On a problem of Nogura about the product of Fréchet-Urysohn $\langle\alpha_4\rangle$-spaces

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Abstract. Assuming Martin's Axiom, we provide an example of two Fréchet-Urysohn $\langle\alpha_4\rangle$-spaces, whose product is a non-Fréchet-Urysohn $\langle\alpha_4\rangle$-space. This gives a consistent negative answer to a question raised by T. Nogura.

Keywords: Fréchet-Urysohn space, $\langle\alpha_4\rangle$-space, Martin's Axiom, almost disjoint functions, double iterated power

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0. Introduction

The classes of $\langle\alpha_i\rangle$-spaces, with $1 \leq i \leq 4$, were introduced by Arhangel'skii in [Ar1], to study the product of Fréchet-Urysohn spaces (Arhangel'skii also introduced the class of $\langle\alpha_5\rangle$-spaces, which turned out to coincide with that of $\langle\alpha_2\rangle$-spaces: see [No, Theorem 2.1]). Each $\langle\alpha_i\rangle$-space is also an $\langle\alpha_{i+1}\rangle$-space for $1 \leq i \leq 3$, and each first countable space is an $\langle\alpha_1\rangle$-space.

The above mentioned paper gave rise, in the following twenty years, to a wide literature, where several problems concerning this kind of spaces are investigated (see, for example, [Do] and related bibliography); often, in these articles, the Fréchet-Urysohn $\langle\alpha_i\rangle$-spaces are briefly called $\langle\alpha_i\text{-FU}\rangle$-spaces. For $i = 1, 2, 3$, Nogura [No] proved that the product of two $\langle\alpha_i\rangle$-spaces is still an $\langle\alpha_i\rangle$-space. Also, the product of an $\langle\alpha_3\text{-FU}\rangle$-space and of a countably compact, regular Fréchet space (which is always an $\langle\alpha_4\rangle$-space, see [Ol]) is a Fréchet space [Ar2]; this is one of the best results about preservation of the Fréchet property under products. Recall that, without additional assumptions, even the product of two compact (T_2) Fréchet spaces may fail to be Fréchet; the first, celebrated example in ZFC of this fact is due to Simon [Si1].

As for $\langle\alpha_4\rangle$-spaces and $\langle\alpha_4\text{-FU}\rangle$-spaces (which coincide with the *strongly Fréchet* spaces — see [Ar2] and the remarks after Theorem 1.4 of [No]), their product is not very well behaved. The product of two $\langle\alpha_4\text{-FU}\rangle$-spaces may fail both to be Fréchet and to be an $\langle\alpha_4\rangle$-space (cf. [No, Example 1.2 and Theorem 3.10]). Thus, Nogura put the following questions [No, Problem 3.15 and 3.18]:

(a) Let X and Y be $\langle\alpha_4\text{-FU}\rangle$-spaces. If $X \times Y$ is Fréchet, then is it an $\langle\alpha_4\rangle$-space?

(b) Let X and Y be $\langle\alpha_4\text{-FU}\rangle$-spaces. If $X \times Y$ is an $\langle\alpha_4\rangle$-space, then is it Fréchet?

Very recently, the first question was solved in the negative by Simon, under the Continuum Hypothesis ([Si2]). In this paper, we give under Martin's Axiom (MA) a negative answer to the second question — actually, our X and Y will turn out to be countable (paracompact) T_2 spaces, where each point, except one, is isolated. We point out that, after this paper had been written, a ZFC example for the same problem was found by Simon and the author (see [CS]).

1. Notations and basic facts

Throughout the paper, the left exponentiation ${}^{A}B$ among sets will denote the set of all functions $f\colon A \to B$, while the right exponentiation ξ^{κ} among cardinals will denote the cardinal number: $|{}^{\kappa}\xi|$. The ordered pairs, triples, and so on are denoted, respectively, by $\langle a, b\rangle$, $\langle a, b, c\rangle$, etc. For every function f, we denote by $\operatorname{dom} f$ its domain and by $\operatorname{Im} f$ its image $\{f(x) \mid x \in \operatorname{dom} f\}$.

We say that a topological space X has the property $\langle\alpha_4\rangle$ at a point $\bar{x}$ if for every family $\{\psi_m \mid m \in \omega\}$ of functions from ω to X such that $\lim_{n\to+\infty} \psi_m(n) = \bar{x}$, there exists a $\psi \in {}^{\omega}X$ such that $\lim_{m\to+\infty} \psi(m) = \bar{x}$ and $|\{m \in \omega \mid \operatorname{Im} \psi \cap \operatorname{Im} \psi_m \neq \emptyset\}| = \omega$. We say that X is an $\langle\alpha_4\rangle$-space if it has the property $\langle\alpha_4\rangle$ at each of its points.

$\tilde{\Phi}$ is the set of all one-to-one functions from ω to ω (throughout the paper, *one-to-one* does not ever involve *onto*, unless explicitly stated). To every $\Phi \subseteq \tilde{\Phi}$ a topological space X_Φ is associated, where $X_\Phi = \omega \cup \{\infty_\Phi\}$, $\infty_\Phi \notin \omega$, the points of ω are isolated and the point ∞_Φ has a local base given by $\{W_\zeta \mid \zeta \in {}^{\Phi}\omega\}$, with

$$W_\zeta = \{\infty_\Phi\} \cup \{\varphi(n) \mid \varphi \in \Phi \ \wedge \ n \geq \zeta(\varphi)\}$$

for every $\zeta \in {}^{\Phi}\omega$. In particular, it is clear that for every $\varphi \in \Phi$ (and for every subsequence of it) we have that $\lim_{n\to+\infty} \varphi(n) = \infty_\Phi$.

Observe that for every $\Phi \subseteq \tilde{\Phi}$, X_Φ is a T_2 paracompact Fréchet space. To prove the latter property, let A be any subset of ω such that $\infty_\Phi \in \overline{A}$. Then for at least one $\tilde{\varphi} \in \Phi$ we have that $|\operatorname{Im} \tilde{\varphi} \cap A| = \omega$ (if, by contradiction, $\forall \varphi \in \Phi\colon \exists\, \zeta(\varphi) \in \omega\colon \forall n \geq \zeta(\varphi)\colon \varphi(n) \notin A$, then W_ζ would be a nbhd of ∞_Φ in X_Φ which does not meet A). Then there is a subsequence φ^* of φ whose image is entirely contained in A, and we have $\lim_{n\to+\infty} \varphi^*(n) = \infty_\Phi$.

Remark 1. It is easy to prove, using an analogous argument, that whenever $\varphi' \in {}^{\omega}\omega$ is such that $\lim_{n\to+\infty} \varphi'(n) = \infty_\Phi$ in X_Φ, there exists $\varphi \in \tilde{\Phi}$ such that $|\operatorname{Im} \varphi' \cap \operatorname{Im} \varphi| = \omega$. We will often use this fact in the sequel.

We say that two elements φ', φ'' of $\tilde{\Phi}$ are *almost disjoint* (briefly, φ' a.d. φ'') if $\operatorname{Im} \varphi'$ and $\operatorname{Im} \varphi''$ are almost disjoint (i.e., if $|\operatorname{Im} \varphi' \cap \operatorname{Im} \varphi''| < \omega$). We say that a subcollection Φ of $\tilde{\Phi}$ is almost disjoint if φ a.d. φ' for distinct $\varphi, \varphi' \in \Phi$. Clearly, φ' a.d. φ'' if and only if $\exists\, n \in \omega\colon \{\varphi'(n') \mid n' \geq n\} \cap \operatorname{Im} \varphi'' = \emptyset$.

We denote by Θ the set ${}^{\omega}\tilde{\Phi}$. For $\vartheta, \theta \in \Theta$ we will often abuse notation and write $\vartheta \circ \theta$ to denote the element of Θ defined by

$$(\vartheta \circ \theta)(m) = \big(\vartheta(m)\big) \circ \big(\theta(m)\big)$$

for every $m \in \omega$. Of course, $|\Theta| = 2^\omega$; in all the paper, we suppose to have fixed a one-to-one indexing

$$\{\theta_\beta \,|\, \beta \in 2^\omega\}$$

of Θ, and a one-to-one indexing

$$(\spadesuit) \qquad \{\hat{\jmath}_\alpha \,|\, \alpha \in 2^\omega \setminus \omega\}$$

of ${}^\omega\omega$.

2. Auxiliary results

Lemma 2 (MA). *Let $\Phi^* \subseteq \tilde{\Phi}$ be an almost disjoint collection, with $|\Phi^*| = \kappa < 2^\omega$. Suppose to have $\vartheta^0, \vartheta^1 \in \Theta$ such that it is possible to associate to every $\langle\iota, m\rangle \in 2 \times \omega$ a $\varphi^\iota_m \in \Phi^*$ in such a way that $\langle\iota, m\rangle \mapsto \varphi^\iota_m$ is one-to-one and*

$$\forall\, \iota \in 2 \colon \forall\, m \in \omega \colon \mathrm{Im}\, \big(\vartheta^\iota(m)\big) \subseteq \mathrm{Im}\, \varphi^\iota_m.$$

Then there exists $j \in {}^\omega\omega$ such that, defining $\varphi^\iota \in {}^\omega\omega$ for $\iota \in 2$ by

$$(1) \qquad \varphi^\iota(m) = \big(\vartheta^\iota(m)\big)\big(j(m)\big),$$

we have:

- (a) $\varphi^\iota \in \tilde{\Phi}$ *for* $\iota = 0, 1$, *and* $\mathrm{Im}\, \varphi^0 \cap \mathrm{Im}\, \varphi^1 = \emptyset$*;*
- (b) φ^ι a.d. φ *for every* $\iota \in 2$ *and* $\varphi \in \Phi^*$.

Proof: Since $\varphi^{\iota'}_{m'}$ a.d. $\varphi^{\iota''}_{m''}$ for $\langle\iota', m'\rangle \neq \langle\iota'', m''\rangle$, for every $m \in \omega$ there exists $j^\star(m)$ such that $\{\varphi^\iota_m(n) \,|\, n \geq j^\star(m)\} \cap \mathrm{Im}\, \varphi^{\iota'}_{m'} = \emptyset$ for every $m' \leq m$ and $\langle\iota', m'\rangle \neq \langle\iota, m\rangle$. For every $m \in \omega$, since $\forall\, \iota \in 2 \colon \big(\vartheta^\iota(m) \in \tilde{\Phi} \;\wedge\; \mathrm{Im}\, \big(\vartheta^\iota(m)\big) \subseteq \mathrm{Im}\, \varphi^\iota_m\big)$, there exists $j^*(m) \in \omega$ such that $\forall\, \iota \in 2 \colon \forall\, n \geq j^*(m) \colon \big(\vartheta^\iota(m)\big)(n) \in \big\{\varphi^\iota_m(n') \,|\, n' \geq j^\star(m)\big\}$. Putting $j^\sharp = \sup\{j^\star, j^*\}$, for every $\langle\iota', m'\rangle, \langle\iota'', m''\rangle \in 2 \times \omega$ with $\langle\iota', m'\rangle \neq \langle\iota'', m''\rangle$ we will have at the same time:

$$(2) \qquad \Big\{\big(\vartheta^{\iota'}(m')\big)(n) \,\Big|\, n \geq j^\sharp(m')\Big\} \cap \Big\{\big(\vartheta^{\iota''}(m'')\big)(n) \,\Big|\, n \geq j^\sharp(m'')\Big\} = \emptyset$$

and

$$(3) \qquad \Big\{\big(\vartheta^{\iota'}(m')\big)(n) \,\Big|\, n \geq j^\sharp(m')\Big\} \cap \Big\{\varphi^{\iota''}_{m''}(n) \,\Big|\, n \geq j^\sharp(m'')\Big\} = \emptyset.$$

We proceed now to a routine application of MA. Put $\Phi^\sharp = \Phi^* \setminus \{\varphi^\iota_m \,|\, \langle\iota, m\rangle \in 2 \times \omega\}$ and define a poset $\langle\mathbf{P}, \leq\rangle$ in the following way:

$$\mathbf{P} = \Big\{\langle g, \mathcal{A}\rangle \,\Big|\, \mathcal{A} \in \big[\Phi^\sharp\big]^{<\omega} \;\wedge\; g \in {}^{<\omega}\omega \;\wedge\; \forall\, m \in \mathrm{dom}\, g \colon g(m) \geq j^\sharp(m)\Big\};$$

for $\langle g', \mathcal{A}' \rangle, \langle g'', \mathcal{A}'' \rangle \in \mathbf{P}$, let $\langle g', \mathcal{A}' \rangle \geq \langle g'', \mathcal{A}'' \rangle$ if $g' \subseteq g''$, $\mathcal{A}' \subseteq \mathcal{A}''$ and $\forall \iota \in 2 \colon \forall m \in \operatorname{dom} g'' \setminus \operatorname{dom} g' \colon \forall \varphi \in \mathcal{A}' \colon \big(\vartheta^\iota(m)\big)\big(g''(m)\big) \notin \operatorname{Im} \varphi$.

Observe that for every $g \in {}^{<\omega}\omega$ and $\mathcal{A}', \mathcal{A}'' \in \big[\Phi^\sharp\big]^{<\omega}$, $\langle g, \mathcal{A}' \cup \mathcal{A}'' \rangle$ is clearly a common extension of $\langle g, \mathcal{A}' \rangle$ and $\langle g, \mathcal{A}'' \rangle$: thus, if $\langle g', \mathcal{A}' \rangle$ and $\langle g'', \mathcal{A}'' \rangle$ are incompatible, then $g' \neq g''$; since $|{}^{<\omega}\omega| = \omega$, we have that $\langle \mathbf{P}, \leq \rangle$ is c.c.c.

For every $\varphi \in \Phi^\sharp$ and $m \in \omega$, the set $D_{\varphi,m} = \{\langle g, \mathcal{A} \rangle \in \mathbf{P} \mid \varphi \in \mathcal{A} \wedge m \in \operatorname{dom} g\}$ is dense in $\mathbf{P}$. Indeed, let $\langle g, \mathcal{A} \rangle$ be any element of $\mathbf{P}$: if $m \in \operatorname{dom} g$, then $\langle g, \mathcal{A} \cup \{\varphi\} \rangle$ is an extension of $\langle g, \mathcal{A} \rangle$ which belongs to $D_{\varphi,m}$. If $m \notin \operatorname{dom} g$, then consider that since $\vartheta^\iota(m)$ a.d. φ' for every $\iota \in 2$ and $\varphi' \in \mathcal{A}$, there exist $n^0, n^1 \in \omega$ such that $\forall \iota \in 2 \colon \forall \varphi' \in \mathcal{A} \colon \big\{\big(\vartheta^\iota(m)\big)(n) \,\big|\, n \geq n^\iota\big\} \cap \operatorname{Im} \varphi' = \emptyset$; define an extension $\tilde{g}$ of g with $\operatorname{dom} \tilde{g} = \operatorname{dom} g \cup \{m\}$ and $\tilde{g}(m) = \max\big\{j^\sharp(m), n^0, n^1\big\}$: then $\langle \tilde{g}, \mathcal{A} \cup \{\varphi\} \rangle \in D_{\varphi,m}$ and $\langle g, \mathcal{A} \rangle \geq \langle \tilde{g}, \mathcal{A} \cup \{\varphi\} \rangle$.

Since $\big|\big\{D_{\varphi,m} \,\big|\, \varphi \in \Phi^\sharp \ \wedge \ m \in \omega\big\}\big| \leq \kappa \cdot \omega = \kappa$, there exists a filter G on $\mathbf{P}$ such that $\forall \varphi \in \Phi^\sharp \colon \forall m \in \omega \colon G \cap D_{\varphi,m} \neq \emptyset$. Let $j = \bigcup \big\{g \in {}^{<\omega}\omega \,\big|\, \exists \mathcal{A} \in \big[\Phi^\sharp\big]^{<\omega} : \langle g, \mathcal{A} \rangle \in G\big\}$: it is easy to see that j is a function and that $j \colon \omega \to \omega$ (of course, we may always suppose that $\Phi^\sharp \neq \emptyset$). We must prove that the functions φ^ι for $\iota = 0, 1$, defined by (1), satisfy (a) and (b).

First of all, observe that $j \geq j^\sharp$. Indeed, let $m \in \omega$: then $\langle m, j(m) \rangle \in j$, i.e., there exists $\langle g, \mathcal{A} \rangle \in G$ such that $\langle m, j(m) \rangle \in g$; thus $g(m) = j(m)$, and by the definition of $\mathbf{P}$ we have that $j(m) = g(m) \geq j^\sharp(m)$. Now, if $m', m'' \in \omega$ with $m' \neq m''$, then $\varphi^\iota(m') = \big(\vartheta^\iota(m')\big)\big(j(m')\big) \in \big\{\big(\vartheta^\iota(m')\big)(n) \,\big|\, n \geq j^\sharp(m')\big\}$ and $\varphi^\iota(m'') = \big(\vartheta^\iota(m'')\big)\big(j(m'')\big) \in \big\{\big(\vartheta^\iota(m'')\big)(n) \,\big|\, n \geq j^\sharp(m'')\big\}$ for $\iota \in 2$, so that $\varphi^\iota(m') \neq \varphi^\iota(m'')$ by (2), and hence φ^0, φ^1 are one-to-one. Moreover, for every $m', m'' \in \omega$ (even, possibly, $m' = m''$), we have that $\varphi^0(m') \in \big\{\big(\vartheta^0(m')\big)(n) \,\big|\, n \geq j^\sharp(m')\big\}$ and $\varphi^1(m'') \in \big\{\big(\vartheta^1(m'')\big)(n) \,\big|\, n \geq j^\sharp(m'')\big\}$, so that $\varphi^0(m') \neq \varphi^1(m'')$ again by (2), and hence $\operatorname{Im} \varphi^0 \cap \operatorname{Im} \varphi^1 = \emptyset$.

To prove (b), let φ^* be any element of Φ^*, and consider first the case where $\varphi^* \in \Phi^\sharp$. Given $\iota \in 2$, suppose by contradiction that $\operatorname{Im} \varphi^* \cap \operatorname{Im} \varphi^\iota$ is infinite. Fix any $\bar{m} \in \omega$ and take $\langle g, \mathcal{A} \rangle \in G \cap D_{\varphi^*, \bar{m}}$, so that $\varphi^* \in \mathcal{A}$. Since $\operatorname{Im} \varphi^* \cap \operatorname{Im} \varphi^\iota$ is infinite, the set $M = (\varphi^\iota)^{-1}(\operatorname{Im} \varphi^* \cap \operatorname{Im} \varphi^\iota) = (\varphi^\iota)^{-1}(\operatorname{Im} \varphi^*)$ is infinite, too: then fix $\hat{m} \in M \setminus \operatorname{dom} g$. Now take $\big\langle \hat{g}, \hat{\mathcal{A}} \big\rangle \in G$ such that $\hat{m} \in \operatorname{dom} \hat{g}$, and let $\big\langle g^\sharp, \mathcal{A}^\sharp \big\rangle \in G$ be a common extension of $\langle g, \mathcal{A} \rangle$ and $\big\langle \hat{g}, \hat{\mathcal{A}} \big\rangle$, so that, in particular, $\hat{m} \in \operatorname{dom} \hat{g} \subseteq \operatorname{dom} g^\sharp$ and $\big(\vartheta^\iota(\hat{m})\big)\big(g^\sharp(\hat{m})\big) = \big(\vartheta^\iota(\hat{m})\big)\big(j(\hat{m})\big) = \varphi^\iota(\hat{m}) \in \operatorname{Im} \varphi^*$ (by the definition of M). This is a contradiction, because $\hat{m} \notin \operatorname{dom} g$, $\varphi^* \in \mathcal{A}$ and $\langle g, \mathcal{A} \rangle \geq \big\langle g^\sharp, \mathcal{A}^\sharp \big\rangle$.

Consider now the case where $\varphi^* = \varphi^{\iota^*}_{m^*}$ for some $\langle \iota^*, m^* \rangle \in 2 \times \omega$. Given any $\iota \in 2$, from $j \geq j^\sharp$ we have that $\varphi^\iota(m) = \big(\vartheta^\iota(m)\big)\big(j(m)\big) \in \big\{\big(\vartheta^\iota(m)\big)(n) \,\big|\, n \geq j^\sharp(m)\big\}$, which implies by (3) that $\forall m \neq m^* \colon \varphi^\iota(m) \notin \big\{\varphi^{\iota^*}_{m^*}(n) \,\big|\, n \geq j^\sharp(m^*)\big\}$ ($m \neq m^*$ entails in any case $\langle \iota, m \rangle \neq \langle \iota^*, m^* \rangle$); therefore, $\operatorname{Im} \varphi^\iota \cap \operatorname{Im} \varphi^{\iota^*}_{m^*} \subseteq \{\varphi^\iota(m^*)\} \cup \big\{\varphi^{\iota^*}_{m^*}(n) \,\big|\, n < j^\sharp(m^*)\big\}$, which is a finite set. $\square$

The following lemma is, in some sense, a "one-dimension" formulation of the previous one; they will both be useful in the sequel.

Lemma 3 (MA). *Let $\hat\Phi \subseteq \tilde\Phi$ be an almost disjoint collection, with $|\hat\Phi| = \kappa < 2^\omega$. Suppose that there exists a $\vartheta \in \Theta$ such that for every $m \in \omega$ there exists an $f_m \in \hat\Phi$ with $\mathrm{Im}\,\big(\vartheta(m)\big) \subseteq \mathrm{Im}\, f_m$; also, suppose that $m \mapsto f_m$ is one-to-one. Then there exists $\rho \in \tilde\Phi$ such that $\rho\,\text{a.d.}\,\varphi$ for every $\varphi \in \hat\Phi$ and $\mathrm{Im}\,\rho \cap \mathrm{Im}\,\big(\vartheta(m)\big) \neq \emptyset$ for every $m \in \omega$.*

The proof may be obtained following the outlines of the previous one; or, alternatively, applying Lemma 2 (after extending $\hat\Phi$ to a collection Φ^* by adding specular elements, which is possible by [Ku, Corollary 2.16]) and then taking as ρ a suitable φ^ι; or, alternatively, applying [Ku, Theorem 2.15] to $\mathcal{C} = \big\{\mathrm{Im}\,\big(\vartheta(m)\big) | m \in \omega\big\}$ and $\mathcal{A} = \big\{\mathrm{Im}\,\varphi \,\big|\, \varphi \in \hat\Phi\big\} \setminus \big\{\mathrm{Im}\, f_m \,\big|\, m \in \omega\big\}$, and then shrinking and indexing the set d.

Now we introduce a set-theoretic operator which will play a crucial role for our further constructions. Let ξ be any infinite cardinal number, and define by transfinite induction the sets M_γ, for $\gamma \in \xi^+$, in the following way. $M_0 = \xi$; if $M_{\gamma'}$ is defined for every $\gamma' < \gamma$, where $\gamma \in \xi^+ \setminus \{0\}$, then

$$M_\gamma = \Big\{\langle \mu^0, \mu^1, \beta^0, \beta^1\rangle \Big| \;\forall\, \iota \in 2\colon \Big(\beta^\iota \in 2^\xi \text{ and } \mu^\iota \text{ is a one-to-one function from } \xi \text{ to } \bigcup_{\gamma'<\gamma} M_{\gamma'}\Big)\Big\}.$$

The set $\bigcup_{\gamma\in\xi^+} M_\gamma$ will be called the *double iterated power* of ξ, and denoted by $\mathrm{DIP}\,(\xi)$. For every $x \in \mathrm{DIP}\,(\xi)$, we also define a subset $\mathrm{supp}\,(x)$ of $\mathrm{DIP}\,(\xi)$, the *support* of x, putting $\mathrm{supp}\,(x) = \emptyset$ if $x \in M_0 = \xi$, and $\mathrm{supp}\,(x) = \mathrm{Im}\,\mu^0 \cup \mathrm{Im}\,\mu^1$ if $x \in \bigcup_{\gamma\in\xi^+\setminus\{0\}} M_\gamma$ and $x = \big\langle \mu^0, \mu^1, \beta^0, \beta^1\big\rangle$.

It is immediate to prove by transfinite induction that $|M_\gamma| = 2^\xi$ for every $\gamma \in \xi^+ \setminus \{0\}$; therefore, $|\mathrm{DIP}\,(\xi)\,| = 2^\xi$. We will say that an indexing $\big\{x_\alpha \,\big|\, \alpha \in 2^\xi\big\}$ of $\mathrm{DIP}\,(\xi)$ is *well founded* if it is one-to-one, $x_\alpha = \alpha$ for every $\alpha \in \xi$, and $\forall\,\alpha \in 2^\omega\colon \mathrm{supp}\,(x_\alpha) \subseteq \big\{x_{\alpha'} \,|\, \alpha' < \alpha\big\}$.

Lemma 4. *For every infinite cardinal ξ there exists a well founded indexing of* $\mathrm{DIP}\,(\xi)$.

Proof: First, fix any one-to-one indexing $\Big\{y_\sigma \,\Big|\, \sigma \in 2^\xi\Big\}$ of $\mathrm{DIP}\,(\xi)$. Then define $j\colon 2^\xi \to 2^\xi$ in the following way:

— $j\,(\alpha) = \alpha$, for $\alpha \in \xi$;
— $j\,(\alpha) = \min\big\{\sigma \in 2^\xi \setminus \big\{j\,(\alpha') \,\big|\, \alpha' < \alpha\big\} \,\big|\, \mathrm{supp}\,(y_\sigma) \subseteq \big\{y_{j(\alpha')} \,\big|\, \alpha' < \alpha\big\}\big\}$, for $\alpha \geq \xi$.

Observe that the above set cannot be empty. Indeed, for every $\beta \in 2^\xi$, we have $\big\langle \mathrm{id}_\xi, \mathrm{id}_\xi, \beta, 0\big\rangle \in M_1 \subseteq \mathrm{DIP}\,(\xi)$, hence there exists $\sigma_\beta \in 2^\xi$ such that

$\langle \mathrm{id}_\xi, \mathrm{id}_\xi, \beta, 0\rangle = y_{\sigma_\beta}$. Since $\beta \mapsto \sigma_\beta$ is one-to-one, there must exist $\hat\beta \in 2^\xi$ such that $\sigma_{\hat\beta} \notin \{j(\alpha') \,|\, \alpha' < \alpha\}$, and for such a $\sigma_{\hat\beta}$ we have that $\operatorname{supp}(y_{\sigma_{\hat\beta}}) = \operatorname{supp}(\langle \mathrm{id}_\xi, \mathrm{id}_\xi, \beta, 0\rangle) = \xi \subseteq \{j(\alpha') \,|\, \alpha' < \alpha\}$.

Now put, for every $\alpha \in 2^\xi$, $x_\alpha = y_{j(\alpha)}$: by the definition of j, $\alpha \mapsto x_\alpha$ is one-to-one and $\operatorname{supp}(x_\alpha) = \operatorname{supp}(y_{j(\alpha)}) \subseteq \{y_{j(\alpha')} \,|\, \alpha' < \alpha\} = \{x_{\alpha'} \,|\, \alpha' < \alpha\}$ for every $\alpha \in 2^\xi \setminus \xi$. Thus, we only need to prove the onto character of $\alpha \mapsto x_\alpha$ over $\mathrm{DIP}(\xi)$, which is clearly equivalent to the onto character of j over 2^ξ.

Suppose j is not onto and let $\hat\gamma = \min\{\gamma \in \xi^+ \,|\, M_\gamma \nsubseteq \{x_\alpha \,|\, \alpha \in 2^\xi\}\}$; fix $\hat\sigma \in 2^\xi$ such that $y_{\hat\sigma} \in M_{\hat\gamma} \setminus \{x_\alpha \,|\, \alpha \in 2^\xi\}$ and put $A = \operatorname{supp}(y_{\hat\sigma})$. Then every $a \in A$ belongs to some M_γ with $\gamma < \hat\gamma$, hence there exists $\alpha(a) \in 2^\xi$ such that $x_{\alpha(a)} = a$; as $|A| \le \xi$ and $\operatorname{cof} 2^\xi > \xi$, there exists $\hat\alpha \in 2^\xi$ such that $\hat\alpha > \alpha(a)$ for every $a \in A$. Then for every $\alpha \in 2^\xi$ with $\alpha \ge \hat\alpha$, since $\hat\sigma \in \{\sigma \in 2^\xi \setminus \{j(\alpha') \,|\, \alpha' < \alpha\} \,|\, \operatorname{supp}(y_\sigma) \subseteq \{y_{j(\alpha')} \,|\, \alpha' < \alpha\}\}$, we have that $j(\alpha) \le \hat\sigma$; this is in contrast with the one-to-one character of j. $\square$

3. The main construction

Henceforth, we assume MA. We will associate by transfinite induction to every $\alpha \in 2^\omega$, a pair $\langle \varphi^0_\alpha, \varphi^1_\alpha\rangle$ of elements of $\tilde\Phi$. We adopt the following notation: for every $x \in \mathrm{DIP}(\omega)$, let $\alpha^\sharp(x)$ denote the unique $\alpha \in 2^\omega$ such that $x_\alpha = x$ (so that $\alpha^\sharp(x_\alpha) = \alpha$ for every $\alpha \in 2^\omega$).

Also, we denote by K the set of all strictly increasing functions $k\colon \omega \to \omega$ and by Λ the set of all functions $\lambda\colon \omega \to K$.

Let $\{F_{\iota,m}\}_{\langle\iota,m\rangle \in 2\times\omega}$ be a partition of ω — where $\langle\iota,m\rangle \mapsto F_{\iota,m}$ is one-to-one — such that $|F_{\iota,m}| = \omega$ for every $\langle\iota,m\rangle \in 2\times\omega$. For every $\langle\iota,m\rangle \in 2\times\omega$, let f^ι_m be an element of $\tilde\Phi$ such that $\operatorname{Im} f^\iota_m = F_{\iota,m}$. For every $\alpha \in \omega$ and $\iota \in 2$, we put $\varphi^\iota_\alpha = f^\iota_\alpha$.

Suppose now to have defined $\varphi^\iota_{\alpha'}$ for every $\iota \in 2$ and $\alpha' < \alpha$, where $\alpha \in 2^\omega \setminus \omega$, in such a way that $\varphi^{\iota'}_{\alpha'}$ a.d. $\varphi^{\iota''}_{\alpha''}$ for $\langle\iota',\alpha'\rangle \ne \langle\iota'',\alpha''\rangle$. Let $x_\alpha = \langle \mu^0, \mu^1, \beta^0, \beta^1\rangle$ and define $\vartheta^0_\alpha, \vartheta^1_\alpha \in \Theta$ by $\vartheta^\iota_\alpha(m) = \varphi^\iota_{\alpha^\sharp(\mu^\iota(m))}$ for $\iota \in 2$. Consider the two elements $\vartheta^\iota_\alpha \circ \theta_{\beta^\iota}$ of Θ ($\iota = 0,1$): since $\vartheta^\iota_\alpha(m)$ a.d. $\vartheta^{\iota'}_\alpha(m')$ for $\langle\iota,m\rangle \ne \langle\iota',m'\rangle$, we also have that $\vartheta^\iota_\alpha(m) \circ \theta_{\beta^\iota}(m)$ a.d. $\vartheta^{\iota'}_\alpha(m') \circ \theta_{\beta^\iota}(m')$ for $\langle\iota,m\rangle \ne \langle\iota',m'\rangle$. Let $\Phi^* = \{\varphi^\iota_{\alpha'} \,|\, \iota \in 2 \ \wedge\ \alpha' < \alpha\}$: then Φ^* is an almost disjoint family and $|\Phi^*| = |\alpha| < 2^\omega$. Moreover,

$$\forall \langle\iota,m\rangle \in 2\times\omega\colon \operatorname{Im}\left(\vartheta^\iota_\alpha(m) \circ \theta_{\beta^\iota}(m)\right) \subseteq \operatorname{Im}\left(\vartheta^\iota_\alpha(m)\right) = \operatorname{Im} \varphi^\iota_{\alpha^\sharp(\mu^\iota(m))};$$

since $\langle\iota,m\rangle \mapsto \varphi^\iota_{\alpha^\sharp(\mu^\iota(m))}$ is one-to-one from $2\times\omega$ to Φ^*, we may apply Lemma 2 to get a $j \in {}^\omega\omega$ such that the functions $\tilde\varphi^0_\alpha, \tilde\varphi^1_\alpha$, defined by

$$\tilde\varphi^\iota_\alpha(m) = \left(\vartheta^\iota_\alpha(m)\right)\Big(\left(\theta_{\beta^\iota}(m)\right)\left(j(m)\right)\Big) \quad \text{for } \iota \in 2, \tag{4}$$

are such that:

1) $\tilde{\varphi}^{\iota}_{\alpha} \in \tilde{\Phi}$ for $\iota \in 2$ and $\tilde{\varphi}^{0}$ a.d. $\tilde{\varphi}^{1}$;

2) $\tilde{\varphi}^{\iota}_{\alpha}$ a.d. $\varphi^{\iota'}_{\alpha'}$ for every $\iota, \iota' \in 2$ and $\alpha' < \alpha$.

Put $\varphi^{0}_{\alpha} = \tilde{\varphi}^{0}_{\alpha}$, so that φ^{0}_{α} a.d. $\varphi^{\iota}_{\alpha'}$ for every $\langle \iota, \alpha' \rangle \in 2 \times \alpha$. Also, define $\hat{\lambda}_{\alpha} \in \Lambda$ by:

$$(5) \qquad \big(\hat{\lambda}_{\alpha}(m)\big)(n) = n + \hat{\jmath}_{\alpha}(m)$$

for every $m, n \in \omega$ — remember (♠).

Now, consider the almost disjoint collection of functions: $\hat{\Phi} = \Phi^{*} \cup \{\varphi^{0}_{\alpha}\}$: putting

$$(6) \qquad \hat{\vartheta}_{\alpha}(m) = f^{0}_{m} \circ \big(\hat{\lambda}_{\alpha}(m)\big)$$

we get a function $\hat{\vartheta}_{\alpha} \in \Theta$ such that $\hat{\vartheta}_{\alpha}(m)$ a.d. $\hat{\vartheta}_{\alpha}(m')$ for $m \neq m'$ and $\mathrm{Im}\ \big(\hat{\vartheta}_{\alpha}(m)\big) \subseteq \mathrm{Im}\ f^{0}_{m}$ for every $m \in \omega$. Since $m \mapsto f^{0}_{m}$ is one-to-one (from ω to $\hat{\Phi}$), we have by Lemma 3 that there exists $\rho_{\alpha} \in \tilde{\Phi}$ such that ρ_{α} a.d. φ for every $\varphi \in \hat{\Phi}$ and that

$$(7) \qquad \mathrm{Im}\ \rho_{\alpha} \cap \mathrm{Im}\ \big(\hat{\vartheta}_{\alpha}(m)\big) \neq \emptyset \quad \text{for every} \quad m \in \omega.$$

Put $S_{\alpha} = \mathrm{Im}\ \tilde{\varphi}^{1}_{\alpha} \cup \mathrm{Im}\ \rho_{\alpha}$ and let φ^{1}_{α} be an element of $\tilde{\Phi}$ such that $\mathrm{Im}\ \varphi^{1}_{\alpha} = S_{\alpha}$. Since both ρ_{α} and $\tilde{\varphi}^{1}_{\alpha}$ are a.d. from every $\varphi \in \hat{\Phi}$, the same holds for φ^{1}_{α}. This completes the inductive definition.

Thus the family $\{\varphi^{\iota}_{\alpha} \mid \langle \iota, \alpha \rangle \in 2 \times 2^{\omega}\}$ is such that φ^{ι}_{α} a.d. $\varphi^{\iota'}_{\alpha'}$ for $\langle \iota, \alpha \rangle \neq \langle \iota', \alpha' \rangle \in 2 \times 2^{\omega}$. Moreover, by our construction we have that for every $\alpha \in 2^{\omega} \setminus \omega$ there exist $\tilde{\varphi}^{0}_{\alpha}, \tilde{\varphi}^{1}_{\alpha}, \rho_{\alpha} \in \tilde{\Phi}$ such that $\tilde{\varphi}^{0}_{\alpha} = \varphi^{0}_{\alpha}$, $\mathrm{Im}\ \tilde{\varphi}^{1}_{\alpha} \subseteq \varphi^{1}_{\alpha}$, $\mathrm{Im}\ \rho_{\alpha} \subseteq \varphi^{1}_{\alpha}$, and (4), (7) are fulfilled (with $\hat{\lambda}_{\alpha}$ and $\hat{\vartheta}_{\alpha}$ defined by (5) and (6)).

We put $\Phi^{\iota} = \{\varphi^{\iota}_{\alpha} \mid \alpha \in 2^{\omega}\}$ for $\iota = 0, 1$. We claim that X_{Φ^0} and X_{Φ^1} are the required spaces X and Y.

4. Proof of the main result

First, we want to prove that X_{Φ^0}, X_{Φ^1} and $X_{\Phi^0} \times X_{\Phi^1}$ are $\langle\alpha_4\rangle$-spaces. In accordance with [En], for $f, g \colon A \to X, Y$ we denote by $f \Delta g$ the function from A to $X \times Y$ defined by: $(f \Delta g)(a) = \langle f(a), g(a) \rangle$ for every $a \in A$.

Lemma 5. *Let X^0, X^1 be two topological spaces, such that $X^{\iota} = D^{\iota} \cup \{\infty^{\iota}\}$ for $\iota \in 2$, where D^{ι} is discrete and $\infty^{\iota} \notin D^{\iota}$. Suppose that for every $\iota \in 2$ there is at least a $\rho^{\iota} \colon \omega \to D^{\iota}$ such that $\lim_{n \to +\infty} \rho^{\iota}(n) = \infty^{\iota}$. Also, suppose that whenever for every $\langle \iota, i \rangle \in 2 \times \omega$, $\hat{\psi}^{\iota}_{i}$ is a sequence in D^{ι} such that $\lim_{n \to +\infty} \hat{\psi}^{\iota}_{i}(n) = \infty^{\iota}$, then there exist $\hat{\psi}^{\iota} \colon \omega \to D^{\iota}$ for $\iota \in 2$ such that $\lim_{i \to +\infty} \hat{\psi}^{\iota}(i) = \infty^{\iota}$ and*

$$\left|\left\{ i \in \omega \,\middle|\, \mathrm{Im}\ \left(\hat{\psi}^{0} \Delta \hat{\psi}^{1}\right) \cap \mathrm{Im}\ \left(\hat{\psi}^{0}_{i} \Delta \hat{\psi}^{1}_{i}\right) \neq \emptyset \right\}\right| = \omega.$$

Then X^0, X^1 and $X^0 \times X^1$ are all $\langle\alpha_4\rangle$-spaces.

Proof: We first prove that, for $\iota \in 2$, X^ι is an $\langle\alpha_4\rangle$-space. Let $\iota = 0$ (the proof for $\iota = 1$ is symmetric). Since the points of D^0 trivially have the property $\langle\alpha_4\rangle$, suppose to have for every $i \in \omega$ a $\tilde{\psi}_i\colon \omega \to X^0$ such that $\lim_{n\to+\infty} \tilde{\psi}_i(n) = \infty^0$. If for infinitely many $i \in \omega$ the sequence $\tilde{\psi}_i$ takes on the value ∞^0, then the $\tilde{\psi}\colon \omega \to X^0$ having constant value ∞^0 is such that $\left|\left\{i \in \omega \,\middle|\, \mathrm{Im}\ \tilde{\psi}_i \cap \mathrm{Im}\ \tilde{\psi}\right\}\right| = \omega$. Thus, we may suppose $\tilde{\psi}_i\colon \omega \to D^0$ for every $i \in \omega$. Putting $\hat{\psi}^0_i = \tilde{\psi}_i$ and $\hat{\psi}^1_i = \rho^1$ for every $i \in \omega$, we get by hypothesis $\hat{\psi}^0, \hat{\psi}^1\colon \omega \to D^0, D^1$ such that $\lim_{n\to+\infty} \hat{\psi}^\iota(n) = \infty^\iota$ for $\iota \in 2$ and $\left|\left\{i \in \omega \,\middle|\, \mathrm{Im}\ \left(\hat{\psi}^0 \Delta \hat{\psi}^1\right) \cap \mathrm{Im}\ \left(\hat{\psi}^0_i \Delta \hat{\psi}^1_i\right) \neq \emptyset\right\}\right| = \omega$; thus $\hat{\psi}^0$ is such that $\lim_{n\to+\infty} \hat{\psi}^0(n) = \infty^0$ and $\left|\left\{i \in \omega \,\middle|\, \mathrm{Im}\ \hat{\psi}^0 \cap \mathrm{Im}\ \hat{\psi}^0_i \neq \emptyset\right\}\right| = \omega$, i.e., $\left|\left\{i \in \omega \,\middle|\, \mathrm{Im}\ \hat{\psi}^0 \cap \mathrm{Im}\ \tilde{\psi}_i \neq \emptyset\right\}\right| = \omega$.

Now we prove that $X^0 \times X^1$ is an $\langle\alpha_4\rangle$-space. Property $\langle\alpha_4\rangle$ is trivial at the points of $D^0 \times D^1$, while at the points of $\left(D^0 \times \left\{\infty^1\right\}\right) \cup \left(\left\{\infty^0\right\} \times D^1\right)$ it easily comes from the $\langle\alpha_4\rangle$ character of X^0 and X^1. Then consider the point $\left\langle\infty^0, \infty^1\right\rangle$ and suppose to have, for every $\langle\iota, i\rangle \in 2 \times \omega$, a $\tilde{\psi}^\iota_i\colon \omega \to X^\iota$ such that $\lim_{n\to+\infty} \tilde{\psi}^\iota_i(n) = \infty^\iota$. Let $M^\iota = \left\{i \in \omega \,\middle|\, \tilde{\psi}^\iota_i \text{ is frequently equal to } \infty^\iota\right\}$ for $\iota \in 2$: if $|M^0| = \omega$, then the property $\langle\alpha_4\rangle$ at the point ∞^1 of X^1 easily gives the property $\langle\alpha_4\rangle$ at $\left\langle\infty^0, \infty^1\right\rangle$, in this case; if $|M^1| = \omega$, the situation is symmetric. If $|M^\iota| < \omega$ for every $\iota \in 2$, then we may suppose that $\tilde{\psi}^\iota_i\colon \omega \to D^\iota$ for every $i \in \omega$; hence the hypothesis gives the property $\langle\alpha_4\rangle$ at $\left\langle\infty^0, \infty^1\right\rangle$, in this case. □

Lemma 6. *Let $a \in X$, where X is any topological space, and $(a_n)_{n\in\omega}$ be a sequence in X with $\lim_{n\to+\infty} a_n = a$. For every $m \in \omega$, let k_m be an element of K — so that $\left(a_{k_m(i)}\right)_{i\in\omega}$ is a subsequence of $(a_n)_{n\in\omega}$; then there exists $j \in {}^\omega\omega$ such that for every $j' \in {}^\omega\omega$ with $j' \geq j$, $\lim_{m\to+\infty} a_{k_m(j'(m))} = a$.*

Proof: Define j by induction: let $j(0)$ be arbitrary; if $j(m)$ is defined, let $j(m+1)$ be such that $k_{m+1}\big(j(m+1)\big) > k_m\big(j(m)\big)$ (this is possible because $\lim_{n\to+\infty} k_{m+1}(n) = +\infty$). Suppose now $j' \geq j$: given any nbhd V of a, we know that there exists $\bar{n} \in \omega$ such that $\forall\, n \geq \bar{n}\colon a_n \in V$; since $m \mapsto k_m\big(j(m)\big)$ is strictly increasing, there exists $\bar{m} \in \omega$ such that $k_{\bar{m}}\big(j\,(\bar{m})\,\big) \geq \bar{n}$; then for every $m \geq \bar{m}$ we have $k_m\big(j'(m)\big) \geq k_m\big(j(m)\big) \geq k_{\bar{m}}\big(j\,(\bar{m})\,\big) \geq \bar{n}$ (because k_m is strictly increasing) and hence $a_{k_m(j'(m))} \in V$. □

Lemma 7. *Let η^ι, for $\iota \in 2$, be a one-to-one function from ω to 2^ω, and for every $m \in \omega$ let $\tilde{\Psi}_m\colon \omega \to \omega \times \omega$ be such that $\tilde{\Psi}_m = \tilde{\psi}^0_m \Delta \tilde{\psi}^1_m$, with $\mathrm{Im}\ \tilde{\psi}^\iota_m \subseteq \mathrm{Im}\ \varphi^\iota_{\eta^\iota(m)}$ and $\tilde{\psi}^\iota_m \in \tilde{\Phi}$ for $\iota \in 2$. Then there exists $\tilde{\Psi}\colon \omega \to \omega \times \omega$ such that $\lim_{m\to+\infty} \tilde{\Psi}(m) = \langle\infty_{\Phi^0}, \infty_{\Phi^1}\rangle$ and $\tilde{\Psi}(m) \in \mathrm{Im}\ \tilde{\Psi}_m$ for every $m \in \omega$.*

Proof: For $\iota \in 2$, let $\mu^\iota\colon \omega \to \mathrm{DIP}\,(\omega)$ be defined by $\mu^\iota(m) = x_{\eta^\iota(m)}$: then μ^ι is one-to-one. For every $\langle\iota, m\rangle \in 2 \times \omega$, there exists $\gamma^\iota_m \in \omega_1$ such that $\mu^\iota(m) \in M_{\gamma^\iota_m}$ (remember the definition of $\mathrm{DIP}\,(\omega)$): take $\hat{\gamma} \in \omega_1$ such that $\gamma^\iota_m < \hat{\gamma}$ for every

$\langle\iota,m\rangle\in 2\times\omega$. Also, for every $\langle\iota,m\rangle\in 2\times\omega$ there exists a $\phi^\iota_m\in\tilde\Phi$ such that

$$\tilde\psi^\iota_m=\varphi^\iota_{\eta^\iota(m)}\circ\phi^\iota_m$$

— namely, $\phi^\iota_m=\left(\varphi^\iota_{\eta^\iota(m)}\right)^{-1}\circ\tilde\psi^\iota_m$; define $\hat\theta^\iota\in\Theta$, for $\iota\in 2$, by $\hat\theta^\iota(m)=\phi^\iota_m$, and take $\beta^\iota\in 2^\omega$ such that $\hat\theta^\iota=\theta_{\beta^\iota}$. Then $\left\langle\mu^0,\mu^1,\beta^0,\beta^1\right\rangle\in M_{\hat\gamma}\subseteq \mathrm{DIP}\,(\omega)$ and hence there exists $\hat\alpha\in 2^\omega\setminus\omega$ such that $\left\langle\mu^0,\mu^1,\beta^0,\beta^1\right\rangle=x_{\hat\alpha}$; we claim that $\tilde\Psi=\tilde\varphi^0_{\hat\alpha}\Delta\tilde\varphi^1_{\hat\alpha}=\varphi^0_{\hat\alpha}\Delta\tilde\varphi^1_{\hat\alpha}$ has the desired properties.

Indeed, since $\tilde\varphi^1_{\hat\alpha}\in\tilde\Phi$, $\mathrm{Im}\;\tilde\varphi^1_{\hat\alpha}\subseteq\mathrm{Im}\;\varphi^1_{\hat\alpha}$, and $\lim_{m\to+\infty}\varphi^1_{\hat\alpha}(m)=\infty_{\Phi^1}$, we also have that $\lim_{m\to+\infty}\tilde\varphi^1_{\hat\alpha}(m)=\infty_{\Phi^1}$; since $\tilde\varphi^0_{\hat\alpha}=\varphi^0_{\hat\alpha}$, we get:

$$\lim_{m\to+\infty}\left(\tilde\varphi^0_{\hat\alpha}\Delta\tilde\varphi^1_{\hat\alpha}\right)(m)=\langle\infty_{\Phi^0},\infty_{\Phi^1}\rangle\,.$$

On the other hand, by (4) we know that there exists a $j\in{}^\omega\omega$ such that

$$\tilde\varphi^\iota_{\hat\alpha}(m)=\big(\vartheta^\iota_{\hat\alpha}(m)\big)\Big(\big(\theta_{\beta^\iota}(m)\big)\big(j(m)\big)\Big)\ \text{ for every }\ \langle\iota,m\rangle\in 2\times\omega,$$

where $\vartheta^\iota_{\hat\alpha}(m)=\varphi^\iota_{\alpha^\sharp(\mu^\iota(m))}=\varphi^\iota_{\alpha^\sharp(x_{\eta^\iota(m)})}=\varphi^\iota_{\eta^\iota(m)}$. Since $\theta_{\beta^\iota}(m)=\hat\theta^\iota(m)=\phi^\iota_m$ for $\langle\iota,m\rangle\in 2\times\omega$, we have that $\tilde\varphi^\iota_{\hat\alpha}(m)=\big(\varphi^\iota_{\eta^\iota(m)}\circ\phi^\iota_m\big)\big(j(m)\big)=\tilde\psi^\iota_m\big(j(m)\big)$, and hence for every $m\in\omega$: $\tilde\Psi(m)=\left\langle\tilde\varphi^0_{\hat\alpha}(m),\tilde\varphi^1_{\hat\alpha}(m)\right\rangle=\left\langle\tilde\psi^0_m\big(j(m)\big),\tilde\psi^1_m\big(j(m)\big)\right\rangle\in\mathrm{Im}\;\tilde\Psi_m$. □

Corollary 8. *Let η be a one-to-one function from ω to 2^ω, $\iota\in 2$ and for every $m\in\omega$ let $\tilde\psi_m$ be an element of $\tilde\Phi$ such that* $\mathrm{Im}\;\tilde\psi_m\subseteq\mathrm{Im}\;\varphi^\iota_{\eta(m)}$*. Then there exists $\tilde\psi\in{}^\omega\omega$ such that* $\lim_{m\to+\infty}\tilde\psi(m)=\infty_{\Phi^\iota}$ *and $\tilde\psi(m)\in\mathrm{Im}\;\tilde\psi_m$ for every $m\in\omega$.*

Proof: We may suppose $\iota=0$. Put $\eta^0=\eta^1=\eta$ and, for every $m\in\omega$, let $\tilde\psi^0_m=\tilde\psi_m$, $\tilde\psi^1_m=\varphi^1_{\eta(m)}$ and $\tilde\Psi_m=\tilde\psi^0_m\Delta\tilde\psi^1_m$. If $\tilde\Psi=\tilde\psi^0\Delta\tilde\psi^1$ satisfies the thesis of Lemma 7, then $\tilde\psi^0$ is the required $\tilde\psi$. □

Lemma 9. *If φ',φ'' are functions from ω to any set E such that $|\mathrm{Im}\;\varphi'\cap\mathrm{Im}\;\varphi''|=\omega$, then there exist $k',k''\in K$ such that $\varphi'\circ k'=\varphi''\circ k''$ (i.e., φ' and φ'' have a common subsequence), and such a function is one-to-one.*

Proof: We will construct simultaneously k' and k'' by induction. Put $F=\mathrm{Im}\;\varphi'\cap\mathrm{Im}\;\varphi''$ and fix $a_0\in F$: let $k'(0)$ be an element of $\left(\varphi'\right)^{-1}(a_0)$ and $k''(0)$ an element of $\left(\varphi''\right)^{-1}(a_0)$, so that $\varphi'\big(k'(0)\big)=a_0=\varphi''\big(k'(0)\big)$.

Suppose now to have defined $k'(m')$, $k''(m')$ for every $m'\leq m$: since F is infinite, the set $F\setminus\big(\{\varphi'(n)\,|\,n\leq k'(m)\}\cup\{\varphi''(n)\,|\,n\leq k''(m)\}\big)$ contains a point a_{m+1}. Then choose $k'(m+1)\in\left(\varphi'\right)^{-1}(a_{m+1})$ and $k''(m+1)\in\left(\varphi''\right)^{-1}(a_{m+1})$: thus $k'(m{+}1)>k'(m)$, $k''(m{+}1)>k(m)$, $\varphi'\big(k'(m{+}1)\big)=a_{m+1}=\varphi''\big(k''(m{+}1)\big)$ and $\varphi'\big(k'(m+1)\big)\neq\varphi'\big(k'(m')\big)$ for every $m'\leq m$. □

We prove now that X_{Φ^0}, X_{Φ^1} and $X_{\Phi^0} \times X_{\Phi^1}$ are $\langle\alpha_4\rangle$-spaces. By Lemma 5, it is sufficient to show that whenever $\big(\hat{\Psi}_i\big)_{i\in\omega}$ is a sequence of functions from ω to $\omega\times\omega$ such that

$$\forall\, i\in\omega\colon \lim_{n\to+\infty} \hat{\Psi}_i(n) = \langle \infty_{\Phi^0}, \infty_{\Phi^1}\rangle\,,$$

there exists a $\hat{\Psi}\colon\omega\to\omega\times\omega$ such that $\big|\big\{i\in\omega\,\big|\,\mathrm{Im}\ \hat{\Psi}\cap\mathrm{Im}\ \hat{\Psi}_i\neq\emptyset\big\}\big| = \omega$.

For every $i\in\omega$, we have that $\hat{\Psi}_i = \hat{\psi}^0_i\Delta\hat{\psi}^1_i$, where $\lim_{n\to+\infty}\hat{\psi}^\iota_i(n) = \infty_{\Phi^\iota}$ for $\iota\in 2$. By Remark 1, for every $i\in\omega$ there exists $\alpha^0_i\in 2^\omega$ such that $|\,\mathrm{Im}\ \varphi_{\alpha^0_i}\cap \mathrm{Im}\ \hat{\psi}^0_i| = \omega$; now use Lemma 9 to get a $\hat{k}^0_i\in K$ such that $\hat{\psi}^0_i\circ\hat{k}^0_i$ is a one-to-one subsequence of $\varphi^0_{\alpha^0_i}$. Of course, for every $i\in\omega$ we still have that $\lim_{m\to+\infty}\big(\hat{\psi}^1_i\circ \hat{k}^0_i\big)(m) = \infty_{\Phi^1}$, hence by Remark 1 there exists α^1_i such that $\big|\,\mathrm{Im}\ \varphi^1_{\alpha^1_i}\cap\mathrm{Im}\ \big(\hat{\psi}^1_i\circ \hat{k}^0_i\big)\big| = \omega$; using again Lemma 9, we get a $\tilde{k}^1_i\in K$ such that $\hat{\psi}^1_i\circ\hat{k}^0_i\circ\hat{k}^1_i$ is a one-to-one subsequence of $\varphi^1_{\alpha^1_i}$.

Putting, for $\langle\iota,i\rangle\in 2\times\omega$, $\psi^\iota_i = \tilde{\psi}^\iota_i\circ\hat{k}^0_i\circ\hat{k}^1_i$ and $\Psi_i = \psi^0_i\Delta\psi^1_i = \hat{\Psi}\circ\hat{k}^0_i\circ\hat{k}^1_i$, for every $\langle\iota,i\rangle\in 2\times\omega$ we have at the same time that Ψ_i is a subsequence of $\hat{\Psi}_i$ and that ψ^ι_i is a one-to-one subsequence of $\varphi^\iota_{\alpha^\iota_i}$. In particular, if we can find a $\Psi\colon\omega\to\omega\times\omega$ with $\lim_{m\to+\infty}\Psi(m) = \langle\infty_{\Phi^0},\infty_{\Phi^1}\rangle$, such that $|\,\{i\in\omega\,|\,\mathrm{Im}\ \Psi\cap\mathrm{Im}\ \Psi_i\neq\emptyset\}\,| = \omega$, we will also have that

$$\big|\big\{i\in\omega\,\big|\,\mathrm{Im}\ \Psi\cap\mathrm{Im}\ \hat{\Psi}_i\neq\emptyset\big\}\big| = \omega.$$

Let $A^0 = \big\{\alpha^0_i\,\big|\,i\in\omega\big\}$: we have two cases.

1st case. A^0 is infinite.
Fix $H^0\subseteq\omega$ such that $\big\{\alpha^0_i\,|\,i\in H^0\big\} = A^0$ and $\alpha^0_{i'}\neq\alpha^0_{i''}$ for $i',i''\in H^0$ with $i'\neq i''$. Consider now $\tilde{A}^1 = \big\{\alpha^1_i\,|\,i\in H^0\big\}$.

1st subcase. $\tilde{A}^1$ is infinite.
Then there exists an (infinite) $\tilde{H}\subseteq H^0$ such that $\big\{\alpha^1_i\,\big|\,i\in\tilde{H}\big\} = \tilde{A}^1$ and $\alpha^1_{i'}\neq\alpha^1_{i''}$ for $i',i''\in\tilde{H}$ with $i'\neq i''$. Let $\tilde{A}^0 = \big\{\alpha^0_i\,\big|\,i\in\tilde{H}\big\}$: since $\tilde{H}\subseteq H^0$, we also have that $\alpha^0_{i'}\neq\alpha^0_{i''}$ for $i',i''\in\tilde{H}$ with $i'\neq i''$.

As $|\tilde{H}| = \omega$, there exists a (unique) $\tilde{k}\in K$ such that $\mathrm{Im}\ \tilde{k} = \tilde{H}$; then $\big\{\alpha^\iota_{\tilde{k}(m)}\,\big|\,m\in\omega\big\} = \tilde{A}^\iota$ for $\iota\in 2$. Define $\eta^\iota\colon\omega\to 2^\omega$, for $\iota\in 2$, by $\eta^\iota(m) = \alpha^\iota_{\tilde{k}(m)}$: since each η^ι is one-to-one and $\mathrm{Im}\ \psi^\iota_{\tilde{k}(m)}\subseteq\mathrm{Im}\ \varphi^\iota_{\eta^\iota(m)}$ for every $\langle\iota,m\rangle\in 2\times\omega$ (because $\psi^\iota_{\tilde{k}(m)}$ is a subsequence of $\varphi^\iota_{\eta^\iota(m)}$), by Lemma 7 there exists $\tilde{\Psi}\colon\omega\to\omega\times\omega$ such that $\lim_{m\to+\infty}\tilde{\Psi}(m) = \langle\infty_{\Phi^0},\infty_{\Phi^1}\rangle$ and $\mathrm{Im}\ \tilde{\Psi}\cap\mathrm{Im}\ \Psi_{\tilde{k}(m)}\neq\emptyset$ for every $m\in\omega$, which implies that $\big|\big\{i\in\omega\,\big|\,\mathrm{Im}\ \tilde{\Psi}\cap\mathrm{Im}\ \Psi_i\neq\emptyset\big\}\big| = \omega$.

2^nd^ subcase. $\tilde{A}^1$ is finite.

Then there exists an infinite subset $\tilde{H}$ of H^1 and an $\hat{\alpha} \in 2^\omega$ such that $\forall i \in \tilde{H}\colon \alpha_i^1 = \hat{\alpha}$. Again, let $\tilde{k} \in K$ be such that $\mathrm{Im}\ \tilde{k} = \tilde{H}$: since $\tilde{H} \subseteq H^0$, we have that $\eta\colon \omega \to 2^\omega$ defined by $\eta(m) = \alpha^0_{\tilde{k}(m)}$ is one-to-one.

For every $m \in \omega$ we have that $\psi^1_{\tilde{k}(m)}$ is a one-to-one subsequence of $\varphi^1_{\alpha^1_{\tilde{k}(m)}}$, which coincides with $\varphi^1_{\hat{\alpha}}$ because $\tilde{k}(m) \in \tilde{H}$; hence by Lemma 6 there exists $j \in {}^\omega\omega$ such that

$$\forall j' \geq j\colon \lim_{m\to+\infty} \psi^1_{\tilde{k}(m)}\big(j'(m)\big) = \infty_{\Phi^1}. \tag{8}$$

Now define, for every $m \in \omega$, a $\tilde{\psi}_m \in \tilde{\Phi}$ by:

$$\tilde{\psi}_m(n) = \psi^0_{\tilde{k}(m)}\big(n + j(m)\big). \tag{9}$$

Observe that, for every $m \in \omega$, $\mathrm{Im}\ \tilde{\psi}_m \subseteq \mathrm{Im}\ \psi^0_{\tilde{k}(m)} \subseteq \mathrm{Im}\ \varphi^0_{\alpha^0_{\tilde{k}(m)}} = \mathrm{Im}\ \varphi^0_{\eta(m)}$. Then by Corollary 8 there exists $\psi^0 \in {}^\omega\omega$ such that

$$\lim_{m\to+\infty} \psi^0(m) = \infty_{\Phi^0} \quad \text{and} \quad \forall m \in \omega\colon \psi^0(m) \in \mathrm{Im}\ \tilde{\psi}_m;$$

using (9), we have that for every $m \in \omega$ there exists $\tilde{n}(m) \in \omega$ such that $\psi^0(m) = \psi^0_{\tilde{k}(m)}\big(\tilde{n}(m) + j(m)\big)$.

Put $j'(m) = \tilde{n}(m) + j(m)$ and define $\psi^1 \in {}^\omega\omega$ by $\psi^1(m) = \psi^1_{\tilde{k}(m)}\big(j'(m)\big)$: then $\lim_{m\to+\infty} \psi^1(m) = \infty_{\Phi^1}$ by (8). Thus, putting $\Psi = \psi^0 \Delta \psi^1$, we have that

$$\lim_{m\to+\infty} \Psi(m) = \langle \infty_{\Phi^0}, \infty_{\Phi^1} \rangle\,;$$

moreover, for every $m \in \omega$,

$$\begin{aligned}\Psi(m) = \Big\langle \psi^0(m), \psi^1(m) \Big\rangle &= \Big\langle \psi^0_{\tilde{k}(m)}\big(j'(m)\big), \psi^1_{\tilde{k}(m)}\big(j'(m)\big) \Big\rangle \\ &= \Psi_{\tilde{k}(m)}\big(j'(m)\big) \in \mathrm{Im}\ \Psi_{\tilde{k}(m)},\end{aligned}$$

so that $\big|\big\{ i \in \omega \,\big|\, \mathrm{Im}\ \Psi \cap \mathrm{Im}\ \Psi_i \neq \emptyset \big\}\big| = \omega$.

2^nd^ case. A^0 is finite.

Then there exists an infinite subset H^0 of ω and an $\hat{\alpha}^0 \in 2^\omega$ such that $\forall i \in H^0\colon \alpha_i^0 = \hat{\alpha}^0$. Again, let $\tilde{A}^1 = \big\{ \alpha_i^1 \,|\, i \in H^0 \big\}$.

1^st^ subcase. $\tilde{A}^1$ is infinite.

Then there exists an infinite subset $\tilde{H}$ of H^1 such that $\big\{ \alpha_i^1 \,\big|\, i \in \tilde{H} \big\} = \tilde{A}^1$ and $\alpha^1_{i'} \neq \alpha^1_{i''}$ for distinct $i', i'' \in \tilde{H}$. The situation is symmetric to the 2^nd^ subcase of the 1^st^ case.

2nd subcase. $\tilde{A}^1$ is finite.

Then there exists an infinite $\tilde{H} \subseteq H^0$ and an $\hat{\alpha}^1 \in 2^\omega$ such that $\forall\, i \in \tilde{H}\colon \alpha^1_i = \hat{\alpha}^1$; clearly, since $\tilde{H} \subseteq H^0$, we also have that $\forall\, i \in \tilde{H}\colon \alpha^0_i = \hat{\alpha}^0$. Let $\tilde{k} \in K$ such that $\operatorname{Im} \tilde{k} = \tilde{H}$: then for every $\langle \iota, m\rangle \in 2 \times \omega$ we have that $\psi^\iota_{\tilde{k}(m)}$ is a subsequence of $\varphi^\iota_{\alpha^\iota_{\tilde{k}(m)}} = \varphi^\iota_{\hat{\alpha}^\iota}$. Applying Lemma 6, we get $j^0, j^1 \in {}^\omega\omega$ such that

$$\forall\, \iota \in 2\colon \forall\, j' \in {}^\omega\omega\colon \bigl(j' \geq j^\iota \Longrightarrow \lim_{m\to+\infty} \psi^\iota_{\tilde{k}(m)}\bigl(j'(m)\bigr) = \infty_{\Phi^\iota}\bigr).$$

Let $j = \sup\bigl\{j^0, j^1\bigr\}$ and define $\psi^\iota \in {}^\omega\omega$ for $\iota \in 2$ by:

$$\psi^\iota(m) = \psi^\iota_{\tilde{k}(m)}\bigl(j(m)\bigr)$$

for every $m \in \omega$. Putting $\Psi = \psi^0 \Delta \psi^1$, we have that $\lim_{m\to+\infty} \Psi(m) = \langle \infty_{\Phi^0}, \infty_{\Phi^1}\rangle$ and that

$$\forall\, m \in \omega\colon \Psi(m) = \Bigl\langle \psi^0(m), \psi^1(m)\Bigr\rangle = \Bigl\langle \psi^0_{\tilde{k}(m)}\bigl(j(m)\bigr), \psi^1_{\tilde{k}(m)}\bigl(j(m)\bigr)\Bigr\rangle \in \operatorname{Im}\, \Psi_{\tilde{k}(m)},$$

whence $\Bigl|\bigl\{i \in \omega \bigm| \operatorname{Im}\, \Psi \cap \operatorname{Im}\, \Psi_i \neq \emptyset\bigr\}\Bigr| = \omega$.

Now we proceed to show that $X_{\Phi^0} \times X_{\Phi^1}$ is not Fréchet-Urysohn. First of all, we prove that putting $D = \{\langle \ell, \ell\rangle \mid \ell \in \omega\}$, we have that $\langle \infty_{\Phi^0}, \infty_{\Phi^1}\rangle \in \overline{D}$ in $X_{\Phi^0} \times X_{\Phi^1}$.

Indeed, let V^0, V^1 be arbitrary nbhds of $\infty_{\Phi^0}, \infty_{\Phi^1}$ in X_{Φ^0}, X_{Φ^1}, respectively. For every $m \in \omega$, $\varphi^0_m = f^0_m$ belongs to Φ^0, and hence there exists $j \in \omega$ such that

$$\forall\, m \in \omega\colon \forall\, n \geq j(m)\colon f^0_m(n) \in V^0. \tag{10}$$

Take $\hat{\alpha} \in 2^\omega \setminus \omega$ such that $j = \hat{\jmath}_{\hat{\alpha}}$: then (5), (6) and (7) (for $\alpha = \hat{\alpha}$) combine to show that

$$\forall\, m \in \omega\colon \exists\, n' \geq \hat{\jmath}_{\hat{\alpha}}(m)\colon f^0_m(n') \in \operatorname{Im}\, \rho_{\hat{\alpha}};$$

hence we can associate to every $m \in \omega$ a $\tilde{n}(m) \geq \hat{\jmath}_{\hat{\alpha}}(m)$ such that

$$f^0_m\bigl(\tilde{n}(m)\bigr) \in \operatorname{Im}\, \rho_{\hat{\alpha}} \subseteq \operatorname{Im}\, \varphi^1_{\hat{\alpha}}. \tag{11}$$

Since $\lim_{n\to+\infty} \varphi^1_{\hat{\alpha}}(n) = \infty_{\Phi^1}$ in X_{Φ^1}, there exists $n^\sharp \in \omega$ such that

$$\forall\, n \geq n^\sharp\colon \varphi^1_{\hat{\alpha}}(n) \in V^1. \tag{12}$$

Observe that $m \mapsto f^0_m\bigl(\tilde{n}(m)\bigr)$ is one-to-one from ω to ω (because $\operatorname{Im}\, f^0_{m'} \cap \operatorname{Im}\, f^0_{m''} = F_{0,m'} \cap F_{0,m''} = \emptyset$ for $m' \neq m''$); therefore the set $\bigl\{f^0_m\bigl(\tilde{n}(m)\bigr) \bigm| m \in \omega\bigr\}$

cannot be contained into $\{\varphi^1_{\hat\alpha}(n) \mid n < n^\sharp\}$, and hence by (11) there exists $n^* \ge n^\sharp$ such that

$$\varphi^1_{\hat\alpha}(n^*) \in \{f^0_m(\tilde n(m)) \mid m \in \omega\}.$$

Since $\varphi^1_{\hat\alpha}(n^*) \in V^1$ by (12), and $f^0_m(\tilde n(m)) \in V^0$ for every $m \in \omega$ (because of (10) and the fact that $\tilde n(m) \ge \hat\jmath_{\hat\alpha}(m) = j(m)$), we conclude that for some $\ell \in \omega$, $\langle \ell, \ell\rangle \in V^0 \times V^1$.

Now, if $X_{\Phi^0} \times X_{\Phi^1}$ were Fréchet, there would exist a sequence in D which converges to $\langle \infty_{\Phi^0}, \infty_{\Phi^1}\rangle$, and clearly it could be supposed to be one-to-one. Thus, there would exist $\tilde\varphi \in \tilde\Phi$ such that $\lim_{n\to+\infty} \tilde\varphi(n) = \infty_{\Phi^0}$ in X_{Φ^0} and $\lim_{n\to+\infty} \tilde\varphi(n) = \infty_{\Phi^1}$ in X_{Φ^1}. From the former relation we have that $|\mathrm{Im}\ \tilde\varphi \cap \mathrm{Im}\ \varphi^0_{\hat\alpha}| = \omega$ for some $\hat\alpha \in 2^\omega$; by Lemma 9, there exists $\varphi^* \in \tilde\Phi$ which is a common subsequence of $\tilde\varphi$ and $\varphi^0_{\hat\alpha}$. In particular, since $\lim_{n\to+\infty} \tilde\varphi(n) = \infty_{\Phi^1}$ in X_{Φ^1}, we also have that $\lim_{n\to+\infty} \varphi^*(n) = \infty_{\Phi^1}$ in X_{Φ^1}, so that there exists $\alpha^* \in 2^\omega$ such that $|\mathrm{Im}\ \varphi^* \cap \mathrm{Im}\ \varphi^1_{\alpha^*}| = \omega$, and hence $|\mathrm{Im}\ \varphi^0_{\hat\alpha} \cap \mathrm{Im}\ \varphi^1_{\alpha^*}| = \omega$ (because $\mathrm{Im}\ \varphi^* \subseteq \mathrm{Im}\ \varphi^0_{\hat\alpha}$). This contradicts the fact that every element of Φ^0 is almost disjoint from every element of Φ^1.

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