Closure rings

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Abstract. We consider rings equipped with a closure operation defined in terms of a collection of commuting idempotents, generalising the idea of a topological closure operation defined on a ring of sets. We establish the basic properties of such rings, consider examples and construction methods, and then concentrate on rings which have a closure operation defined in terms of their lattice of central idempotents.

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1. Basic properties

The notion of a *closure algebra* as defined in [3] has been around for some time in one form or another. They are Boolean algebras with identity on which a closure operation is defined, satisfying the usual Kuratowski conditions: $C(0) = 0$, $aC(a) = a$, $C(C(a)) = C(a)$, and $C(a \vee b) = C(a) \vee C(b)$ for all a, b. These objects are also of significance in modal logic, where they provide the algebraic models for the so-called S_4 form of modal logic, probably the most important of the non-classical modal logics. In fact we shall view all Boolean algebras as Boolean rings, and call closure algebras so viewed Boolean closure rings.

We now generalise the notion of a closure ring. If R is a ring, we have the *adjoint operation* $\circ$ given by $a \circ b = a + b - ab$ for all $a, b \in R$, which is associative, has 0 as an identity, and is commutative if and only if R is. (In a Boolean ring, this is exactly the join operation in the corresponding Boolean algebra.) We say R is a *closure ring* if it has an additional unary operation C such that for all $a, b \in R$:

1. $C(0) = 0$;
2. $a \circ C(a) = C(a)$;
3. $C(C(a)) = C(a)$;
4. $C(a) \circ C(b) = C(b) \circ C(a)$; and
5. $C(a) \circ C(b) = C(a \circ b) \circ C(b)$.

In this case C is a *closure operation* on R. Note that if R has an identity, $C(1) = 1$ is immediate from the second condition.

The above definition generalises the Boolean case: if R is Boolean, then letting $b = a \vee c = a \circ c$ where $a, c \in R$, we have that $C(a) \vee C(a \vee c) = C(a \vee c)$ so C is order-preserving, so $C(a \vee b) \geq C(a) \vee C(b) \geq C(a \vee b)$, and we recover the previously considered Boolean closure ring definition.

We say R is a *strong closure ring* if $a \circ b \circ C(a) = b \circ C(a)$.

Lemma 1. *The closure ring R is strong if and only if $abC(a) = ab$ for all $a, b \in R$.*

PROOF: $a \circ b \circ C(a) = b \circ C(a)$ expands out to $a + b + C(a) - bC(a) - ab - aC(a) + abC(a) = b + C(a) - bC(a)$, that is, $ab = abC(a)$ upon use of the rule $aC(a) = a$ (equivalent to rule 2 above). $\square$

Clearly if the closure ring R is commutative then it is strong. Every ring with identity is a strong closure ring if one defines $C(0) = 0$ with $C(a) = 1$ otherwise.

If the closure ring R has an identity, then one may define $I(a) = 1 - C(1 - a)$ and obtain rules for I in terms of the multiplicative monoid of R analogous to those for C in terms of the adjoint monoid. Indeed one may make I primitive in such cases and recover the closure operation from it, in a sense generalising the familiar equivalence between the open and closed set approaches to topology.

Let R be a closure ring. Define $L_R = \{a \in R \mid C(a) = a\} = \{C(a) \mid a \in R\}$.

Proposition 2. *L_R is a submonoid of $(R, \circ)$ which is a semilattice and, viewing $\circ$ as meet in L_R, $C(a) = \max\{\alpha \mid \alpha \in L_R, a \circ \alpha = \alpha\} = \max\{\alpha \mid \alpha \in L_R, a\alpha = a\}$ for all $a \in R$.*

PROOF: For all $a \in R$, $C(a) \circ C(a) = C(a) \circ C(C(a)) = C(a)$; thus the elements of L_R are idempotent. Also $\circ$ is idempotent on L_R by the fourth rule. Furthermore, $0 = C(0) \in L_R$, and for all $\alpha, \beta \in L_R$

$$\begin{aligned}
\alpha \circ \beta &= \alpha \circ \alpha \circ \beta \\
&= \alpha \circ \beta \circ \alpha \\
&= C(\alpha) \circ C(\beta) \circ \alpha \\
&= C(\alpha \circ \beta) \circ C(\beta) \circ \alpha \\
&= \alpha \circ C(\beta) \circ C(\alpha \circ \beta) \\
&= \alpha \circ \beta \circ C(\alpha \circ \beta) \\
&= C(\alpha \circ \beta).
\end{aligned}$$

So L_R is a submonoid which is a semilattice.

Further, for $a \in R$, $a \circ C(a) = C(a)$, and if $a \circ \alpha = \alpha$ for some $\alpha \in L_R$, then

$$C(a) \circ \alpha = C(a) \circ C(\alpha) = C(a \circ \alpha) \circ C(\alpha) = C(\alpha) \circ C(\alpha) = \alpha,$$

so $\alpha \leq C(a)$. Hence $C(a) = \max\{\alpha \in L_R \mid a \circ \alpha = \alpha\}$. $\square$

The converse works also.

Proposition 3. *If R is a ring in which $(L_R, \circ)$ is a semilattice which is a submonoid of $(R, \circ)$ with $\circ$ viewed as meet in L_R, and if $C(a) = \max\{\alpha \in L_R \mid a\alpha = a\}$ exists for all $a \in R$, then R together with C is a closure ring.*

PROOF: The only closure ring rule that is not immediate is the last one. Letting $\alpha = C(a)$, $\beta = C(b)$ and $\gamma = C(a \circ b)$, we have that

$$(a \circ b) \circ (\alpha \circ \beta) = a \circ (b \circ \beta) \circ \alpha = a \circ \beta \circ \alpha = (a \circ \alpha) \circ \beta = \alpha \circ \beta,$$

so $C(a) \circ C(b) \leq C(a \circ b)$. Conversely,

$$a \circ (\gamma \circ \beta) = a \circ \beta \circ \gamma = a \circ b \circ \beta \circ \gamma = (a \circ b) \circ \gamma \circ \beta = \gamma \circ \beta.$$

So $C(a \circ b) \circ C(b) \leq C(a)$. Hence $C(a) \circ C(b) = C(a \circ b)$. □

Viewing $\circ$ as meet is dual to the convention in the Boolean case where $\circ$ is viewed as join and the order models set inclusion. Note that C is order-preserving on L_R: if $\alpha \leq \beta$ then $\alpha \circ \beta = \alpha$ then

$$C(\alpha) = C(\alpha) \circ C(\alpha) = C(\beta \circ \alpha) \circ C(\alpha) = C(\beta) \circ C(\alpha),$$

so $C(\alpha) \leq C(\beta)$.

For any closure ring R, L_R is closed under multiplication.

Proposition 4. *L_R is a Brouwerian lattice in which $\alpha \vee \beta = \alpha\beta$ and $\alpha \wedge \beta = \alpha \circ \beta$ for all $\alpha, \beta \in L_R$.*

Proof: Let $\alpha, \beta, \gamma \in L_R$. It is routine to verify that $(\alpha\beta) \circ \gamma = (\alpha \circ \gamma)(\beta \circ \gamma)$. Thus from the five defining properties of C and the order-preserving property,

$$\begin{aligned} C(\alpha\beta) &= (\alpha\beta) \circ C(\alpha\beta) \\ &= (\alpha \circ C(\alpha\beta))(\beta \circ C(\alpha\beta)) \\ &= (C(\alpha) \circ C(\alpha\beta))(C(\beta) \circ C(\alpha\beta)) \\ &= C(\alpha)C(\beta) \\ &= \alpha\beta. \end{aligned}$$

Thus L_R is closed under ring multiplication. Moreover, because $\alpha^2 = \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in L_R$, L_R is a semilattice with respect to the ring product; indeed $\alpha \circ \beta = \alpha$ if and only if $\alpha\beta = \beta$ for any such α, β, L_R is a lattice with meet the adjoint operation, join the ring product and top element 0. Finally, and omitting a couple of easily checked steps,

$$\begin{aligned} C(\beta - \alpha\beta) &= \max\{\gamma \in L_R \mid (\beta - \alpha\beta) \circ \gamma = \gamma\} \\ &= \max\{\gamma \in L_R \mid (\alpha \circ \gamma)\beta = \beta\} \\ &= \max\{\gamma \in L_R \mid \alpha \circ \gamma \circ \beta = \alpha \circ \gamma\} \\ &= \max\{\gamma \in L_R \mid \alpha \circ \gamma \leq \beta\}, \end{aligned}$$

the relative pseudo-complement of α with respect to β, so L_R is Brouwerian. □

Recall that every ring with identity is a strong closure ring if one defines $C(0) = 0$ with $C(a) = 1$ otherwise. Because $C(a) = 0$ implies $a = a \circ 0 = a \circ C(a) = C(a) = 0$, this condition is equivalent to saying that $L_R = \{0, 1\}$. In the Boolean case, such closure rings arise as models of so-called S_5 modal logic.

Lemma 5. *The closure ring R is strong if and only if $\alpha \circ a \circ \alpha = a \circ \alpha$ (that is, $\alpha a \alpha = \alpha a$) for all $a \in R$ and $\alpha \in L_R$.*

Proof: If R is strong then because $C(\alpha) = \alpha$ for all $\alpha \in L_R$, it is immediate that $\alpha \circ a \circ \alpha = a \circ \alpha$ for all $a \in R$ (and from Lemma 1 that $\alpha a \alpha = \alpha a$). Conversely, if for all $a \in R$ and $\alpha \in L_R$ in the closure ring R, $\alpha \circ a \circ \alpha = a \circ \alpha$ (which expands out to $\alpha a \alpha = \alpha a$ as is almost immediate), then $a \circ b \circ C(a) = a \circ C(a) \circ b \circ C(a) = C(a) \circ b \circ C(a) = b \circ C(a)$ and so R is strong. □

Thus if L_R is contained in the centre of R, then R is strong.

Let R be a ring with identity and let X be a topological space. Let A be the ring of functions $X \to R$ with pointwise operations and let L_A consist of all characteristic functions of closed subsets of X. Then L is closed under the adjoint operation (because the union of two closed sets is closed) and in fact L is a semilattice. Under its natural order (viewed as a meet-semilattice), $C(f) = \max\{h \mid h \in L, f \circ h = h\}$ is defined for all $f \in A$ and is the (characteristic function of) the closure of the subset on which f is non-zero. (Again note that the natural order on L is the opposite of set inclusion.) Clearly L in this example is a subset of the centre of A, so A is a strong closure ring in which $L_A = L$. This idea can be generalised in the obvious way to Cartesian products indexed by a topological space. (We see some non-strong examples shortly.)

Theorem 6. *Let R be a ring having an additional unary operation C. Then*

1. *R is a closure ring if and only if, for all $a, b, c \in R$: $C(0) = 0$, $C(a) \circ C(b) = C(b) \circ C(a)$, $a \circ C(a) = C(a)$ and $f(b) \circ C(a) = f(b \circ C(a)) \circ C(a)$ for every derived unary operation f on R not involving the adjoint operation (equivalently, the ring multiplication);*
2. *R is a strong closure ring if and only if, for all $a, b, c \in R$, $C(0) = 0$, $C(a) \circ C(b) = C(b) \circ C(a)$ and $f(a) \circ C(a) = f(0) \circ C(a)$ where $f(x)$ is an arbitrary derived unary operation on R.*

Proof: First note that in any ring, $(-a) \circ b = -(a \circ b) + 2b$ and $(a + b) \circ c = a \circ c + b \circ c - c$.

Suppose R is a closure ring. We must show that $f(b) \circ C(a) = f(b \circ C(a)) \circ C(a)$ holds, where $f(x)$ is a derived unary operation on R not involving the adjoint operation. We do this by induction on the size of any expression representing $f(x)$.

If $f(x) = x$, the result is immediate. Now suppose the identity holds for $g(x)$, $h(x)$.

If $f(x) = -g(x)$, then

$$\begin{aligned} f(b \circ C(a)) \circ C(a) &= (-g(b \circ C(a))) \circ C(a) \\ &= -(g(b \circ C(a)) \circ C(a)) + 2C(a) \\ &= -(g(b) \circ C(a)) + 2C(a) \\ &= (-g(b)) \circ C(a) \\ &= f(b) \circ C(a). \end{aligned}$$

If $f(x) = g(x) + h(x)$, then

$$\begin{aligned} f(b \circ C(a)) \circ C(a) &= (g(b \circ C(a)) + h(b \circ C(a))) \circ C(a) \\ &= g(b \circ C(a)) \circ C(a) + h(b \circ C(a)) \circ C(a) - C(a) \\ &= g(b) \circ C(a) + h(b) \circ C(a) - C(a) \\ &= (g(b) + h(b)) \circ C(a) \\ &= f(b) \circ C(a). \end{aligned}$$

If $f(x) = C(g(x))$, then

$$\begin{aligned} f(b \circ C(a)) \circ C(a) &= C(g(b \circ C(a))) \circ C(a) \\ &= C(g(b \circ C(a)) \circ C(a)) \circ C(a) \\ &= C(g(b) \circ C(a)) \circ C(a) \\ &= C(g(b)) \circ C(a) \\ &= f(b) \circ C(a). \end{aligned}$$

It follows that the identity holds for all possible $f(x)$ of the required form.

Conversely, suppose that R is a ring having an additional unary operation C for which, for all $a, b, c \in R$: $C(0) = 0$, $C(a) \circ C(b) = C(b) \circ C(a)$, $a \circ C(a) = C(a)$ and $f(b) \circ C(a) = f(b \circ C(a)) \circ C(a)$ where $f(x)$ is a derived unary operation on R not involving the adjoint operation. Then also $C(C(a)) = C(a) \circ C(C(a)) = C(C(a)) \circ C(a) = C(0 \circ C(a)) \circ C(a) = C(0) \circ C(a) = 0 \circ C(a) = C(a)$, and $C(a) \circ C(b) = C(a \circ C(b)) \circ C(b) = C(a \circ (b \circ C(b))) \circ C(b) = C((a \circ b) \circ C(b)) \circ C(b) = C(a \circ b) \circ C(b)$, so R is a closure ring.

The proof for the strong case follows similar lines, although the recursive step in the proof of the identity scheme $f(a) \circ C(a) = f(0) \circ C(a)$, where $f(x)$ is an arbitrary derived unary operation on R, requires consideration of a case of the form $f(x) = g(x) \circ h(x)$ where the identity holds for $g(x)$ and $h(x)$. Then

$$\begin{aligned} (g(a) \circ h(a)) \circ C(a) &= (g(a) \circ C(a)) \circ (h(a) \circ C(a)) \\ &= (g(0) \circ C(a)) \circ (h(0) \circ C(a)) \\ &= (g(0) \circ h(0)) \circ C(a) \\ &= f(0) \circ C(a). \end{aligned}$$

□

Corollary 7. *For the closure ring R and for all $a, b \in R$, $f(a) \circ C(a - b) = f(b) \circ C(a - b)$, where $f(x)$ is any derived unary operation on R not involving the adjoint operation.*

PROOF: For such an $f(x)$, $f(a) \circ C(a - b) = f(b + (a - b)) \circ C(a - b) = f(b + 0) \circ C(a - b)$ by the previous theorem applied to $g(x) = f(b + x)$ with $x = a - b$. □

2. Normal filters and closed ideals

Every closure ring R is a ring; consequently congruences of R respecting all operations including C correspond to certain ideals of R. In the case of Boolean closure rings, it is shown in [5] that these are exactly ideals closed under C, a fact which we generalise below. Let us call an ideal J of the closure ring R *closed* if $C(i) \in J$ for all $i \in J$. Note that both R and $\{0\}$ are closed ideals of R.

We begin with a useful result.

Lemma 8. *If R is a closure ring, then for all $a, b \in R$,*

- $C(-a) = C(a)$,
- $C(a+b) \geq C(a) \circ C(b)$,
- $C(ab) \geq C(b)$, *and*
- *if R is strong then $C(ab) \geq C(a)$.*

PROOF: Let $a, b \in R$. For $\alpha \in L_R$, $a\alpha = a$ if and only if $(-a)\alpha = -a$, and the first part follows from Proposition 2. Similarly, because

$$(a+b)(C(a) \circ C(b)) = (a+b)(C(a) + C(b) - C(a)C(b)) = a + b,$$

it follows that $C(a+b) \geq C(a) \circ C(b)$, proving the second part. Further, $(ab)C(b) = a(bC(b)) = ab$ so $C(b) \leq C(ab)$, and if R is strong then $(ab)C(a) = ab$ so $C(a) \leq C(ab)$, and the final two parts are proved. □

Theorem 9. *The ring ideal J of the closure ring R induces a congruence respecting C if and only if J is closed.*

PROOF: Let J be an ideal of the closure ring R.

Suppose J is closed, with a and b congruent modulo J: thus $a - b \in J$, so because J is closed, $C(a-b) \in J$. But by Corollary 7, $C(a) \circ C(a-b) = C(b) \circ C(a-b)$, from which it follows immediately that $C(a) - C(b) = (C(a) - C(b))C(a-b) \in J$, so the ring congruence induced by J respects C also.

Conversely, if J respects C, then $C(j) \in J$ for all $j \in J$ since $C(0) = 0$. □

Let R be a closure ring, with F a filter of L_R: thus for all $\alpha, \beta \in F$, $\alpha \circ \beta \in F$ and if $\alpha \in F$ and $\beta \geq \alpha$ for some $\beta \in L_R$, then $\beta \in F$. Note that if $C(a) \in F$ then because $C(ra) \geq C(a)$, $C(ra) \in F$ from Lemma 8; indeed if R is strong then $C(ar) \in F$ also. In general, we shall say a filter F of L_R is *normal* if $C(a) \in F$ for some $a \in R$ implies $C(ar) \in F$ for all $r \in R$; thus all filters of L_R are normal in the strong case, by the final part of Lemma 8. Note that the collection of normal filters of L_R is closed under arbitrary intersections; hence it is a complete lattice.

For any closed ideal J of R, if $C(a) \in J$ then $a = aC(a) \in J$ also, and it follows easily that $J \cap L_R$ is a normal filter of L_R. Conversely, for any normal filter F of L_R, define $J_F = \{a \mid C(a) \in F\}$.

Lemma 10. *J_F is a closed ideal of R.*

PROOF: Suppose $a, b \in J_F$. Then $C(a), C(b) \in F$, so $C(a) \circ C(b) \in F$. Because $C(a+b) \geq C(a) \circ C(b)$ from the second part of Lemma 8, $C(a+b) \in F$ and so $a+b \in J_F$. Also, $C(a) = C(-a)$ from the first part of Lemma 8, so $-a \in J_F$. Finally, if $r \in R$, then $C(ra) \geq C(a)$ by the fourth part of Lemma 8, so $C(ra) \in F$ and $ra \in J_F$; also, $C(ar) \in F$ by normality, so $ar \in J_F$. So J_F is an ideal of R, which is closed, since if $a \in J_F$ then $C(C(a)) = C(a) \in F$ so $C(a) \in J_F$. □

Theorem 11. *The lattice of normal filters of L_R is isomorphic to the lattice of closed ideals of R, under the correspondence*

$$J \leftrightarrow J \cap L_R, F \leftrightarrow J_F.$$

PROOF: The correspondences are clearly order-preserving; we show they are mutually inverse.

Suppose F is a normal filter of L_R. If $\alpha \in F$ then $C(\alpha) = \alpha \in F$, so $\alpha \in J_F \cap L_R$. Conversely, if $\alpha \in J_F \cap L_R$ then $\alpha = C(\alpha) \in F$. Thus $J_F \cap L_R = F$. On the other hand, suppose I is a closed ideal of R. If $a \in J_{I \cap L_R}$ then $C(a) \in I$ so $a \in I$. Conversely, if $a \in I$ then $C(a) \in I \cap L_R$ and so $a \in J_{I \cap L_R}$. Thus $I = J_{I \cap L_R}$. □

Theorem 12. *Let R be a strong closure ring. Then R is simple if and only if $L_R = \{0, e\}$ for some $e \in R$; in this case e is a right identity in R.*

PROOF: If R is strong and simple then the only filters in L_R are $\{0\}$ and L_R itself by the previous theorem, so $L_R = \{0, e\}$ for some $e \in R$. Now for $a \neq 0$, the fact that $a = aC(a)$ implies that $C(a) = e$, so $ae = a$; trivially $0e = 0$.

Conversely, if $L_R = \{0, e\}$ for some $e \in R$ then for non-zero $a \in R$, arguing as above shows that $C(a) = e$, so for any non-zero closed ideal I of R and any non-zero $b \in I$, $C(a) = e = C(b) \in I$, so $a \in I$. Thus $I = R$ and R is simple. □

Any Cartesian product R of simple closure rings is such that the closure of an element represents the "characteristic function" of where an element is non-zero, with $C(a)$ equal to zero where a is zero and e where it is non-zero, for any $a \in R$; moreover the semilattice L_R of all such "characteristic functions" is a Boolean algebra, isomorphic to the power set of the index set. Conversely, we have the following

Theorem 13. *Let R be a strong closure ring for which L_R is a Boolean algebra. Then R is a subdirect product of simple closure rings.*

PROOF: Recall from Proposition 4 that in any closure ring R, $(L_R, \vee, \wedge)$ is a Brouwerian lattice in which $\vee$ is ring multiplication and $\wedge$ is $\circ$. Suppose L_R is a Boolean algebra with bottom element e. Then the complement α' of $\alpha \in L_R$ is the relative pseudo-complement of α with respect to e which, by the proof of Proposition 4, is $C(e - \alpha e) = C(e - \alpha) = C(\alpha - e)$ by the first part of Lemma 8.

Let F be an ultra-filter of L_R; thus $\alpha \in F$ or $\alpha' \in F$ for any $\alpha \in L_R$, and $e \notin F$. Now if $a \notin J_F$ then $C(a) \notin F$, so $C(C(a)-e) = C(a)' \in F$, and so $C(a)-e \in J_F$. Thus $L_{R/J_F} = \{0+J_F, e+J_F\}$, and R/J_F is simple by Theorem 12.

Finally, let $J = \bigcap\{J_F \mid F \text{ an ultrafilter of } L_R\} = \{a \mid C(a) \in F \text{ for all ultrafilters } F \text{ of } L_R\}$. Suppose $a \neq 0$. If $C(a) = e$ and $a \in J$ then $L_R = \{e\}$, whence $e = 0$ and so $R = \{0\}$, a contradiction, so $a \notin J$. If $C(a) \neq e$ then because $C(a) \neq 0$, $\{\beta \mid C(a)' \leq \beta\}$ is a filter which extends to an ultrafilter G containing $C(a)'$ and therefore not containing $C(a)$, so $a \notin J_G$ and $a \notin J$. Thus $J = \{0\}$ and so R is a subdirect product of the R/J_F. □

Next we consider an interesting family of examples. Let R be a ring with identity $1 \neq 0$. Then in $M_2(R)$, let $L = \{0, e, f, 1\}$, where $0, 1$ are as usual and $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $e^2 = e$, $f^2 = f$ and L is a subsemigroup of R which is a semilattice with $1 < e,\ f < 0$. Moreover if $ae = a$ and $af = a$ then $a = a1 = a(e+f) = 2a$ so $a = 0$. It follows easily that $C(a) = \max\{\alpha \in L \mid a \circ \alpha = \alpha\} = \max\{\alpha \in L \mid a\alpha = a\}$ exists for all $a \in M_2(R)$, which is therefore a closure ring in which $L_{M_2(R)} = L$. Letting $a = e$ and $b = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, observe that $ab = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) = b$, but $abC(a) = bC(a) = be = e$, so $M_2(R)$ is not strong. Note that if R is a simple ring then so is $M_2(R)$ which is therefore simple as a closure ring, and hence subdirectly irreducible, although $|L| > 2$ and L is a Boolean algebra (in which $e' = f$).

Now consider the subring A of $M_2(R)$ consisting of lower triangular matrices of the form $\left(\begin{smallmatrix} x & 0 \\ y & z \end{smallmatrix}\right)$, $x, y, z \in R$. Clearly $L' = C(A) = \{C(a) \mid a \in A\} \subseteq A$, so A is a sub-closure ring of $M_2(R)$. Moreover, letting $a = \left(\begin{smallmatrix} x & 0 \\ y & z \end{smallmatrix}\right) \in A$, $ea = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & 0 \\ y & z \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$ so $eae = \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right) = ea$; of course $1a1 = a1$ and $0a0 = a0$. Thus $\alpha a \alpha = a\alpha$ for all $a \in A$ and $\alpha \in L'$, so A is strong. Hence closed ideals of A correspond to filters of L_A. But there is only one proper filter of the chain $L_A = \{1, e, 0\}$, namely $F = \{e, 0\}$. So $J_F = \left(\begin{smallmatrix} R & 0 \\ R & 0 \end{smallmatrix}\right)$ is the unique proper closed ideal of A which is therefore subdirectly irreducible, although non-simple since $|L_A| > 2$. This is consistent with Theorem 13 because L is not Boolean. Note also that J_F is strong (since A is) and is simple with right identity e by Theorem 12.

3. Constructions

We now consider various ways of building closure rings out of simpler ones. Of course, viewing the class of closure rings as a variety of rings with an additional unary operation, three obvious ways to do this are by taking products, subobjects and homomorphic images within this variety. As we have seen, factoring a closed ideal out of a closure ring gives a quotient closure ring structure. Given a closure ring, taking a subring closed under the closure operation again yields a closure ring. More relevantly here, any Cartesian product of closure rings is a closure ring if one defines the closure operation component-wise (like the other operations).

Of more interest is to try to define a closure operation on the ring of matrices over a closure ring, or on the ring of polynomials in arbitrarily many (perhaps non-commuting) indeterminates over a closure ring. Both things can be done in a natural manner.

For finitely many elements $a_1, a_2, \dots, a_k$ of the ring R, denote by $\Pi_i^\circ a_i$ the element $a_1 \circ a_2 \circ \cdots \circ a_k$ of R. We extend to multiple subscripts in the obvious way. For $a_1, a_2, \dots, a_n \in R$, let $diag(a_1, a_2, \dots, a_n)$ be the matrix in $M_n(R)$ with $a_1, a_2, \dots, a_n$ down the main diagonal and zeros elsewhere.

Theorem 14. *If C is a closure operation on the ring R, then so is C on $M_n(R)$ defined by setting $C(A) = diag(\alpha_A, \alpha_A, \dots, \alpha_A)$ where $A = (a_{ij})$ and $\alpha = \Pi_{k,l}^\circ C(a_{ij})$. Moreover $M_n(R)$ is strong if and only if R is strong.*

Proof: Suppose C is a closure operation on R and extend to $M_n(R)$ as in the theorem statement.

First note that for all $a_1, a_2, \dots, a_n \in R$ and all $i = 1, 2, \dots, n$, $a_i \circ (\Pi_i^\circ C(a_i)) = (a_i \circ C(a_i)) \circ C(a_1) \circ C(a_2 \circ \cdots \circ a_n) = C(a_1) \circ C(a_2) \circ \cdots \circ C(a_n) = (\Pi_i^\circ C(a_i))$.

Let $L = \{diag(\alpha, \alpha, \dots, \alpha) \mid \alpha \in L_R\} \subseteq M_n(R)$. Then L is a submonoid of $(M_n(R), \circ)$, isomorphic to L_R in $(R, \circ)$. For $A = (a_{ij}) \in M_n(R)$, define $\alpha_A \in L_R$ and $C(A)$ as in the theorem statement. Then $AC(A) = (a_{ij}\alpha_A) = (a_{ij} + \alpha_A - a_{ij} \circ \alpha_A) = (a_{ij} + \alpha_A - \alpha_A) = (a_{ij}) = A$. Moreover, letting $A \cdot diag(\alpha) = A$ for some $\alpha \in L_R$, we have $a_{ij}\alpha = a_{ij} + \alpha - a_{ij} \circ \alpha = a_{ij}$, so $a_{ij} \circ \alpha = \alpha$, so

$$C(a_{ij}) \circ \alpha = C(a_{ij}) \circ C(\alpha) = C(a_{ij} \circ \alpha) \circ C(\alpha) = C(\alpha) \circ C(\alpha) = C(\alpha) = \alpha.$$

So $\alpha \circ \alpha_A = \alpha \circ (\Pi_{ij}^\circ C(a_{ij})) = \Pi_{ij}^\circ (\alpha \circ C(a_{ij})) = \Pi_{ij}^\circ \alpha = \alpha$, so $C(A) diag(\alpha, \alpha, \dots, \alpha) = diag(\alpha_A \alpha, \alpha_A \alpha, \dots, \alpha_A \alpha) = diag(\alpha_A, \alpha_A, \dots, \alpha_A) = C(A)$, so $C(A) = \max\{\beta \in L \mid A\beta = \beta\}$, which is therefore a closure operation on $M_n(R)$ for which $L_{M_n(R)} = L$ by Proposition 3.

If R is strong then $\alpha a \alpha = \alpha a$ for all $\alpha \in L_R$ and $a \in R$ by Lemma 5, so $diag(\alpha, \alpha, \dots, \alpha) A \cdot diag(\alpha, \alpha, \dots, \alpha) = (\alpha a_{ij} \alpha) = (\alpha a_{ij}) = diag(\alpha, \alpha, \dots, \alpha) A$ for all $\alpha \in L_R$ and $A \in M_n(R)$, so $M_n(R)$ is strong. Conversely if $M_n(R)$ is strong then so must the copy of R consisting of all matrices of the form $diag(a)$, $a \in R$. □

Note that the closure operation on $M_n(R)$ extends the original closure operation on the usual copy of R in $M_n(R)$, $\{diag(a, a, \dots, a) \mid a \in R\}$.

If R has an identity, it is well-known that ideals of R correspond to ideals of $M_n(R)$ under the lattice isomorphism $I \leftrightarrow M_n(I)$. Indeed closed ideals correspond under the same mapping. For if I is closed then for $A = (a_{ij}) \in M_n(I)$, $C(A) = \Pi_{ij}^\circ C(a_{ij}) I_n \in M_n(I)$ since $\Pi_{ij}^\circ C(a_{ij}) \in I$. Conversely, if $M_n(I)$ is closed, then because $I = \{a \mid aI_n \in M_n(I)\}$, and because given $aI_n \in M_n(I)$ we have $C(a)I_n = C(aI_n) \in M_n(I)$, it follows that $C(a) \in I$.

There is a similar result to Theorem 14 for polynomial rings.

Theorem 15. *Let S be a set of indeterminates (commuting or otherwise). If R has closure operation C, then $R[S]$ has closure operation C obtained by setting $C(p) = \Pi^\circ\{C(a) \mid a$ a coefficient in $p\}$ for all $p \in R[S]$.*

Proof: Now R is a subring of $R[S]$ in the usual way, and so the lattice L_R associated with C on R is also embedded in $R[S]$. Moreover, for $p \in R[S]$ and $\alpha \in L_R$, $p \circ \alpha = \alpha$ if and only if $p\alpha = p$ if and only if $a\alpha = a$ for all coefficients a of p, if and only if $a \circ \alpha = \alpha$ for all coefficients a appearing in p. Arguing as in the previous proof, defining $C(p) = \Pi^\circ\{C(a) \mid a$ is a coefficient in $p\}$ for all $p \in R[S]$ makes C a closure operation on $R[S]$. $\square$

4. Central closure rings

Let R be a ring. The set of *central idempotents* $E(R) = \{e \in R \mid e^2 = e, er = re$ for all $r \in R\}$ is a submonoid of $(R, \circ)$, and is closed under ring multiplication as well. If R has identity 1 then $1 \in E(R)$ and it is well known that $E(R)$ is a Boolean algebra under ring multiplication and the adjoint operation, with $e' = 1 - e$. (Note that we view the ring product as join, the opposite of what is usually done.) In general, $E(R)$ is a Brouwerian lattice with top element 0. We next consider closure rings for which $L_R = E(R)$, or in other words, rings for which $\max\{\alpha \in E(R) \mid a\alpha = a\}$ exists for all $a \in R$. Let us call such a ring a *central closure ring*; trivially such closure rings are strong.

If $E(R) = \{0, 1\}$ for some ring R, then R is a central closure ring, with $C(0) = 0$ and $C(a) = 1$ otherwise.

The class of central closure rings is closed under the constructions discussed in the previous section (forming Cartesian products of closure rings, and forming matrix and polynomial rings over a given closure ring), and indeed the closure operations defined on the constructed rings are exactly those defined in terms of its central idempotents. To show this we need a preliminary result of some independent interest.

Proposition 16. *Let A be a ring, $(0 : A) = \{r \in A \mid ra = ar = 0$ for all $a \in A\}$ the two-sided annihilator of A. The centre $Z(M_n(A))$ of the $n \times n$ matrix ring A is $\{(a_{ij}) \mid a_{ij} \in (0 : A)$ whenever $i \neq j$, $a_{ii} \in Z(A)$ for all $i, a_{ii} - a_{jj} \in (0 : A)$ for all $i, j\}$.*

Proof: We use the notation $[x]_{ij}$ to mean the matrix with (i, j) entry equal to x and all others zero.

Let $(a_{ij}) = \sum [a_{ij}]_{ij}$ be in $Z(M_n(A))$. For every $x \in A$ and every k, l, we have

$$\sum_j [xa_{lj}]_{kj} = [x]_{kl}(a_{ij}) = (a_{ij})[x]_{kl} = \sum_i [a_{ik}x]_{il}.$$

If $j \neq l$ then $xa_{lj} = 0$ and if $i \neq k$ then $a_{ik}x = 0$. Since k, l and x are arbitrary, we conclude that all off-diagonal entries if (a_{ij}) are in $(0 : A)$. Now $(0 : M_n(A)) \subseteq Z(M_n(A))$ so $\sum_{i \neq j} [a_{ij}]_{ij}$ (being clearly in $(0 : M_n(A))$) is in $Z(M_n(A))$, whence $\sum_{i=1}^n [a_{ii}]_{ii} \in Z(M_n(A))$.

Again, for $x \in A$ and any k,

$$[a_{kk}x]_{kk} = \sum_{i=1}^{n} [a_{ii}]_{ii}[x]_{kk} = [x]_{kk} \sum_{i=1}^{n} [a_{ii}] = [xa_{kk}]_{kk},$$

so $a_{kk}x = xa_{kk}$ and $a_{kk} \in Z(A)$.

For every $x \in A$ and every k, l, we have $\sum_{i=1}^{n}[a_{ii}]_{ii}[x]_{kl} = [x]_{kl}\sum_{i=1}^{n}[a_{ii}]_{ii} = [xa_{ll}]_{kl}$ so $a_{kk}x = xa_{ll}$. But a_{kk} and a_{ll} are central, so $(a_{kk} - a_{ll})x = 0 = x(a_{kk} - a_{ll})$, that is, $a_{kk} - a_{ll} \in (0 : A)$ for all k, l.

Conversely, if (a_{ij}) satisfies all the stated conditions, then for every $(b_{ij}) \in M_n(A)$, the (r, s) entry of $(a_{ij})(b_{ij})$ is $\sum_{j=1}^{n} a_{rj}b_{js} = a_{rr}b_{rs}$ (since off-diagonal entries are in $(0 : A)$), and the (r, s) entry of $(b_{ij})(a_{ij})$ is

$$\begin{aligned}\sum_{j=1}^{n} b_{rj}a_{js} &= b_{rs}a_{ss} \text{ since off-diagonal entries are in } (0 : A)\\ &= a_{ss}b_{rs} \text{ since diagonal entries are central}\\ &= a_{rr}b_{rs} \text{ since } a_{ss} - a_{rr} \in (0 : A).\end{aligned}$$

The result follows. □

Corollary 17. *If* $(0 : A) = 0$ *then* $Z(M_n(A))$ *consists of the matrices of the form* $diag(a, a, \dots, a)$, $a \in Z(A)$.

Corollary 18. *A matrix is in* $E(M_n(A))$ *if and only if it equals* $diag(e, e, \dots, e)$ *for some* $e \in E(A)$.

Proof: Any matrix of the stated form is obviously in $E(M_n(A))$. Conversely, every matrix in $E(M_n(A))$ is also in $Z(M_n(A))$ and so must be as described in Proposition 16: it has the form $diag(a_{11}, a_{22}, \dots, a_{nn}) + \alpha$, where each $a_{ii} \in Z(A)$ and $\alpha \in (0 : M_n(A))$ has all diagonal entries zero. So

$$\begin{aligned}diag(a_{11}, a_{22}, \dots, a_{nn}) + \alpha &= (diag(a_{11}, a_{22}, \dots, a_{nn}) + \alpha)^2\\ &= diag(a_{11}, a_{22}, \dots, a_{nn})^2 + \text{ terms containing } \alpha\\ &= diag(a_{11}, a_{22}, \dots, a_{nn})^2\\ &= diag(a_{11}^2, a_{22}^2, \dots, a_{nn}^2),\end{aligned}$$

so $\alpha = 0$ and each a_{ii} is idempotent (as well as central) and hence in $E(A)$. Now for all i, j, we have $a_{ii} - a_{jj} \in (0 : A)$, so

$$0 = (a_{ii} - a_{jj})a_{jj} = a_{ii}a_{jj} - a_{jj}^2 = a_{ii}a_{jj} - a_{jj}$$

whence

$$a_{jj} = a_{ii}a_{jj} = a_{jj}a_{ii} = a_{ii}$$

since $a_{ii}, a_{jj} \in Z(A)$. So the matrix equals $diag(e, e, \dots, e)$ where $e = a_{ii}$ for each i. □

Proposition 19. • *If $\{R_\lambda \mid \lambda \in \Lambda\}$ is a set of central closure rings, then $R = \Pi_\Lambda R_\lambda$ is a central closure ring, and viewed as a Cartesian product in the variety of closure rings, the product closure operation is that determined by $E(R)$.*

- *If R is a central closure ring, then so is $M_n(R)$ for any $n > 0$, and the resulting operation is exactly that defined in Theorem 14 in terms of the closure operation on R associated with $E(R)$.*
- *If R is a central closure ring, then so is $R[S]$ for any set of indeterminates S, and the resulting closure operation is exactly that defined in Theorem 15 in terms of the closure operation on R associated with $E(R)$.*

PROOF: Suppose $\{R_\lambda \mid \lambda \in \Lambda\}$ is a set of central closure rings and $R = \Pi_\Lambda R_\lambda$. Now $E(R) = \{(\alpha_\lambda)_\Lambda \mid \alpha_\lambda \in E(R_\lambda)\}$. For $(a_\lambda) \in \Pi_\Lambda R_\lambda$, note that

$$(a_\lambda)_\Lambda \circ (C(a_\lambda))_\Lambda = (a_\lambda \circ C(a_\lambda))_\Lambda = C(a_\lambda)_\Lambda,$$

and moreover for any $(\alpha_\lambda)_\Lambda \in E(R)$, if $(a_\lambda)_\Lambda \circ (\alpha_\lambda)_\Lambda = (\alpha_\lambda)_\Lambda$, then $a_\lambda \circ \alpha_\lambda = \alpha_\lambda$ for all $\lambda \in \Lambda$, so by definition, $\alpha_\lambda \circ C(a_\lambda) = \alpha_\lambda$ for all such λ. Hence $(\alpha_\lambda)_\Lambda \circ C((a_\lambda)_\Lambda) = (C(\alpha_\lambda))_\Lambda \circ (C(a_\lambda))_\Lambda = (C(\alpha_\lambda) \circ C(a_\lambda))_\Lambda = (C(\alpha_\lambda \circ a_\lambda))_\Lambda = (C(\alpha_\lambda))_\Lambda = (\alpha_\lambda)_\Lambda$, so $(C(a_\lambda))_\Lambda = C((a_\lambda)_\Lambda)$, where the latter is defined in terms of $E(R)$.

For the second part, suppose R is a central closure ring. Then $L_R = E(R)$, $E(M_n(R)) = \{diag(e, e, \dots, e) \mid e \in E(R)\}$ by Corollary 18, and the result now follows as in the proof of Theorem 14.

For the third part, again note that if R is a central closure ring and S is a set of indeterminates, then $E(R[S]) = E(R)$, since all idempotents of $R[S]$ are in R and the centre of R is contained in the centre of $R[S]$. Again, the result follows as in the proof of Theorem 15. □

An example of a ring which is not a central closure ring is as follows. Let $R = CS(\mathbf{R})$ be the ring of convergent real sequences, a subring of the ring of all real sequences $\mathbf{R}^N$. This is a commutative ring with identity, and $E(R)$ consists of those sequences all of whose entries are 0 or 1 and which become constant. Let $\theta = (0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, \dots)$. Any $e \in E(R)$ for which $\theta e = \theta$ is ultimately 1, as it cannot be ultimately zero. But $\theta e_n = \theta$ if $e_n = (0, 1, 0, 1, 0, 1, \dots, 1, 1, 1, 1, 1, \dots)$, where e_n contains exactly n zero entries. Note that $e_n \circ e_{n+1} = e_n$ so $e_1, e_2, \dots$ is an infinite ascending chain in $E(R)$ having no upper bound in $E(R)$, so $C(\theta) = \max\{e \mid e \in E(R), \theta e = \theta\}$ does not exist. Letting L_n consist of all elements of $E(R)$ which are zero after the n'th term, it is clear that L_n is a sublattice of $E(R)$ which is itself a Boolean algebra, and that there exists a closure operation C_n defined relative to L_n for each n. Note also that $\mathbf{R}^N$ itself is a central closure ring since $E(\mathbf{R}) = \{0, 1\}$ by Proposition 19; thus a subring (with the same identity) of a central closure ring is not in general a central closure ring itself. Also, $\mathbf{R}^{(N)}$ is a subring of $CS(\mathbf{R})$ which is therefore a subdirect product, so the class of central closure rings is not closed under subdirect products.

A *strongly regular ring* R is a regular ring (one for which for all $a \in R$ there exists $b \in R$ such that $aba = a$) satisfying the following equivalent (for regular rings) conditions:

1. there are no non-zero nilpotent elements;
2. all idempotent elements are central;
3. for all $a \in R$ there exists $b \in R$ for which $a^2 b = a$.

Thus a commutative ring is strongly regular if and only if it is regular. The class of strongly regular rings contains all division rings and is closed under direct sums, direct products and homomorphic images, whence also under filtered products.

Proposition 20. *Every strongly regular ring is a central closure ring.*

PROOF: Let R be strongly regular, $a \in R$. Then there exists $b \in R$ such that $aba = a$, so $(ab)^2 = (aba)b = ab$ so $ab \in E(R)$ and $a(ab) = (ab)a = a$. If $ae = a$ for some $e \in E(R)$, then $eab = aeb = ab$ so $C(A)$ exists relative to $E(R)$ and is ab. □

For the remainder of the section we consider only rings with identity. For R such a ring and $a \in R$, define $(a : 0) = \{r \in R \mid ar = 0\}$, the right annihilator of a in R. For $X \subseteq R$, define $(X : 0) = \{r \in R \mid xr = 0 \text{ for all } x \in X\}$, the right annihilator of X in R.

Proposition 21. *For any ring with identity R, $F(a) = (a : 0) \cap E(R)$ is a filter of $E(R)$, and R is a central closure ring if and only if each such filter is principal.*

PROOF: If $e, f \in F(a)$ then $ae = af = 0$ so $a(e \circ f) = ae + af - aef = 0$, and if also $g \in E(R)$, then $a(eg) = (ae)g = 0$, so $F(a)$ is a filter. Moreover $F(a)$ is principal if and only if there is a smallest $e \in E(R)$ for which $ae = 0$; that is, there is a largest $f = e' = 1 - e \in E(R)$ for which $af = a(1 - e) = a$. □

Thus if the Boolean algebra $E(R)$ is Noetherian as a Boolean ring (for instance, if R is finite), then all ideals are principal (since finitely many generators can be replaced by their meet), which implies that all filters are principal as well by duality, and so R is a central closure ring.

Corollary 22. *Let R be a ring with identity. If for all $a \in R$, $(a : 0)$ is generated as a right ideal by a central idempotent, then R is a central closure ring.*

PROOF: If $(a : 0) = e_a R$ for some $e_a \in E(R)$, then $(a : 0) \cap E(R) = e_a E(R)$ which is therefore principal in $E(R)$. □

A ring R with identity is called a *Baer ring* if for any $X \subseteq R$, $(X : 0) = eR$ for some idempotent e. So by the previous corollary, every commutative Baer ring is a central closure ring. Indeed Kaplansky ([2, Theorem 9, p. 9]) has proved that all Baer rings are central closure rings. (An important example of a Baer ring is the ring R of all linear transformations of a vector space of arbitrary dimension ([1, Proposition 1, p. 179]), but because $E(R) = \{0, 1\}$, this is not interesting.)

Picavet [4] calls a commutative ring R with identity a *weak Baer ring* if for every $a \in R$, $(a : 0) = eR$ for some (necessarily central) idempotent e. So for commutative rings, every Baer ring is a weak Baer ring, and by the previous corollary, every weak Baer ring is a central closure ring. Neither of these implications is reversible.

Let R be the subring of the ring of all real sequences $\mathbf{R}^N$ consisting of all ultimately constant sequences. (The argument works with any integral domain in place of $\mathbf{R}$.) If $a = (a_i) \in R$, let $e_a = (u_i)$ where $u_i = 1$ if $a_i = 0$ and is zero otherwise; thus e_a is ultimately zero and so is in R. Clearly $ae_a = 0$. If also $ab = 0$ for some $b = (b_i) \in R$, then $b_i = 0$ whenever $a_i \neq 0$ and so $b_i = 0 = b_i u_i$ for such i. If $a_i = 0$ then $b_i u_i = b_i 1 = b_i$, so $b(= be) = eb \in eR$. Hence $(a : 0) \subseteq eR$, and the reverse inclusion is clear, so $(a : 0) = eR$. It follows that R is a weak Baer ring. Now consider $S = \{e_2, e_4, e_6, \dots\}$ where $e_1 = (1, 0, 0, \dots), e_2 = (0, 1, 0, 0, \dots), \dots$. If $c = (c_i) \in (S : 0)$ then for all i we have $0 = e_2 c = (0, c_2, 0, 0, \dots)$, so $c_2 = 0$ and in the same way, $c_{2i} = 0$ for all i. But $c \in R$ so c is ultimately zero, and there exists l such that $c_i = 0$ for all $i > l$. Suppose c is idempotent with $(S : 0) = cR$. Then $c_i = 1$ for finitely many i and is zero elsewhere, so cR cannot contain all of $e_3, e_5, e_7, \dots$, all of which are in $(S : 0)$. So $(S : 0)$ is not generated by an idempotent and R is not a Baer ring. Note that R is a unital subring of $R' = CS(\mathbf{R})$, $E(R) = E(R')$ and R is a central closure ring while R' is not.

Not all commutative central closure rings are weak Baer rings. If $R = \mathbf{Z}^0 * \mathbf{Z}$ denotes the zero ring on $\mathbf{Z}$ with the identity adjoined (that is, $R = \{(m, n) \mid m, n \in \mathbf{Z}\}$, with $(m, n) + (k, l) = (m + k, n + l)$ and $(m, n)(k, l) = (ml + nk, nl)$), then $E(R) = \{0, 1\}$ so R is a central closure ring. However, $(1, 0)(1, 0) = (0, 0)$ so $0R = \{0\} \subset ((1, 0) : 0) \subset R = 1R$, where the inclusions are strict, so $((1, 0) : 0)$ is not generated by an idempotent and R is not a weak Baer ring.

Recall that if $E(R) = \{0, 1\}$ for some ring R, then R is a central closure ring, with $C(0) = 0$ and $C(a) = 1$ otherwise; moreover R is simple as a closure ring. Conversely, if a central closure ring is simple as a closure ring then $E(R) = L_R = \{0, e\}$ where e is a right identity and hence an identity for R. So a central closure ring is simple as a closure ring if and only if R has an identity and $E(R)$ is simple as a Boolean algebra; in such cases we say R is a *simple central closure ring.*

A central closure ring R with identity is a strong closure ring and $E(R)$ is a Boolean algebra, so Theorem 13 applies. The question arises as to whether the simple closure rings in the subdirect product representation of R are simple central closure rings.

Theorem 23. *Every commutative central closure ring with identity is embeddable in a subdirect product of simple central closure rings.*

Proof: Suppose R is a central closure ring. For F an ultrafilter of the Boolean algebra $E(R)$, if $e + J_F$ is idempotent in R/J_F for some $e \in R$, then $e^2 - e \in J_F$, so $\alpha = C(e^2 - e) \in F$. Now $(e^2 - e)\alpha = e^2 - e$, so $(e^2 - e)(1 - \alpha) = 0$, so $e^2\beta = e\beta$, $\beta = 1 - \alpha \in E(R)$. Thus $(e\beta)^2 = e\beta e\beta = e^2\beta = e\beta$. Hence in R/J_F, $e\beta + J_F =$

$C(e\beta)+J_F = C(e\beta+J_F) \in L_{R/J_F} = \{0+J_F, 1+J_F\}$ by the proof of Theorem 13 and by Theorem 12. So either $e\beta \in J_F$ or $e\beta - 1 \in J_F$. Now $\alpha \in F \subseteq J_F$, and if $e\beta - 1 \in J_F$ then $e-1 = (e\beta-1)-e(\beta-1) = (e\beta-1)+e\alpha \in J_F$, while if $e\beta \in J_F$ then $e = e\beta - e(\beta - 1) = e\beta + e\alpha \in J_F$. Thus $E(R/J_F) = \{0 + J_F, 1 + J_F\}$. $\square$

This theorem may generalise to the non-commutative case: we know of no counterexample. From Proposition 20 and the theorem, we can recover the well-known fact that every commutative regular ring is a subdirect product of (strongly) regular rings having only 0 and 1 as idempotents, that is, fields.

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