Vector integral equations with discontinuous right-hand side

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Abstract. We deal with the integral equation $u(t) = f(\int_I g(t,z)\,u(z)\,dz)$, with $t \in I = [0,1]$, $f : \mathbf{R}^n \to \mathbf{R}^n$ and $g : I \times I \to [0,+\infty[$. We prove an existence theorem for solutions $u \in L^\infty(I,\mathbf{R}^n)$ where the function f is not assumed to be continuous, extending a result previously obtained for the case $n = 1$.

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1. Introduction

Let $I := [0,1]$. Consider the integral equation

$$u(t) = f\Big(\int_I g(t,z)\,u(z)\,dz\Big) \quad \text{for a.a.} \quad t \in I, \tag{1}$$

where $f : \mathbf{R} \to \mathbf{R}$ and $g : I \times I \to [0,+\infty[$ are given functions. Recently, in the paper [4], the authors proved an existence theorem for solutions of (1) in the space $L^\infty(I,\mathbf{R})$, where, unlike other recent results in the field (see [3], [5], [6], [7], to which we also refer for motivations for studying equation (1)), the continuity of f was not assumed. More precisely, f was assumed to be a.e. equal in a suitable interval $[0,\sigma]$ to a function $f_0 : [0,\sigma] \to \mathbf{R}$ such that the set $\{x \in [0,\sigma] : f_0$ is discontinuous at $x\}$ has null 1-dimensional Lebesgue measure. Consequently, a function f satisfying the assumptions of [4] can be discontinuous at each point of its domain.

In this note we are interested in the study of equation (1) in the more general case where $f : \mathbf{R}^n \to \mathbf{R}^n$. We prove an existence result for solutions $u \in L^\infty(I,\mathbf{R}^n)$ which, in the explicit case, extends the main result of [4]. In particular, the above assumption on f is extended by assuming that there exist a function $\overline{f} : \prod_{i=1}^n [0,\sigma_i] \to \mathbf{R}^n$ (with suitable positive σ_i) and subsets $E_1,\dots,E_n$ of $\prod_{i=1}^n [0,\sigma_i]$ such that the projection of each E_i over the i-th axis has null 1-dimensional Lebesgue measure and

$$\{x \in \prod_{i=1}^n [0,\sigma_i] : \overline{f} \text{ is discontinuous at } x\} \cup \{x \in \prod_{i=1}^n [0,\sigma_i] : \overline{f}(x) \neq f(x)\} \subseteq \bigcup_{i=1}^n E_i.$$

* Born on August 4, 1968. This clarification is needed because of a complete coincidence of names within the same Department.

We also prove by a counterexample that the set $\bigcup_{i=1}^{n} E_i$ cannot be replaced by an arbitrary set $E \subseteq \prod_{i=1}^{n} [0, \sigma_i]$ with null n-dimensional Lebesgue measure.

2. Preliminaries

Let $n \in \mathbf{N}$. We shall denote by m_n the n-dimensional Lebesgue measure in the space $\mathbf{R}^n$. If $x \in \mathbf{R}^n$, then x_i shall denote the i-th component of x. Moreover, we shall denote by $p_i : \mathbf{R}^n \to \mathbf{R}$ the projection over the i-th axis, namely we put $p_i(x) = x_i$.

If $x, y \in \mathbf{R}^n$, we say that $x < y$ (resp., $x \leq y$) if and only if one has $x_i < y_i$ (resp., $x_i \leq y_i$) for each $i = 1, \dots n$. If $x, y \in \mathbf{R}^n$, with $x \leq y$, we put

$$[x, y] := \prod_{i=1}^{n} [x_i, y_i],$$

$$]x, y[:= \prod_{i=1}^{n}]x_i, y_i[\qquad (\text{if } x < y).$$

We shall denote by 0_n the origin of the space $\mathbf{R}^n$, which, in turn, will be considered with its Euclidean norm $\|\cdot\|_n$.

If $x \in \mathbf{R}^n$, $\varepsilon > 0$, $A \subseteq \mathbf{R}^n$, $A \neq \emptyset$, we put

$$B(x, \varepsilon) := \{y \in \mathbf{R}^n : \|x - y\|_n < \varepsilon\},$$

$$d(x, A) := \inf_{v \in A} \|x - v\|_n.$$

Moreover, we shall denote by $\overline{\text{co}}\, A$ the closed convex hull of A.

If $p \in [1, +\infty]$, we shall denote by $L^p(I, \mathbf{R}^n)$ the space of all (equivalence classes of) measurable functions $u : I \to \mathbf{R}^n$ such that

$$\int_I \|u(t)\|_n^p \, dt < +\infty \qquad \text{if } \ p < +\infty,$$

$$\operatorname*{ess\,sup}_{t \in I} \|u(t)\|_n < +\infty \qquad \text{if } \ p = +\infty,$$

with the usual norm

$$\|u\|_{L^p(I, \mathbf{R}^n)} := \left(\int_I \|u(t)\|_n^p \, dt \right)^{\frac{1}{p}} \qquad \text{if } \ p < +\infty,$$

$$\|u\|_{L^\infty(I, \mathbf{R}^n)} := \operatorname*{ess\,sup}_{t \in I} \|u(t)\|_n \qquad \text{if } \ p = +\infty.$$

We shall denote by $\mathcal{B}(I, \mathbf{R}^n)$ the set of all $u \in L^\infty(I, \mathbf{R}^n)$ for which there exists some function $v : I \to \mathbf{R}^n$ such that $u(t) = v(t)$ a.e. in I and also

$$m_1(\{t \in I : v \ \text{ is discontinuous at } t\}) = 0.$$

Moreover, we shall put $L^p(I) := L^p(I, \mathbf{R})$. As usual, we denote by $C^0(I, \mathbf{R}^n)$ the space of all continuous functions $v : I \to \mathbf{R}^n$.

For the definitions and basic facts about multifunctions, we refer to [2], [11]. Finally, we put $I_0 :=]0, 1[$.

3. The result

The following is our result.

Theorem 1. *Let $\alpha, \beta, \sigma \in \mathbf{R}^n$, with $0_n < \alpha < \beta$ and $0_n < \sigma$. Let $f : [0_n, \sigma] \to \mathbf{R}^n$ and $g : I \times I \to [0, +\infty[$ be given functions. Assume that:*

(i) *for each $i = 1, \dots, n$, one has*

$$\alpha_i < \operatorname*{ess\,inf}_{x \in [0_n,\sigma]} f_i(x) \leq \operatorname*{ess\,sup}_{x \in [0_n,\sigma]} f_i(x) < \beta_i\,;$$

(ii) *there exist sets $E_1, \dots, E_n \subseteq [0_n, \sigma]$, with $m_1(p_i(E_i)) = 0$ for all $i = 1, \dots, n$, and a function $\overline{f} : [0_n, \sigma] \to \mathbf{R}^n$ such that for each $x \in [0_n, \sigma] \setminus (\bigcup_{i=1}^n E_i)$ one has $\overline{f}(x) = f(x)$ and $\overline{f}$ is continuous at x;*

(iii) *for each $t \in I$, the function $g(t, \cdot\,)$ is measurable.*

Moreover, assume that there exist $\phi_0 \in L^j(I)$, with $j > 1$ and

$$0 < \|\phi_0\|_{L^1(I)} \leq \min_{1 \leq i \leq n} \frac{\sigma_i}{\beta_i}\,,$$

and $\phi_1 \in L^1(I)$ such that:

(iv) *for a.a. $z \in I$, the function $g(\cdot\,, z)$ is continuous in I, differentiable in I_0 and*

$$g(t,z) \leq \phi_0(z), \quad 0 < \frac{\partial g}{\partial t}(t,z) \leq \phi_1(z) \ \text{ for all } \ t \in I_0.$$

Then there exists $u \in \mathcal{B}(I, \mathbf{R}^n)$ such that

$$u(t) = f\Big(\int_I g(t,z)\,u(z)\,dz\Big) \ \text{ for a.a. } \ t \in I. \tag{2}$$

Before proving Theorem 1, we need the following preliminary result.

Lemma 1. *Let $\sigma, \gamma, \delta \in \mathbf{R}^n$, with $0_n < \sigma$ and $\delta < \gamma$, and let $f : [0_n, \sigma] \to \mathbf{R}^n$ be such that for each $i = 1, \dots, n$ one has*

$$\delta_i < \operatorname*{ess\,inf}_{x \in [0_n,\sigma]} f_i(x) \leq \operatorname*{ess\,sup}_{x \in [0_n,\sigma]} f_i(x) < \gamma_i\,.$$

Assume that there exists a function $\overline{f} : [0_n, \sigma] \to \mathbf{R}^n$ and a set $E \subseteq [0_n, \sigma]$, with $m_n(E) = 0$, such that

$$\overline{f}(x) = f(x) \ \text{ for all } \ x \in [0_n, \sigma] \setminus E \tag{3}$$

and

$$\big\{x \in [0_n, \sigma] : \overline{f} \ \text{ is discontinuous at } x\big\} \subseteq E. \tag{4}$$

Then there exists $\hat{f} : [0_n, \sigma] \to \mathbf{R}^n$ such that

(i) *$\hat{f}([0_n, \sigma]) \subseteq [\delta, \gamma]$;*

(ii) *$\hat{f}(x) = f(x)$ for all $x \in [0_n, \sigma] \setminus E$; and*

(iii) *$\big\{x \in [0_n, \sigma] : \hat{f}$ is discontinuous at $x\big\} \subseteq E$.*

Proof: For each $i \in \{1, \dots, n\}$, put

$$A_i := \{x \in [0_n, \sigma] : \overline{f}_i(x) \le \delta_i\}, \quad B_i := \{x \in [0_n, \sigma] : \overline{f}_i(x) \ge \gamma_i\}.$$

Let

$$T := \bigcup_{i=1}^{n} (A_i \cup B_i).$$

If $T = \emptyset$, our claim follows by taking $\hat{f} = \overline{f}$. Assume $T \neq \emptyset$. We claim that $T \subseteq E$. Arguing by contradiction, assume that there exists $x^* \in T \setminus E$, and let $i^* \in \{1, \dots, n\}$ be such that $x^* \in A_{i^*} \cup B_{i^*}$. Assume $x^* \in A_{i^*}$ (if $x^* \in B_{i^*}$, the argument is analogous). Therefore, one has

$$\overline{f}_{i^*}(x^*) \le \delta_{i^*} < \operatorname*{ess\,inf}_{x \in [0_n, \sigma]} f_{i^*}(x). \tag{5}$$

By (4), the function $\overline{f}$ is continuous at x^*. Therefore, taking into account (5), there exists $\mu \in \mathbf{R}^n$, with $0_n < \mu$, such that

$$\overline{f}_{i^*}(u) < \operatorname*{ess\,inf}_{x \in [0_n, \sigma]} f_{i^*}(x) \quad \text{for all} \quad u \in U := [0_n, \sigma] \cap [x^* - \mu, x^* + \mu],$$

which contradicts (3) since $m_n(U) > 0$. Such a contradiction implies $T \subseteq E$, as claimed. Now, let $\hat{f} : [0_n, \sigma] \to \mathbf{R}^n$ be defined by

$$\hat{f}(x) = \begin{cases} \delta & \text{if } x \in T \\ \overline{f}(x) & \text{if } x \in [0_n, \sigma] \setminus T. \end{cases} \tag{6}$$

By the definition of T we immediately get $\hat{f}([0_n, \sigma]) \subseteq [\delta, \gamma]$. To prove conclusions (ii) and (iii), let $\overline{x} \in [0_n, \sigma] \setminus E$ be fixed. Since $T \subseteq E$, we have $\overline{x} \in [0_n, \sigma] \setminus T$, hence by (3) and (6) we get $\hat{f}(\overline{x}) = \overline{f}(\overline{x}) = f(\overline{x})$. Now we prove that $\hat{f}$ is continuous at $\overline{x}$. Since $\overline{x} \notin T$, we have

$$\delta_i < \overline{f}_i(\overline{x}) < \gamma_i \quad \text{for all} \quad i = 1, \dots, n.$$

Since by (4) the function $\overline{f}$ is continuous at $\overline{x}$, there exists a neighborhood V of $\overline{x}$ in $[0_n, \sigma]$ such that

$$\delta_i < \overline{f}_i(x) < \gamma_i \quad \text{for all} \quad i = 1, \dots, n \quad \text{and all} \quad x \in V.$$

Therefore, $V \cap T = \emptyset$ and thus $\hat{f}(x) = \overline{f}(x)$ for all $x \in V$. Consequently, the continuity of $\overline{f}$ at $\overline{x}$ implies the continuity of $\hat{f}$ at $\overline{x}$. The proof is complete. □

Proof of Theorem 1: We can assume $j < +\infty$. Put $E := \bigcup_{i=1}^{n} E_i$. By (ii) we get $m_n(E) = 0$. By Lemma 1, there exists a function $\hat{f} : [0_n, \sigma] \to \mathbf{R}^n$ such that

$$\alpha_i \le \hat{f}_i(x) \le \beta_i \quad \text{for all} \quad x \in [0_n, \sigma], \quad \text{and all} \quad i = 1, \dots, n, \tag{7}$$

$$\hat{f}(x) = f(x) \quad \text{for all} \quad x \in [0_n, \sigma] \setminus E, \tag{8}$$

and

$$\{x \in [0_n, \sigma] : \hat{f} \text{ is discontinuous at } x\} \subseteq E. \tag{9}$$

Let $\psi : \mathbf{R}^n \to \mathbf{R}^n$ be defined by

$$\psi(x) = \begin{cases} \hat{f}(x) & \text{if } x \in [0_n, \sigma] \\ \beta & \text{otherwise.} \end{cases} \tag{10}$$

Of course, one has

$$\psi(\mathbf{R}^n) \subseteq [\alpha, \beta]. \tag{11}$$

Now we want to apply Theorem 1 of [13] by taking $T = I$, $X = Y = \mathbf{R}^n$, $p = s = +\infty$, $q = j'$ (the conjugate exponent of j), $V = L^\infty(I, \mathbf{R}^n)$, $\Psi(u) = u$, $r = \|\beta\|_n$, $\varphi(\lambda) \equiv +\infty$,

$$\Phi(u)(t) = \int_I g(t, z)\, u(z)\, dz,$$

and $F : \mathbf{R}^n \to 2^{\mathbf{R}^n}$ as the multifunction defined by

$$F(x) = \bigcap_{\varepsilon > 0} \bigcap_{m_n(N) = 0} \overline{\text{co}}\, \psi(B(x, \varepsilon) \setminus N).$$

To this aim, observe what follows.

(a) $\Phi(L^\infty(I, \mathbf{R}^n)) \subseteq C^0(I, \mathbf{R}^n)$. This follows easily from our assumptions and Lebesgue's dominated convergence theorem.

(b) If $\{v^k\}$ is a sequence in $L^\infty(I, \mathbf{R}^n)$ and $v \in L^\infty(I, \mathbf{R}^n)$, with $\{v^k\}$ weakly convergent to v in $L^{j'}(I, \mathbf{R}^n)$, then the sequence $\{\Phi(v^k)\}$ converges to $\Phi(v)$ strongly in $L^1(I, \mathbf{R}^n)$. This follows by Theorem 2 at p. 359 of [10], observing that g is measurable in $I \times I$ by the classical Scorza Dragoni's theorem (see [14] or also [9]).

(c) The multifunction F has closed graph and nonempty convex values (see Proposition 1 at p. 102 of [1]). Moreover, by (11) we have

$$F(x) \subseteq [\alpha, \beta] \text{ for all } x \in \mathbf{R}^n. \tag{12}$$

Consequently, one has

$$\sup_{x \in \mathbf{R}^n} d(0_n, F(x)) \leq \|\beta\|_n.$$

Therefore, all the assumptions of Theorem 1 of [13] are satisfied. Thus, there exist a function $\hat{u} \in L^\infty(I, \mathbf{R}^n)$ and a set $K \subseteq I$, with $m_1(K) = 0$, such that

$$\hat{u}(t) \in F(\Phi(\hat{u})(t)) \text{ for all } t \in I \setminus K. \tag{13}$$

By (12), this implies

$$\hat{u}(t) \in [\alpha, \beta] \quad \text{for all} \quad t \in I \setminus K. \tag{14}$$

Therefore, for each $i = 1, \dots, n$ and each $t \in I$ one gets

$$0 \leq \left[\Phi(\hat{u})(t)\right]_i = \int_I g(t,z)\, \hat{u}_i(z)\, dz \leq \beta_i \, \|\phi_0\|_{L^1(I)} \leq \beta_i \, \frac{\sigma_i}{\beta_i} = \sigma_i,$$

hence $\Phi(\hat{u})(I) \subseteq [0_n, \sigma]$. For each fixed $i = 1, \dots, n$, let $h_i : I \to [0, \sigma_i]$ be defined by

$$h_i(t) := \left[\Phi(\hat{u})(t)\right]_i .$$

Taking into account (14) and assumption (iv), it is easily seen that the function h_i is strictly increasing. Moreover, by assumptions (iii), (iv) and Lemma 2.2 at p. 226 of [12], we have

$$\frac{d}{dt}\, h_i(t) = \int_I \frac{\partial g}{\partial t}(t,z)\, \hat{u}_i(z)\, dz > 0 \quad \text{for all} \quad t \in I_0.$$

By Theorem 2 of [15] (taking into account (a)), each function h_i^{-1} is absolutely continuous. For each $i = 1, \dots, n$, put

$$S_i := h_i^{-1}\left[(p_i(E_i) \cup \{0, \sigma_i\}) \cap h_i(I)\right].$$

By assumption (ii) and Theorem 18.25 of [8], we get $m_1(S_i) = 0$. Now, let

$$S := (\bigcup_{i=1}^{n} S_i) \cup K.$$

Of course, $m_1(S) = 0$. Let $t^* \in I \setminus S$ be fixed. Since $t^* \notin K$, by (13) we have

$$\hat{u}(t^*) \in F(\Phi(\hat{u})(t^*)). \tag{15}$$

Moreover, one has

$$\Phi(\hat{u})(t^*) \in]0_n, \sigma[\setminus E. \tag{16}$$

To see this, observe that for each $i = 1, \dots, n$, since $t^* \notin S_i$, we have $h_i(t^*) \notin p_i(E_i) \cup \{0, \sigma_i\}$. In particular, the last fact implies that $\Phi(\hat{u})(t^*) \notin E_i$ for all $i = 1, \dots, n$. Therefore, (16) follows. Now, observe that by (10) we have $\hat{f} = \psi$ in $]0_n, \sigma[$. Since by (9) and (16) the function $\hat{f}$ is continuous at the point $\Phi(\hat{u})(t^*)$, it follows that ψ is continuous at the same point $\Phi(\hat{u})(t^*)$. Hence, by Proposition 1 at p. 102 of [1], and taking into account (8), we get

$$F(\Phi(\hat{u})(t^*)) = \{\psi(\Phi(\hat{u})(t^*))\} = \{\hat{f}(\Phi(\hat{u})(t^*))\} = \{f(\Phi(\hat{u})(t^*))\},$$

hence by (15) we have

$$\hat{u}(t^*) = f(\Phi(\hat{u})(t^*)).$$

As t^* was any point in $I \setminus S$, the function $\hat{u}$ satisfies equation (2). Moreover, if $v : I \to \mathbf{R}^n$ is defined by $v(t) = \hat{f}(\Phi(\hat{u})(t))$, it follows easily from above that $v(t) = \hat{u}(t)$ for all $t \in I \setminus S$, and also

$$\{t \in I : v \text{ is discontinuous at } t\} \subseteq S.$$

Hence we have $\hat{u} \in \mathcal{B}(I, \mathbf{R}^n)$, as claimed. This completes the proof. $\square$

The next example shows that Theorem 1 is no longer true if in assumption (ii) the sets $E_1, \ldots, E_n$ are replaced by an arbitrary set $E \subseteq [0_n, \sigma]$ with $m_n(E) = 0$.

Example. Let $n = 2$, $\alpha_1 = \alpha_2 = \frac{1}{2}$, $\beta_1 = \beta_2 = 3$, $\sigma_1 = \sigma_2 = 4$, $g(t,z) = t$, $\phi_0(t) \equiv 1$, $\phi_1(t) \equiv 1$, and

$$f(u,v) = \begin{cases} (1,1) & \text{if } u \neq v \\ (2,1) & \text{if } u = v. \end{cases} \tag{17}$$

It is immediate to check that all the assumptions of Theorem 1 are satisfied, with the exception of assumption (ii). Moreover, f is almost everywhere equal to the constant $(1,1)$ in $[0_2, \sigma]$ (or also, observe that $m_2(\{(u,v) \in \mathbf{R}^2 : f \text{ is discontinuous at } (u,v)\}) = 0$). Now, assume that there exists a solution $u \in L^1(I, \mathbf{R}^2)$ to the equation (2). By (17) we have

$$u_1(t) \in \{1,2\} \quad \text{and} \quad u_2(t) = 1 \quad \text{for a.a.} \quad t \in I,$$

and thus

$$u(t) = f\big(t\,\|u_1\|_{L^1(I)}, t\big) \quad \text{for a.a.} \quad t \in I. \tag{18}$$

If we suppose $\|u_1\|_{L^1(I)} = 1$, by (17) and (18) we get $u_1(t) = 2$ for a.a. $t \in I$, a contradiction. If, on the contrary, we suppose $\|u_1\|_{L^1(I)} > 1$, by (17) and (18) we get $u_1(t) = 1$ for a.a. $t \in I$, another contradiction. Consequently, there is no solution $u \in L^1(I, \mathbf{R}^2)$ to problem (2).

Remark. The example at p. 245 of [4] shows that Theorem 1 is no longer true if in assumption (iv) we assume $0 \le \frac{\partial g}{\partial t}(t,z) \le \phi_1(z)$.

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