot and Quittner in the case where $\Omega \subset \mathbb{R}^1$, $\varphi(u) = u$, $L = \Delta$, $a(x,t) = a = const$, $b(x,t) = 1$. In [1], we have generalized some results of [2] concerning the localization of blow-up set in $\Omega \subset \mathbb{R}^n$ with $n \geq 1$.

In this paper, we generalize the results of [1] considering the problem of the form (1.1)–(1.3). We also describe the asymptotic behavior of some solutions of (1.1)–(1.3) which tend to zero as $t \to \infty$ in the case where $\varphi(u) \neq u$, $f(u) \neq g(u)$ and precise some results of Egorov and Kondratiev ([3]) in the case of blow-up solutions.

The paper is written in the following manner. In Section 2, some conditions of blow-up are given. In Section 3, we obtain some conditions under which the solutions of the problem (1.1)–(1.3) tend to zero as $t \to \infty$. In Section 4, we give the asymptotic behavior of the solutions which tend to zero as $t \to \infty$. In Section 5, we obtain the asymptotic behavior near the blow-up time of certain blow-up solutions and finally in Section 6, we describe their blow-up set.

2. Blow-up solutions

In this section, we suppose that

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big).$$

We give some conditions under which the solutions of the problem (1.1)–(1.3) blow up in a finite time for any positive initial data.

The following lemma will be useful in the proofs of some theorems below.

Comparison lemma 2.1. *Let $u, v \in C^1(\overline{\Omega} \times [0,T]) \cap C^2(\Omega \times (0,T))$ satisfy the following inequalities:*

$$\frac{\partial \varphi(u)}{\partial t} - Lu + a(x,t)f(u) > \frac{\partial \varphi(v)}{\partial t} - Lv + a(x,t)f(v) \quad \text{in} \quad \Omega \times (0,T),$$
$$\frac{\partial u}{\partial N} - b(x,t)g(u) > \frac{\partial v}{\partial N} - b(x,t)g(v) \quad \text{on} \quad \partial\Omega \times (0,T),$$
$$u(x,0) > v(x,0) \quad \text{in} \quad \Omega.$$

Then we have

$$u(x,t) > v(x,t) \quad \text{in} \quad \Omega \times (0,T).$$

Proof: The function $w(x,t) = u(x,t) - v(x,t)$ is continuous in $\overline{\Omega} \times [0,T]$. Then its minimum value m is attained at a point $(x_o,t_o) \in \overline{\Omega} \times [0,T]$. Suppose that $u(x_o,t_o) \le v(x_o,t_o)$. If $t_o = 0$, then $m > 0$ which is a contradiction. If $0 < t_o \le T$, then there exists a t_1 such that $0 < t_1 \le t_o$ with $u(x,t) > v(x,t)$ in $\Omega \times [0,t_1[$ but $u(x_1,t_1) = v(x_1,t_1)$ for some $x_1 \in \overline{\Omega}$.

If $x_1 \in \Omega$, then we obtain

$$\frac{\partial(\varphi(u) - \varphi(v))}{\partial t}(x_1,t_1) \le 0, Lw(x_1,t_1) \ge 0, f(u(x_1,t_1)) = f(v(x_1,t_1)),$$

which implies that

$$\frac{\partial(\varphi(u) - \varphi(v))}{\partial t}(x_1,t_1) - Lw(x_1,t_1) + a(x_1,t_1)[f(u(x_1,t_1)) - f(v(x_1,t_1))] \le 0.$$

But, this contradicts the first inequality of the lemma. Finally if $x_1 \in \partial\Omega$, then $\frac{\partial w}{\partial N}(x_1,t_1) \le 0$. It follows that

$$\frac{\partial w}{\partial N}(x_1,t_1) - b(x_1,t_1)[g(u(x_1,t_1)) - g(v(x_1,t_1))] \le 0,$$

which contradicts the second inequality of the lemma. Therefore, we have $m > 0$. $\square$

Theorem 2.2. *Suppose that for positive values of s, $\varphi(s)$ is positive, increasing, convex and $\frac{\varphi'(s)}{g(s)}$ is decreasing. Suppose also that $\int^{+\infty}\frac{\varphi'(s)ds}{g(s)}<+\infty$ and there exist $k\geq 0$, $T_*>0$ such that*

$$f(s)\leq kg(s) \quad \text{for} \quad s>0$$

and

$$\int_0^{T_*}[-k\int_\Omega a(x,t)\,dx+\int_{\partial\Omega}b(x,t)\,dS_x]\,dt>\int_\Omega\int_{u_o(x)}^{+\infty}\frac{\varphi'(s)ds}{g(s)}\,dx.$$

Then any solution u of the problem (1.1)–(1.3) blows up in a finite time for $u_o(x)>0$.

Proof: Let $(0,T)$ be the maximum time interval in which the solution u of (1.1)–(1.3) exists. Our aim in this proof is to show that T is finite. Since $u_o(x)>0$ in Ω, from the maximum principle we have $u(x,t)>0$ in $\Omega\times(0,T)$. Put

$$v(x,t)=F(u(x,t))=\int_u^{+\infty}\frac{\varphi'(s)ds}{g(s)}. \tag{2.1}$$

The function v is well defined because $\int^{+\infty}\frac{\varphi'(s)ds}{g(s)}<\infty$. Moreover, for positive values of u, the function $F(u)$ is positive and decreasing. We have

(2.2)
$$\frac{\partial v}{\partial t}-\frac{1}{\varphi'(u)}Lv=-\frac{1}{g(u)}((\varphi(u))_t-Lu)+\frac{1}{\varphi'(u)}\frac{d}{du}(\frac{\varphi'(u)}{g(u)})\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}.$$

Since $\varphi(u)$ is increasing and $\frac{\varphi'(u)}{g(u)}$ is decreasing, from (1.1) and (2.2) we obtain

$$\frac{\partial v}{\partial t}-\frac{1}{\varphi'(u)}Lv-a(x,t)\frac{f(u)}{g(u)}\leq 0 \quad \text{in} \quad \Omega\times(0,T). \tag{2.3}$$

From (1.2) and (2.1), we also have

$$\frac{\partial v}{\partial N}=-\frac{\varphi'(u)}{g(u)}\frac{\partial u}{\partial N}=-b(x,t)\varphi'(u) \quad \text{on} \quad \partial\Omega\times(0,T). \tag{2.4}$$

Put

$$w(t)=\int_\Omega v(x,t)\,dx. \tag{2.5}$$

From (2.3) and (2.5), we get

$$w'(t) = \int_\Omega v_t(x,t)\,dx \le \int_\Omega [\frac{1}{\varphi'(u)} Lv(x,t) + a(x,t)\frac{f(u)}{g(u)}]\,dx. \tag{2.6}$$

Using Green's formula, (2.4) and (2.6), we obtain

$$\begin{aligned} w'(t) \le & -\int_{\partial\Omega} b(x,t)\,dS_x \\ & - \int_\Omega \frac{\varphi''(u)\varphi'(u)}{(\varphi'(u))^2 g(u)} \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx + \int_\Omega a(x,t)\frac{f(u)}{g(u)}\,dx. \end{aligned} \tag{2.7}$$

Since by hypotheses $f(u) \le kg(u)$ and $\varphi(u)$ is increasing and convex, from (2.7) it follows that

$$w'(t) \le -\int_{\partial\Omega} b(x,t)\,dS_x + k\int_\Omega a(x,t)\,dx. \tag{2.8}$$

Integrating (2.8) over $(0,s)$, we deduce that

$$w(s) \le w(0) + \int_0^s [-\int_{\partial\Omega} b(x,t)\,dS_x + k\int_\Omega a(x,t)\,dx]\,dt. \tag{2.9}$$

Since $v(x,t)$ is nonnegative and defined in $\Omega\times(0,T)$, then in virtue of (2.5), $w(t)$ is also nonnegative and defined for every $t \in (0,T)$. This implies that $T \le T_* < \infty$. In fact, if $T_* < T$ then by hypothesis, we have

$$w(T_*) \le \int_0^{T_*} [-\int_{\partial\Omega} b(x,t)\,dS_x + k\int_\Omega a(x,t)\,dx]\,dt + w(0) < 0,$$

which is a contradiction. Therefore u blows up in a finite time, which yields the result. □

Corollary 2.3. *Suppose that* $f(u) = 0$, $\int^{+\infty} \frac{\varphi'(z)}{g(z)}\,dz < +\infty$ *and for positive values of* s, $\varphi(s)$ *is positive, increasing, convex and* $\frac{\varphi'(s)}{g(s)}$ *is decreasing. Suppose also that there exists* $T_* > 0$ *such that*

$$\int_0^{T_*}\int_{\partial\Omega} b(x,t)\,dx\,dt > \int_\Omega \int_{u_o(x)}^{+\infty} \frac{\varphi'(s)ds}{g(s)}\,dx.$$

Then any solution u *of the problem* (1.1)–(1.3) *blows up in a finite time for* $u_o(x) > 0$.

Corollary 2.4. *Suppose that* $\int^{+\infty} \frac{\varphi'(z)}{g(z)}\,dz < +\infty$ *and for positive values of* s, $f(s) = g(s)$, $\varphi(s)$ *is positive, increasing, convex and* $\frac{\varphi'(s)}{g(s)}$ *is decreasing. Suppose also that there exists* $T_* > 0$ *such that*

$$\int_0^{T_*} [-\int_\Omega a(x,t)\,dx + \int_{\partial\Omega} b(x,t)\,dS_x]\,dt > \int_\Omega \int_{u_o(x)}^{+\infty} \frac{\varphi'(s)ds}{g(s)}\,dx.$$

Then any solution u *of the problem (1.1)–(1.3) blows up in a finite time for* $u_o(x) > 0$.

Corollary 2.5. *Suppose that* $\varphi(u) = u^m$, $f(u) = u^p$, $g(u) = u^q + u^s$ *where* $q \geq p \geq s \geq m-1$ *and* $q > m \geq 1$. *Suppose also that there exists* $T_* > 0$ *such that*

$$\int_0^{T_*} [-\int_\Omega a(x,t)\,dx + \int_{\partial\Omega} b(x,t)\,dS_x]\,dt > \int_\Omega \int_{u_o(x)}^{+\infty} \frac{\varphi'(s)ds}{g(s)}\,dx.$$

Then any solution u *of the problem (1.1)–(1.3) blows up in a finite time for* $u_o(x) > 0$. *If* $a(x,t) = a(x) \geq 0$, $b(x,t) = b(x) > 0$, *then the last hypothesis is satisfied when*

$$-\int_\Omega a(x)\,dx + \int_{\partial\Omega} b(x)\,dS_x > 0.$$

3. Global solutions

In this section, we give some conditions under which the solutions of the problem (1.1)–(1.3) exist globally and tend to zero as $t \to \infty$.

Theorem 3.1. *Suppose that* $0 \leq b(x,t) \leq b_o < \infty$, $0 < a(x,t) \leq a_o < \infty$, $c(x,t) \leq 0$, $d(x,t) \leq 0$, $f'(0) = g'(0) = 0$ *and* $\lim_{s\to 0} \frac{g(s)}{f(s)} \in \{0, \beta\}$ *where* β *is a positive constant. Suppose also that there exist a function* $\psi(x) > 0$ *and positive constants* A, B *such that*

$$-L_1\psi = -\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial^2\psi}{\partial x_i \partial x_j} - \sum_{i=1}^n a_i(x,t)\frac{\partial\psi}{\partial x_i} \geq -a(x,t) + A,$$

$$\frac{\partial\psi}{\partial N} \geq \varepsilon_g^{(f)} b(x,t) + B,$$

where $\varepsilon_g^{(f)} = 0$ *if* $\lim_{s\to 0} \frac{g(s)}{f(s)} = 0$ *and* $\varepsilon_g^{(f)} = \beta$ *if* $\lim_{s\to 0} \frac{g(s)}{f(s)} = \beta$. *Finally suppose that for positive values of* s, *the function* $\frac{f(s)}{\varphi'(s)}$ *is positive, increasing,* $\lim_{s\to 0} \frac{\varphi''(s)f(s)}{\varphi'(s)} = 0$ *and* $\int_0 \frac{\varphi'(z)dz}{f(z)} = \infty$. *Then there exists a positive function*

$v(x,t)$ continuous in $\overline{\Omega} \times [0,\infty[$ and tending to zero as $t \to \infty$ uniformly in $x \in \overline{\Omega}$ such that, if u is a solution of the problem (1.1)–(1.3), *the inequality $u(x,0) < v(x,t_o)$ $(t_o \geq 0)$ implies that $u(x,t) < v(x,t+t_o)$ and*

$$\lim_{t\to\infty} \sup_{x\in\Omega} u(x,t) = 0.$$

Remark 3.2. We have

$$\lim_{s\to 0}\{\varepsilon_g^{(f)} - \frac{g(s)}{f(s)}\} = 0.$$

Proof of Theorem 3.1: Put $v(x,t) = \alpha(t) + \psi(x)f(\alpha(t))$ with

$$\varphi^{'}(\alpha(t))\alpha^{'}(t) = -\lambda f(\alpha(t)),\ \alpha(0) = 1, \tag{3.1}$$

where $\lambda = A - \delta$ and $\delta < A$ is a positive constant. Since $\int_0 \frac{\varphi^{'}(z)dz}{f(z)} = +\infty$, then the function $\alpha(t)$ is defined for $0 \leq t < \infty$ and $\lim_{t\to+\infty} \alpha(t) = 0$. In fact $\alpha(t)$ satisfies the following relation:

$$\int_{\alpha(t)}^{1} \frac{\varphi^{'}(s)ds}{f(s)} = \lambda t. \tag{3.2}$$

Suppose that there is a finite time T such that $\alpha(T) = 0$. But this contradicts (3.2), because $\int_0 \frac{\varphi^{'}(s)ds}{f(s)} = \infty$. Therefore, we have $\lim_{t\to\infty} \alpha(t) = 0$. We also have

$$\begin{gathered}(\varphi(v))_t - Lv + a(x,t)f(v) = \varphi^{'}(\alpha(t))\alpha^{'}(t)\\ +\varphi^{'}(\alpha(t))\alpha^{'}(t)f^{'}(\alpha(t))\psi(x) + \psi(x)f(\alpha(t))\varphi^{''}(z)\alpha^{'}(t)\\ +\psi^2(x)f(\alpha(t))f^{'}(\alpha(t))\alpha^{'}(t)\varphi^{''}(z)\\ -f(\alpha(t))L_1\psi(x) - c(x,t)v - d(x,t) + a(x,t)f(\alpha(t)) + a(x,t)\psi(x)f(\alpha(t))f^{'}(y),\\ \frac{\partial v}{\partial N} - b(x,t)g(v) = \frac{\partial \psi(x)}{\partial N}f(\alpha(t)) - b(x,t)g(\alpha(t)) - b(x,t)\psi(x)f(\alpha(t))g^{'}(\tilde{y}),\end{gathered}$$

with $\{y,\tilde{y},z\} \in [\alpha(t), \alpha(t)+\psi(x)f(\alpha(t))]$. Since $\alpha^{'}(s) = -\lambda\frac{f(s)}{\varphi^{'}(s)}$ is a decreasing function, $c(x,t) \leq 0$, $d(x,t) \leq 0$ and $\psi > 0$ satisfies the following inequalities

$$-\lambda - L_1\psi \geq -a(x,t) + \delta, \qquad \frac{\partial\psi}{\partial N} \geq \varepsilon_g^{(f)} b(x,t) + B,$$

we obtain

$$(\varphi(v))_t - Lv + a(x,t)f(v) \geq \delta f(\alpha(t))$$
$$-\lambda f(\alpha(t))f^{'}(\alpha(t))\psi(x) - \lambda\psi(x)f(\alpha(t))|\varphi^{''}(z)|\frac{f(z)}{\varphi^{'}(z)}$$
$$-\lambda\psi^2(x)f(\alpha(t))f^{'}(\alpha(t))\frac{f(z)}{\varphi^{'}(z)}|\varphi^{''}(z)| + a(x,t)\psi(x)f(\alpha(t))f^{'}(y),$$
$$\frac{\partial v}{\partial N} - b(x,t)g(v) \geq (B + \varepsilon_g^{(f)}b(x,t))f(\alpha(t)) - b(x,t)g(\alpha(t))$$
$$- b(x,t)\psi(x)f(\alpha(t))g^{'}(\tilde{y}).$$

Since $f^{'}(0) = g^{'}(0) = 0$, $\lim_{s\to 0}\frac{\varphi^{''}(s)f(s)}{\varphi^{'}(s)} = 0$, by Remark 3.2 there exists $t_1 \geq 0$ such that

$$(\varphi(v))_t - Lv + a(x,t)f(v) > 0 \quad \text{in} \quad \Omega\times(t_1,\infty), \tag{3.3}$$
$$\frac{\partial v}{\partial N} - b(x,t)g(v) > 0 \quad \text{on} \quad \partial\Omega\times(t_1,\infty). \tag{3.4}$$

Then if $u(x,0) < v(x,t_1)$, by Comparison lemma 2.1, we deduce that

$$\lim_{t\to\infty}\sup_{x\in\Omega} u(x,t) = 0$$

because $\lim_{t\to\infty} v(x,t) = 0$ uniformly in $x \in \overline{\Omega}$. $\square$

Corollary 3.3. *Suppose that* $Lu = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + c(x,t)u + d(x,t)$, $f^{'}(0) = g^{'}(0) = 0$, $\lim_{s\to 0}\frac{g(s)}{f(s)} \in \{0,\beta\}$ *where* β *is a positive constant. Suppose also that for positive values of* s, *the function* $\frac{f(s)}{\varphi^{'}(s)}$ *is positive, increasing,* $\lim_{s\to 0}\frac{\varphi^{''}(s)f(s)}{\varphi^{'}(s)} = 0$ *and* $\int_0 \frac{\varphi^{'}(z)dz}{f(z)} = \infty$. *Finally suppose that* $0 \leq b(x,t) \leq b_o(x)$, $0 < a_o(x) \leq a(x,t)$, $c(x,t) \leq 0$, $d(x,t) \leq 0$, $-\varepsilon_g^{(f)}\int_{\partial\Omega} b_o(x)\,ds + \int_\Omega a_o(x)\,dx > 0$, *where* $\varepsilon_g^{(f)} = 0$ *if* $\lim_{s\to 0}\frac{g(s)}{f(s)} = 0$ *and* $\varepsilon_g^{(f)} = \beta$ *if* $\lim_{s\to 0}\frac{g(s)}{f(s)} = \beta$. *Then there exists a positive function* $v(x,t)$ *continuous in* $\overline{\Omega}\times[0,\infty[$ *and tending to zero as* $t\to\infty$ *uniformly in* $x\in\overline{\Omega}$ *such that, if* u *is a solution of the problem* (1.1)–(1.3), *the inequality* $u(x,0) < v(x,t_o)$ ($t_o \geq 0$) *implies that* $u(x,t) < v(x,t+t_o)$ *and*

$$\lim_{t\to\infty}\sup_{x\in\Omega} u(x,t) = 0.$$

PROOF: Let ψ be a positive solution of the following problem:

$$-\lambda - L_1\psi = \delta - a_o(x), \qquad \frac{\partial\psi}{\partial N} = \varepsilon_g^{(f)}b_o(x) + \delta,$$

where $L_1\psi = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial \psi}{\partial x_j})$. Taking

$$\lambda \leq \frac{1}{2(|\Omega| + |\partial\Omega|)}[-\varepsilon_g^{(f)} \int_{\partial\Omega} b_o(x)\, ds + \int_{\Omega} a_o(x)\, dx]$$

and putting

$$\delta = \frac{1}{|\Omega| + |\partial\Omega|}[-\varepsilon_g^{(f)} \int_{\partial\Omega} b_o(x)\, ds + \int_{\Omega} a_o(x)\, dx] - \lambda,$$

we see that the function ψ exists and $\delta > 0$. Take $A = \lambda + \delta$, $B = \delta$. Then all the hypotheses of Theorem 3.1 are satisfied, which yields the result. $\square$

Corollary 3.4. *Suppose that* $Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + c(x,t)u + d(x,t)$, $0 \leq b(x,t) \leq b_o(x)$, $0 < a_o(x) \leq a(x,t)$, $c(x,t) \leq 0$, $d(x,t) \leq 0$. *Suppose also that* $\varphi(u) = u^m$, $f(u) = u^p$, $g(u) = u^q$, $-\varepsilon_q^{(p)} \int_{\partial\Omega} b_o(x)\, ds + \int_{\Omega} a_o(x)\, dx > 0$ *with* $q \geq p > 1$, $p \geq m > 0$ *where* $\varepsilon_q^{(p)} = 0$ *if* $q > p$ *and* $\varepsilon_q^{(p)} = 1$ *if* $q = p$. *Then if* u *is a solution of the problem* (1.1)–(1.3), *there exists a positive constant* b *such that the solution* u *tends to zero as* $t \to \infty$ *uniformly in* $x \in \overline{\Omega}$ *for* $u_o(x) \leq b$.

4. Asymptotic behavior of solutions which tend to zero

In Section 3, we have shown that under some conditions, the solutions of the problem (1.1)–(1.3) tend to zero as $t \to \infty$ uniformly in $x \in \overline{\Omega}$. In this section, we describe the asymptotic behavior of these solutions in the case where $a(x,t) = a(x)$, $b(x,t) = b(x)$ and

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big).$$

Consider the following boundary value problem:

$$\frac{\partial \varphi(u)}{\partial t} - Lu + a(x)f(u) = 0 \quad \text{in} \quad \Omega \times (0,\infty), \tag{4.1}$$

$$\frac{\partial u}{\partial N} - b(x)g(u) = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \tag{4.2}$$

$$u(x,0) = u_o(x) > 0 \quad \text{in} \quad \Omega. \tag{4.3}$$

We are dealing with the asymptotic behavior as $t \to \infty$ of the solutions for the problem (4.1)–(4.3).

Theorem 4.1. *Suppose that* $f^{'}(0) = g^{'}(0) = 0$, $\lim_{s\to 0} \frac{g(s)}{f(s)} \in \{0, \beta\}$ *where* β *is a positive constant and for positive values of* s, *the function* $\frac{f(s)}{\varphi^{'}(s)}$ *is positive, increasing,* $\lim_{s\to 0} \frac{\varphi^{''}(s)f(s)}{\varphi^{'}(s)} = 0$ *and* $\int_0 \frac{\varphi^{'}(z)dz}{f(z)} = \infty$. *Suppose also that*

$-\varepsilon_g^{(f)} \int_{\partial\Omega} b(x)\,ds + \int_\Omega a(x)\,dx > 0$, *where* $\varepsilon_g^{(f)} = 0$ *if* $\lim_{s\to 0} \frac{g(s)}{f(s)} = 0$ *and* $\varepsilon_g^{(f)} = \beta$ *if* $\lim_{s\to 0} \frac{g(s)}{f(s)} = \beta$. *Then there exists a constant* $b > 0$ *such that, if* u *is a solution of the problem* (4.1)–(4.3), *we have*
(i)

$$\lim_{t\to\infty} u(x,t) = 0$$

uniformly in $x \in \overline{\Omega}$ *for* $u_o(x) \le b$.
(ii) *Moreover, if there exists a positive constant* c_1 *such that*

$$\lim_{s\to\infty} \frac{sf(H(s))}{\varphi'(H(s))H(s)} \le c_1,$$

we also have

$$u(x,t) = \alpha(t)(1+o(1)) \quad \text{as} \quad t\to\infty,$$

where $H(s)$ *is the inverse function of* $G(s) = \int_s^1 \frac{\varphi'(\sigma)d\sigma}{f(\sigma)}$ *and*

$$\varphi'(\alpha(t))\alpha'(t) = -c_{ab} f(\alpha(t)), \quad \alpha(0) = 1,$$

with $c_{ab} = \frac{1}{|\Omega|}[\int_\Omega a(x)\,dx - \varepsilon_g^{(f)} \int_{\partial\Omega} b(x)\,ds]$.

The proof of Theorem 4.1(i) is a direct consequence of Corollary 3.3, but that of Theorem 4.1(ii) is based on the following lemmas:

Lemma 4.2. *For any* $\varepsilon > 0$ *small enough, there exist* $\tau > 0$ *and* $t_1 > 0$ *such that*

$$u(x,t+\tau) \le \alpha_1^\varepsilon(t+t_1) + \psi_1(x) f(\alpha_1^\varepsilon(t+t_1)),$$

where $\alpha_1^\varepsilon(t)$ *satisfies the following equation:*

$$\varphi'(\alpha_1^\varepsilon(t))(\alpha_1^\varepsilon)'(t) = -(c_{ab} - \frac{\varepsilon}{2}) f(\alpha_1^\varepsilon(t)), \quad \alpha_1^\varepsilon(0) = 1,$$

and $\psi_1(x)$ *is a certain function.*

Proof: Put $v_1(x,t) = \alpha_1^\varepsilon(t) + \psi_1(x) f(\alpha_1^\varepsilon(t))$, where ψ_1 will be indicated later. We have

$$(\varphi(v_1))_t - Lv_1 + a(x)f(v_1) = \varphi'(\alpha_1^\varepsilon(t))(\alpha_1^\varepsilon)'(t)$$
$$+\varphi'(\alpha_1^\varepsilon(t))(\alpha_1^\varepsilon)'(t) f'(\alpha_1^\varepsilon(t))\psi_1(x) + \psi_1(x) f(\alpha_1^\varepsilon(t))\varphi''(z_1)(\alpha_1^\varepsilon)'(t)$$
$$+\psi_1^2(x) f(\alpha_1^\varepsilon(t)) f'(\alpha_1^\varepsilon(t))(\alpha_1^\varepsilon)'(t)\varphi''(z_1)$$
$$-f(\alpha_1^\varepsilon(t))L\psi_1(x) + a(x)f(\alpha_1^\varepsilon(t)) + a(x)\psi_1(x) f(\alpha_1^\varepsilon(t)) f'(y_1),$$
$$\frac{\partial v_1}{\partial N} - b(x)g(v_1) = \frac{\partial \psi_1(x)}{\partial N} f(\alpha_1^\varepsilon(t)) - b(x)g(\alpha_1^\varepsilon(t)) - b(x)\psi_1(x) f(\alpha_1^\varepsilon(t)) g'(\tilde{y}_1),$$

with $\{y_1, \tilde{y_1}, z_1\} \in [\alpha_1^\varepsilon(t), \alpha_1^\varepsilon(t) + \psi_1(x) f(\alpha_1^\varepsilon(t))]$. Let ψ_1 be a positive solution of the following problem:

$$-(c_{ab} - \frac{\varepsilon}{2}) - L\psi_1 = \delta - a(x), \qquad \frac{\partial \psi_1}{\partial N} = \varepsilon_g^{(f)} b(x) + \delta.$$

ψ_1 exists if and only if

$$\delta = \frac{1}{|\Omega| + |\partial\Omega|}[-\varepsilon_g^{(f)} \int_{\partial\Omega} b(x)\, ds + \int_\Omega a(x)\, dx] - \frac{|\Omega|}{|\Omega| + |\partial\Omega|}(c_{ab} - \frac{\varepsilon}{2}).$$

If $\varepsilon = 0$ then $\delta = 0$. Put

$$\delta(r) = \frac{1}{|\Omega| + |\partial\Omega|}[-\varepsilon_g^{(f)} \int_{\partial\Omega} b(x)\, ds + \int_\Omega a(x)\, dx] - \frac{|\Omega|}{|\Omega| + |\partial\Omega|}(c_{ab} - r).$$

We have $\delta'(0) > 0$. Then for any $\varepsilon > 0$ small enough, it follows that $\delta(\frac{\varepsilon}{2}) > 0$. Consequently, we obtain

$$(\varphi(v_1))_t - Lv_1 + a(x) f(v_1) \geq \delta f(\alpha_1^\varepsilon(t))$$

$$-(c_{ab} - \frac{\varepsilon}{2}) f(\alpha_1^\varepsilon(t)) f'(\alpha_1^\varepsilon(t)) \psi_1(x) - (c_{ab} - \frac{\varepsilon}{2}) \psi_1(x) f(\alpha_1^\varepsilon(t)) |\varphi''(z_1)| \frac{f(z_1)}{\varphi'(z_1)}$$

$$-(c_{ab} - \frac{\varepsilon}{2}) \psi_1^2(x) f(\alpha_1^\varepsilon(t)) f'(\alpha_1^\varepsilon(t)) \frac{f(z_1)}{\varphi'(z_1)} |\varphi''(z_1)| + a(x)\psi_1(x) f(\alpha_1^\varepsilon(t)) f'(y_1),$$

$$\frac{\partial v_1}{\partial N} - b(x) g(v_1) = (\delta + \varepsilon_g^{(f)} b(x)) f(\alpha_1^\varepsilon(t)) - b(x) g(\alpha_1^\varepsilon(t)) - b(x)\psi_1(x) f(\alpha_1^\varepsilon(t)) g'(\tilde{y_1}).$$

Since $f'(0) = g'(0) = 0$, $\lim_{s\to 0} \frac{\varphi''(s) f(s)}{\varphi'(s)} = 0$, by Remark 3.2 there exists $t_1 \geq 0$ such that

$$(\varphi(v_1))_t - Lv_1 + a(x) f(v_1) > 0 \quad \text{in} \quad \Omega \times (t_1, \infty),$$
$$\frac{\partial v_1}{\partial N} - b(x) g(v_1) > 0 \quad \text{on} \quad \partial\Omega \times (t_1, \infty).$$

Since $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $x \in \overline{\Omega}$, then there exists $\tau > 0$ such that

$$u(x, \tau) < v_1(x, t_1) \ \text{ in } \ \Omega.$$

By Comparison lemma 2.1, it follows that

$$u(x, t + \tau) \leq v_1(x, t + t_1) = \alpha_1^\varepsilon(t + t_1) + \psi_1(x) f(\alpha_1^\varepsilon(t + t_1)),$$

which yields the result. $\square$

Lemma 4.3. *For any $\varepsilon > 0$ small enough, there exists $t_2 > 0$ such that:*

$$u(x, t+\tau) \geq \alpha_2^{\varepsilon}(t+t_2) + \psi_2(x) f(\alpha_2^{\varepsilon}(t+t_2)),$$

where $\alpha_2^{\varepsilon}(t)$ satisfies the following equation:

$$\varphi^{'}(\alpha_2^{\varepsilon}(t))(\alpha_2^{\varepsilon})^{'}(t) = -(c_{ab} + \frac{\varepsilon}{2}) f(\alpha_2^{\varepsilon}(t)), \quad \alpha_2^{\varepsilon}(0) = 1,$$

and $\psi_2(x)$ is a certain function.

Proof: Put $v_2(x,t) = \alpha_2^{\varepsilon}(t) + \psi_2(x) f(\alpha_2^{\varepsilon}(t))$, where ψ_2 will be indicated later. We have

$$\begin{gathered}(\varphi(v_2))_t - Lv_2 + a(x) f(v_2) = \varphi^{'}(\alpha_2^{\varepsilon}(t))(\alpha_2^{\varepsilon})^{'}(t) \\ +\varphi^{'}(\alpha_2^{\varepsilon}(t))(\alpha_2^{\varepsilon})^{'}(t) f^{'}(\alpha_2^{\varepsilon}(t))\psi_2(x) + \psi_2(x) f(\alpha_2^{\varepsilon}(t))\varphi^{''}(z_2)(\alpha_2^{\varepsilon})^{'}(t) \\ +\psi_2^2(x) f(\alpha_2^{\varepsilon}(t)) f^{'}(\alpha_2^{\varepsilon}(t))(\alpha_2^{\varepsilon})^{'}(t)\varphi^{''}(z_2) \\ -f(\alpha_2^{\varepsilon}(t)) L\psi_2(x) + a(x) f(\alpha_2^{\varepsilon}(t)) + a(x)\psi_2(x) f(\alpha_2^{\varepsilon}(t)) f^{'}(y_2), \\ \frac{\partial v_2}{\partial N} - b(x) g(v_2) = \frac{\partial \psi_2(x)}{\partial N} f(\alpha_2^{\varepsilon}(t)) - b(x) g(\alpha_2^{\varepsilon}(t)) - b(x)\psi_2(x) f(\alpha_2^{\varepsilon}(t)) g^{'}(\tilde{y_2}),\end{gathered}$$

with $\{y_2, \tilde{y_2}, z_2\} \in [\alpha_2^{\varepsilon}(t), \alpha_2^{\varepsilon}(t) + \psi_2(x) f(\alpha_2^{\varepsilon}(t))]$. Let ψ_2 be a positive solution of the following problem:

$$-(c_{ab} + \frac{\varepsilon}{2}) - L\psi_2 = \mu - a(x), \quad \frac{\partial \psi_2}{\partial N} = \varepsilon_g^{(f)} b(x) + \mu.$$

ψ_2 exists if and only if

$$\mu = \frac{1}{|\Omega| + |\partial\Omega|}[-\varepsilon_g^{(f)} \int_{\partial\Omega} b(x)\, ds + \int_{\Omega} a(x)\, dx] - \frac{|\Omega|}{|\Omega| + |\partial\Omega|}(c_{ab} + \frac{\varepsilon}{2}).$$

Put

$$\mu(r) = \frac{1}{|\Omega| + |\partial\Omega|}[-\varepsilon_g^{(f)} \int_{\partial\Omega} b(x)\, ds + \int_{\Omega} a(x)\, dx] - \frac{|\Omega|}{|\Omega| + |\partial\Omega|}(c_{ab} + r).$$

Since $\mu(\frac{\varepsilon}{2}) = \delta(-\frac{\varepsilon}{2})$ and $\delta^{'}(0) > 0$, then for any $\varepsilon > 0$ small enough, it follows that $\mu(\frac{\varepsilon}{2}) < 0$. Therefore, we obtain

$$(\varphi(v_2))_t - Lv_2 + a(x) f(v_2) \leq \mu f(\alpha_2^{\varepsilon}(t))$$

$$-(c_{ab} + \frac{\varepsilon}{2}) f(\alpha_2^{\varepsilon}(t)) f^{'}(\alpha_2^{\varepsilon}(t))\psi_1(x) + (c_{ab} + \frac{\varepsilon}{2})\psi_2(x) f(\alpha_2^{\varepsilon}(t))|\varphi^{''}(z_2)|\frac{f(z_2)}{\varphi^{'}(z_2)}$$

$$+(c_{ab}+\frac{\varepsilon}{2})\psi_2^2(x)f(\alpha_2^\xi(t))f^{'}(\alpha_2^\xi(t))\frac{f(z_2)}{\varphi^{'}(z_2)}|\varphi^{''}(z_2)|+a(x)\psi_2(x)f(\alpha_2^\xi(t))f^{'}(y_2),$$

$$\frac{\partial v_2}{\partial N}-b(x)g(v_2)\le(\mu+\varepsilon_g^{(f)}b(x))f(\alpha_2^\xi(t))-b(x)g(\alpha_2^\xi(t))-b(x)\psi_2(x)f(\alpha_2^\xi(t))g^{'}(\tilde{y_2}).$$

Since $f^{'}(0)=g^{'}(0)=0$, $\lim_{s\to 0}\frac{\varphi^{''}(s)f(s)}{\varphi^{'}(s)}=0$, by Remark 3.2 there exists $t_*\ge 0$ such that

$$(\varphi(v_2))_t-Lv_2+a(x)f(v_2)<0 \quad \text{in} \quad \Omega\times(t_*,\infty),$$
$$\frac{\partial v_2}{\partial N}-b(x)g(v_2)<0 \quad \text{on} \quad \partial\Omega\times(t_*,\infty).$$

Since $\lim_{t\to\infty}v_2(x,t)=0$ uniformly in $x\in\overline{\Omega}$, there exists $t_2>t_*$ such that

$$u(x,\tau)>v_2(x,t_2) \ \text{ in } \ \Omega.$$

By Comparison lemma 2.1, we deduce that

$$u(x,t+\tau)\ge v_2(x,t+t_2)=\alpha_2^\xi(t+t_2)+\psi_2(x)f(\alpha_2^\xi(t+t_2)),$$

which gives the result. □

Proof of Theorem 4.1(ii): For any $\gamma>0$, we have

$$\lim_{t\to\infty}\frac{\alpha(\gamma+t)}{\alpha(t)}=1. \tag{4.4}$$

In fact, since $\alpha(t)$ is decreasing and convex, it follows that

$$\alpha(t)-\gamma c_{ab}\frac{f(\alpha(t))}{\varphi^{'}(\alpha(t))}\le\alpha(t+\gamma)\le\alpha(t).$$

Moreover, since $\lim_{t\to\infty}\frac{f(\alpha(t))}{\varphi^{'}(\alpha(t))\alpha(t)}=0$, we deduce that $\lim_{t\to\infty}\frac{\alpha(\gamma+t)}{\alpha(t)}=1$. On the other hand, if $\varepsilon>0$ is small enough, we obtain

$$1-\frac{c_1\varepsilon}{2c_{ab}}\le\liminf_{t\to\infty}\frac{\alpha_2^\xi(t)}{\alpha(t)}\le\limsup_{t\to\infty}\frac{\alpha_2^\xi(t)}{\alpha(t)}\le 1. \tag{4.5}$$

In fact

$$1\ge\frac{\alpha_2^\xi(t)}{\alpha(t)}=\frac{H(c_{ab}t+\frac{\varepsilon}{2}t)}{H(c_{ab}t)}\ge\frac{H(c_{ab}t)-\frac{\varepsilon}{2}t\frac{f(H(c_{ab}t))}{\varphi^{'}(H(c_{ab}t))}}{H(c_{ab}t)}.$$

Since $\lim_{s\to\infty}\frac{sf(H(s))}{\varphi^{'}(H(s))H(s)}\le c_1$, we obtain the result. We also have

$$1\le\liminf_{t\to\infty}\frac{\alpha_1^\xi(t)}{\alpha(t)}\le\limsup_{t\to\infty}\frac{\alpha_1^\xi(t)}{\alpha(t)}\le 1+\frac{2c_1\varepsilon}{c_{ab}}. \tag{4.6}$$

In fact

$$1 \leq \liminf_{t\to\infty} \frac{\alpha_1^{\varepsilon}(t)}{\alpha(t)} \leq \limsup_{t\to\infty} \frac{\alpha_1^{\varepsilon}(t)}{\alpha(t)} \leq \frac{1}{1-\frac{c_1\varepsilon}{2(c_{ab}-\frac{\varepsilon}{2})}} \leq 1+\frac{2c_1\varepsilon}{c_{ab}}\,.$$

Then from (4.4)–(4.6), Lemmas 4.2 and 4.3, we deduce that for any $\varepsilon > 0$ small enough

$$1-k_1\varepsilon \leq \liminf_{t\to\infty} \frac{u(x,t)}{\alpha(t)} \leq \limsup_{t\to\infty} \frac{u(x,t)}{\alpha(t)} \leq 1+k_2\varepsilon$$

where k_1 and k_2 are two positive constants. Consequently,

$$u(x,t) = \alpha(t)(1+o(1)) \quad \text{as} \quad t\to\infty,$$

which gives the result. □

Remark 4.4. Let $\varphi(u) = u^m$, $f(u) = u^p$, $g(u) = u^q$ with $p \geq m > 0$, $q \geq p > 1$. Suppose also that $\int_\Omega a(x)\,dx - \varepsilon_q^{(p)} \int_{\partial\Omega} b(x)\,ds > 0$ where $\varepsilon_q^{(p)} = 0$ if $q > p$ and $\varepsilon_q^{(p)} = 1$ if $q = p$. Then there exists a positive constant b such that, if u is a solution of the problem (4.1)–(4.3), u tends to zero as $t \to \infty$ uniformly in $x \in \overline{\Omega}$ for $u_o(x) \leq b$. Moreover,

$$\lim_{t\to\infty} \frac{u(x,t)}{t^{-\frac{1}{p-m}}} = \left(\frac{p-m}{m|\Omega|}\Big[\int_\Omega a(x)\,dx - \varepsilon_q^{(p)} \int_{\partial\Omega} b(x)\,ds\Big]\right)^{\frac{1}{m-p}}.$$

5. Asymptotic behavior near the blow-up time

In this section, we give another condition under which the solutions of the problem (1.1)–(1.3) blow up in a finite time in the case where

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i}\,.$$

We also give the asymptotic behavior near the blow-up time of these solutions.

Theorem 5.1. *Suppose that $a_t(x,t) \leq 0$, $b_t(x,t) \geq 0$. Suppose also that there exists a function $F(s)$ such that $\int^\infty \frac{ds}{F(s)} < \infty$ and for positive values of s, $F(s)$ is positive, increasing, convex satisfying*

$$-f'(s)F(s) + F'(s)f(s) \geq 0 \quad \text{for} \quad s > 0,$$
$$-F'(s)g(s) + F(s)g'(s) \geq 0 \quad \text{for} \quad s > 0.$$

Finally, suppose that $Lu_o(x) + a(x,0)f(u_o(x)) > 0$ and for positive values of s, $\varphi(s)$ is concave. Then any solution u of the problem (1.1)–(1.3) blows up in a *finite time T and there exists a positive constant δ such that the following estimate holds*

$$\sup_{x\in\overline{\Omega}} u(x,t) \leq H(\delta(T-t)),$$

where $H(s)$ is the inverse function of $G(s) = \int_s^\infty \frac{d\sigma}{F(\sigma)}$.

PROOF: Let $(0,T)$ be the maximum time interval in which the solution u of the problem (1.1)–(1.3) exists. Our aim is to show that T is finite and the above estimate holds. Since $u_o(x) > 0$ in Ω, from the maximum principle, we have $u(x,t) \geq 0$ in $\Omega\times(0,T)$. Let $w = u_t$. Since $w(x,0) = Lu_o(x) - a(x,0)f(u_o(x)) > 0$, $a_t(x,t) \leq 0$, $b_t(x,t) \geq 0$, we obtain

$$(\varphi'(u)w)_t - Lw \geq -a(x,t)f'(u)w \quad \text{in} \quad \Omega\times(0,T), \tag{5.1}$$

$$\frac{\partial w}{\partial N} \geq b(x,t)g'(u)w \quad \text{on} \quad \partial\Omega\times(0,T), \tag{5.2}$$

$$w(x,0) > 0 \quad \text{in} \quad \Omega. \tag{5.3}$$

From the maximum principle, there exists a constant $c > 0$ such that

$$u_t(x,t) \geq c \quad \text{in} \quad \Omega\times(\varepsilon_o, T) \tag{5.4}$$

for $\varepsilon_o > 0$. Consider the following function:

$$J(x,t) = u_t - \delta F(u), \tag{5.5}$$

where $\delta > 0$ small enough will be indicated later. We have

$$\begin{aligned}
&(\varphi'(u)J)_t - LJ \\
&= ((\varphi(u))_t - Lu)_t - \delta F'(u)((\varphi(u))_t - Lu) \\
&\quad + \delta F''(u)\sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_{x_j} - \delta\varphi''(u)F(u)u_t \\
&= -a(x,t)f'(u)J - a_t(x,t)f(u) + \delta a(x,t)[F'(u)f(u) - F(u)f'(u)] \\
&\quad + \delta F''(u)\sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_{x_j} - \delta\varphi''(u)F(u)u_t,
\end{aligned} \tag{5.6}$$

$$\frac{\partial J}{\partial N} = b_t(x,t)g(u) + b(x,t)g'(u)J + \delta b(x,t)[g'(u)F(u) - F'(u)g(u)].$$

Since $a_t \leq 0$, $b_t \geq 0$, $u_t \geq 0$ and for positive values of u, $F''(u)$, $-\varphi''(u)$, $-f'(u)F(u) + F'(u)f(u)$ and $g'(u)F(u) - F'(u)g(u)$ are nonnegative by hypotheses, we obtain

$$(\varphi'(u)J)_t - LJ + a(x,t)f'(u)J \geq 0 \quad \text{in} \quad \Omega\times(0,T), \tag{5.7}$$

$$\frac{\partial J}{\partial N} \geq b(x,t)g'(u)J \quad \text{on} \quad \partial\Omega\times(0,T). \tag{5.8}$$

From (5.4) and (5.5), take δ so small that

$$J(x,\varepsilon_o) > 0 \quad \text{in} \quad \Omega. \tag{5.9}$$

Therefore, from the maximum principle, we deduce that

$$u_t \geq \delta F(u) \quad \text{in} \quad \Omega \times (\varepsilon_o, T), \tag{5.10}$$

that is

$$-(G(u))_t = \frac{u_t}{F(u)} \geq \delta. \tag{5.11}$$

Integrating (5.11) over (ε_o, T) we have

$$G(u(x,\varepsilon_o)) \geq G(u(x,\varepsilon_o)) - G(u(x,T)) \geq \delta(T-\varepsilon_o). \tag{5.12}$$

Therefore T is finite and u blows up in a finite time. Integrating again (5.11) over (t,T), we see that

$$G(u(x,t)) \geq G(u(x,t)) - G(u(x,T)) \geq \delta(T-t). \tag{5.13}$$

Since the inverse function H of G is decreasing, from (5.13) we obtain

$$u(x,t) \leq H[\delta(T-t)],$$

which yields the result. □

Corollary 5.2. *Suppose that $a_t \leq 0$, $b_t \geq 0$, $\varphi(u) = u^m$, $f(u) = u^p$, $g(u) = u^q$, $Lu_o - a(x,0)u_o^p > 0$ where $q > 1 \geq m > 0$, $q \geq p > 0$. Then any solution u of the problem* (1.1)–(1.3) *blows up in a finite time T and there exists a positive constant c_2 such that*

$$\sup_{x\in\overline{\Omega}} u(x,t) \leq \frac{c_2}{(T-t)^{\frac{1}{q-m}}}.$$

Remark 5.3. The argument in the proof of Theorem 5.1 is a classical one. It was introduced in [4] and later used and modified by many authors. Unfortunately, this method does not yield optimal results if blow-up occurs on the boundary. More precisely, it is known that Corollary 5.2 is not sharp if $m = 1$. In this case, the blow-up rate is

$$(T-t)^{-\frac{1}{2(q-1)}},$$

see [6].

6. Blow-up set

In this section, we describe the blow-up set of some blow-up solutions for the problem (1.1)–(1.3). More precisely, we show that under some conditions, certain solutions of the problem (1.1)–(1.3) blow up in a finite time and their blow-up set is on the boundary $\partial\Omega$ of the domain Ω.

Theorem 6.1. *Suppose that the hypotheses of Theorem 5.1 are satisfied. Suppose also that there are positive constants C_o, c_o such that*

$$\varphi^{'}(s) \geq c_o \quad \text{for} \quad s > 0 \quad \text{and} \quad sF^{'}(H(s)) \leq C_o \quad \text{for} \quad s > 0.$$

Then any solution u of the problem (1.1)–(1.3) *blows up in a finite time T and $E_B \subset \partial\Omega$, where E_B is the blow-up set of the solution u.*

Remark 6.2. If $F(s) = s^q$ with $q > 1$, then we may take $C_o = \frac{q}{q-1}$.

PROOF: By Theorem 5.1, we know that u blows up in a finite time T. Thus our aim in this proof is to show that $E_B \subset \partial\Omega$. Let $d(x) = \operatorname{dist}(x, \partial\Omega)$ and $v(x) = d^2(x)$ for $x \in N_\varepsilon(\partial\Omega)$ where

$$N_\varepsilon(\partial\Omega) = \{x \in \Omega \quad \text{such that} \quad d(x) < \varepsilon\}.$$

Since $\partial\Omega$ is of class C^2, then the function $v(x) \in C^2(\overline{N_\varepsilon(\partial\Omega)})$ if ε is sufficiently small. On $\partial\Omega$, we have

$$Lv - \frac{C_o}{v} \sum_{i,j=1}^{n} a_{ij}(x) v_{x_i} v_{x_j}$$

$$\begin{aligned}
&= 2\sum_{i,j=1}^{n} a_{ij}(x) d_{x_i} d_{x_j} + 2d \sum_{i,j=1}^{n} d_{x_i x_j} + 2d \sum_{i=1}^{n} a_i(x) d_{x_i} - 4C_o \sum_{i,j=1}^{n} a_{ij}(x) d_{x_i} d_{x_j} \\
&\geq 2\lambda_1 - 2\sum_{i=1}^{n} |a_{ii}(x)| - 2d \sum_{i=1}^{n} |a_i(x)| - 4C_o\lambda_2 - 4\lambda_2 \\
&\geq 2\lambda_1 - 2C_1 - 2d^{'} C_2 - 4C_o\lambda_2 - 4\lambda_2
\end{aligned}$$

where $d^{'} = \sup_{x\in\overline{\Omega}, y\in\overline{\Omega}} \|x - y\|$. Therefore, there exists a positive constant C_1 such that

$$Lv - \frac{C_o}{v} \sum_{i,j=1}^{n} a_{ij}(x) v_{x_i} v_{x_j} \geq -C_1 \quad \text{on} \quad \partial\Omega. \tag{6.1}$$

Since $v \in C^2(\overline{N_\varepsilon(\partial\Omega)})$ for ε sufficiently small, let ε_o be so small that

$$Lv - \frac{C_o}{v} \sum_{i,j=1}^{n} a_{ij}(x) v_{x_i} v_{x_j} \geq -2C_1 \quad \text{in} \quad \overline{N_{\varepsilon_o}(\partial\Omega)}. \tag{6.2}$$

We extend v to a function of class $C^2(\overline{\Omega})$ such that $v \geq C_o^* > 0$ in $\overline{\Omega - N_{\varepsilon_o}(\partial\Omega)}$. Therefore, we deduce that

$$Lv - \frac{C_o}{v}\sum_{i,j=1}^{n} a_{ij}(x)v_{x_i}v_{x_j} \geq -C^* \quad \text{in} \quad \overline{\Omega} \tag{6.3}$$

for some $C^* > 0$. Multiplying (6.3) by ϵ small enough, we may assume without loss of generality that $C^* < 1$. Put $w_*(x,t) = C_1 H(\tau)$ where $\tau = \delta(v(x) + \frac{C^*}{c_o}(T-t))$ and $C_1 > 1$ is a constant which will be indicated later. We get

$$(\varphi(w_*))_t - Lw_* \geq -\delta C_1 H^{'}(\tau)[C^* + Lv + \delta\frac{H^{''}(\tau)}{H^{'}(\tau)}\sum_{i,j=1}^{n} a_{ij}(x)v_{x_i}v_{x_j}]. \tag{6.4}$$

Since $H(s)$ is the inverse function of $G(s)$, we have $H^{'}(s) = -F(H(s))$ and $H^{''}(s) = -H^{'}(s)F^{'}(H(s))$. Consequently,

$$(\varphi(w_*))_t - Lw_* \geq \delta C_1 F(H(s))[C^* + Lv - \delta F^{'}(H(\tau))\sum_{i,j=1}^{n} a_{ij}(x)v_{x_i}v_{x_j}]. \tag{6.5}$$

Since $sF^{'}(H(s)) \leq C_o$ for $s > 0$, using the fact that $F^{'}(H(s))$ is a decreasing function ($F^{'}$ is increasing and H is decreasing), we have

$$(\varphi(w_*))_t - Lw_* \geq \delta C_1 F(H(\tau))[C^* + Lv - \frac{C_o}{v}\sum_{i,j=1}^{n} a_{ij}(x)v_{x_i}v_{x_j}]. \tag{6.6}$$

Therefore from (6.3), we deduce that

$$(\varphi(w_*))_t - Lw_* + a(x,t)f(w_*) \geq 0 \quad \text{in} \quad \Omega \times (\varepsilon_o, T). \tag{6.7}$$

On $\partial\Omega$, we have $w_*(x,t) = C_1 H(\delta C^*(T-t)) > H(\delta(T-t))$ because $C_1 > 1$ and $C^* < 1$. Then by Theorem 5.1, we obtain

$$w_*(x,t) > u(x,t) \quad \text{on} \quad \partial\Omega \times (\varepsilon_o, T). \tag{6.8}$$

Choose C_1 large enough that

$$w_*(x,\varepsilon_o) = C_1 H(\delta(v(x) + C^*(T-\varepsilon_o))) > u(x,\varepsilon_o). \tag{6.9}$$

Consequently, from the maximum principle we deduce that

$$u(x,t) < w_*(x,t) \quad \text{in} \quad \Omega \times (\varepsilon_o, T).$$

Then if $\Omega^{'} \subset\subset \Omega$ we have

$$u(x,t) \leq C_1 H(\delta(v(x) + C^*(T-t))) \leq C_1 H(\delta v(x)).$$

It follows that

$$\sup_{x\in\Omega^{'}, t\in[\varepsilon_o,T)} u(x,t) \leq \sup_{x\in\Omega^{'}} C_1 H(\delta v(x)) < \infty,$$

which yields the result. $\square$

Corollary 6.2. *Suppose that* $a_t \leq 0$, $b_t \geq 0$, $\varphi(u) = au + bu^m$, $f(u) = u^p$, $g(u) = u^q$, $Lu_o - a(x,0)u_o^p > 0$ *where* $a > 0$, $b \geq 0$, $q > 1 \geq m > 0$, $q \geq p > 0$. *Then any solution* u *of the problem* (1.1)–(1.3) *blows up in a finite time and* $E_B \subset \partial\Omega$ *where* E_B *is the blow-up set of the solution* u.

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