Pervasive algebras on planar compacts

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Abstract. We characterize compact sets X in the Riemann sphere $\mathbb{S}$ not separating $\mathbb{S}$ for which the algebra $A(X)$ of all functions continuous on $\mathbb{S}$ and holomorphic on $\mathbb{S} \setminus X$, restricted to the set X, is pervasive on X.

Keywords: compact Hausdorff space X, the sup-norm algebra $C(X)$ of all complex-valued continuous functions on X, its closed subalgebras (called function algebras), pervasive algebras; the algebra $A(X)$ of all functions continuous on $\mathbb{S}$ and holomorphic on $\mathbb{S} \setminus X$

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Let X be a compact Hausdorff topological space. Denote by $C(X)$ the commutative Banach algebra consisting of all continuous complex-valued functions on X (with respect to usual point-wise algebraic operations) endowed with the sup-norm:

$$|f| = \sup_{x \in X} |f(x)|, \quad f \in C(X).$$

By a *function algebra* on X we mean any closed subalgebra of $C(X)$ which contains constant functions on X and which separates points of X. (The last property, more precisely, means: whenever x, y is a couple of distinct points in X, then there exists a function $f \in A$ such that $f(x) \neq f(y)$.)

A function algebra on X is said to be *pervasive* whenever for any nonvoid proper closed subset F of X the algebra A/F of all restrictions f/F of functions $f \in A$ to the set F is dense in $C(F)$ with respect to $|.|_F$, the sup-norm on F.

The notion "pervasive" is due to Hoffman and Singer [1].

In whole the following text let X be a compact subset of the complex plane $\mathbb{C}$ which *does not separate* $\mathbb{C}$, i.e. such that the complement of X in $\mathbb{C}$ *is connected.* Let $A(X)$ be the algebra consisting of all functions which are continuous on the Riemann sphere $\mathbb{S}$ and holomorphic on $\mathbb{S} \setminus X$. It is well known (see Gamelin [2, 1.6]) that if $A(X)$ contains at least one non-constant function, then it separates points of $\mathbb{S}$ (and is a function algebra on $\mathbb{S}$).

By the Maximum Modulus Principle, every function $f \in A(X)$ attains its maximum modulus on X; it follows that $\tilde{A}(X) := A(X)/X$, the algebra of all restrictions to the set X, is a function algebra on X which is in a natural way isometrically isomorphic to $A(X)$.

Fuka [3] has proved that if X is the two-dimensional Cantor discontinuum, then $\tilde{A}(X)$ is pervasive. His idea is as follows.

Let F be a nonvoid proper closed subset of X and $x \in X \setminus F$. Because $\mathbb{C} \setminus X$ a nd then also $\mathbb{C} \setminus F$ is connected, any function in $C(F)$ is a uniform limit of complex polynomials on the set F; it follows from the Mergelyan Theorem (see [2, 9.1]). Now it is enough to approximate the function Z, $Z(z) = z$, uniformly on F by functions in A. But it follows from the classical Runge Theorem (see Saks and Zygmund [4, 2.1]) that Z/F is on F a uniform limit of a sequence f_n, $f_n(z) = p_n(\frac{1}{z-x})$ where p_n are appropriate polynomials. Hence it is sufficient to approximate only the function R, $R(z) = \frac{1}{z-x}$, uniformly on F by functions from $\tilde{A}(X)$.

The last part of Fuka's proof is based on the nice Urysohn construction [5]: let us denote by C_n the set of all centres of all 4^n partial squares of the rank n, in obvious sense, of the two-dimensional Cantor set in $\mathbb{C}$. Then the sequence of functions

$$f_n(z) = \sum_{c \in C_n} \frac{1}{z-c}$$

is pointwise convergent to a continuous nonconstant function f on the whole $\mathbb{S}$ and the convergence is locally uniform in $\mathbb{C} \setminus X$. It follows that $f \in A(X)$. But the Cantor set is a fractal — it is similar, in usual geometrical sense, to its intersection with any partial square of rank n. Thus it is possible to construct the Urysohn's sequence in any partial square. Fuka has shown that, whenever F is a closed subset of X, $x \in X \setminus F$, we can take a partial square containing x so small that Urysohn's limit function is near to the function $R(z) = \frac{1}{z-x}$ in the norm $|.|_F$.

Now we shall show that this idea holds in a **more general setting**, not only in the case of the two-dimensional Cantor set: whenever X is a compact perfect set in the complex plane with a connected complement which is nowhere dense in $\mathbb{C}$, the existence of a function in $\tilde{A}(X)$ which is rather big in x and rather small on F implies that it is possible to approximate the function R by functions from $A(X)$. More precisely, we shall prove the following

Theorem. *Let X be a compact perfect subset of the complex plane $\mathbb{C}$ with connected complement $\mathbb{C} \setminus X$ which is nowhere dense in $\mathbb{C}$ (or, which has empty interior). Let $A(X)$ and $\tilde{A}(X)$ be the algebras defined above. Then the following two properties are equivalent:*

(1) *$\tilde{A}(X)$ is pervasive on X;*
(2) *whenever F is a closed subset of X and x a point in $X \setminus F$, then there exists a function $f \in \tilde{A}(X)$ such that*

$(*)$ $f(x) = 1, \quad |f|_F < 1.$

For the proof we shall use the following Proposition (see [2, 1.8]):

Proposition. *Let f be a continuous function on the Riemann sphere $\mathbb{S}$, which is holomorphic on an open subset U of $\mathbb{S}$. Let $z_0 \in \mathbb{S}$. Then there is a sequence*

$\{f_n\}_{n=1}^{\infty}$ *of continuous functions on* $\mathbb{S}$ *such that* f_n *is holomorphic on* U, f_n *is holomorphic in a neighbourhood of* z_0, *and* $f_n \to f$ *uniformly on* $\mathbb{S}$.

Proof of the Theorem: Let $\tilde{A}(X)$ be pervasive on X; take a closed proper subset F and a point $x \in X \setminus F$. Put $H = F \cup \{x\}$; then H is closed in X and it is a proper subset of X because x is not isolated in X. The function h which is equal to 1 at x and to 0 on F is continuous on H. Then $\tilde{A}(X)$ being pervasive contains a function g such that

$$|h - g|_H < \frac{1}{2}\,.$$

If we put $f = \frac{g}{g(x)}$, then f fulfills $(*)$.

Conversely, suppose that the condition (2) is valid. Let F be a proper closed subset of X, $x \in X \setminus F$, $\varepsilon > 0$. From the above considerations it follows that it is enough to approximate the function R where $R(z) = \frac{1}{z-x}$ or to find a function $g \in \tilde{A}(X)$ such that

$$|g - R|_F < \varepsilon.$$

Now put $\eta = \frac{1}{3} \cdot \varepsilon \cdot \operatorname{dist}(x, F)$. Let $f \in \tilde{A}(X)$ be a function satisfying $(*)$; let n be a natural number so great that

$$|f|_F^n < \eta.$$

Denote by $\tilde{f}$ the function in $A(X)$ for which $\tilde{f}/F = f$ and put $v = 1 - \tilde{f}^n$. Then for $v \in A(X)$ we have

$$v(x) = 0, \quad |v - 1|_F < \eta.$$

The existence of a function $w \in A(X)$ which is holomorphic at the point x and satisfies

$$|v - w|_X < \eta$$

follows from Proposition; if we put $\tilde{w} = w - w(x)$ we have $\tilde{w} \in A(X)$, $\tilde{w}(x) = 0$ and

$$|v - \tilde{w}|_X \le |v - w|_X + |w(x)| < 2\eta;$$

moreover $\tilde{w}$ is holomorphic at the point x. It follows that the function g defined by $g(z) = \frac{\tilde{w}(z)}{z-x}$ is also in $A(X)$ and for any $z \in F$ we have

$$\begin{aligned} |g(z) - R(z)| &= \left|\frac{\tilde{w}(z)}{z-x} - \frac{1}{z-x}\right| \\ &\le \left|\frac{\tilde{w}(z)}{z-x} - \frac{v(z)}{z-x}\right| + \left|\frac{v(z)}{z-x} - \frac{1}{z-x}\right| < \frac{3\eta}{\operatorname{dist}(x, F)} = \varepsilon, \end{aligned}$$

hence Theorem is proved. □

Remark that in the case X is not perfect the notion "to be pervasive" does not make any reasonable sense: in [1] it is proved that for any abstract Hausdorff space X every pervasive function algebra A on X which is a proper part of $C(X)$ is *analytic* which means: whenever a function in A vanishes on a nonvoid open subset it must vanish identically. Then any function which vanishes on the isolated point of X is zero. It follows that $C(X)$ has no proper pervasive function subalgebras.

Remark also that the so called *classical disc algebra*, i.e. the algebra A consisting of all restrictions to the unit circle X of all functions continuous on the closed unit disc and holomorphic on its interior, fulfills both conditions of our theorem:

In fact, it follows immediately from the Mergelyan Theorem that A is pervasive on X. Let $z_0 \in X$; it is enough to find a function $f \in A$ such that $f(z_0) = 1$, $|f(z)| < 1$ for any $z \in X \setminus \{z_0\}$. Put $f(z) = \frac{1}{2z_0}(z_0 + z)$.

Remark at last that the two conditions of the Theorem are **not** equivalent one to the other in general case, neither for X being a compact subset of the complex plane: McKissick [6] has constructed a function algebra A on a planar compact X which is a proper subset of $C(X)$ and which is *normal* on X; it means that for any disjoint couple F, H of closed subsets of X there is a function in A which is equal to 1 on F and to 0 on H. It is clear that every normal function algebra on X fulfills the condition (2) of the Theorem, but it is not pervasive (because it is not analytic) whenever X contains more than one point.

References

[1] Hoffman K., Singer I.M., *Maximal algebras of continuous functions*, Acta Math. **103** (1960), 217–241.

[2] Gamelin T.W., *Uniform Algebras*, Prentice Hall, Inc., Englewood Cliffs, N.J., 1969.

[3] Fuka J., *A remark to maximality of several function algebras* (in Russian), Čas. Pěst. Mat. **93** (1968), 346–348.

[4] Saks S., Zygmund A., *Analytic Functions*, Polskie Towarzystwo Matematyczne, Warszawa, 1952.

[5] Urysohn P.S., *Sur une fonction analytique partout continue*, Fund. Math. **4** (1922), 144–150.

[6] McKissick R., *A nontrivial normal sup norm algebra*, Bull. Amer. Math. Soc. **69** (1963), 391–395.

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