Remarks on fixed points of rotative Lipschitzian mappings

Jarosław Górnicki

Abstract. Let C be a nonempty closed convex subset of a Banach space E and $T : C \to C$ a k-Lipschitzian rotative mapping, i.e. such that $\|Tx - Ty\| \leq k \cdot \|x - y\|$ and $\|T^n x - x\| \leq a \cdot \|x - Tx\|$ for some real k, a and an integer $n > a$. The paper concerns the existence of a fixed point of T in p-uniformly convex Banach spaces, depending on k, a and $n = 2, 3$.

Keywords: rotative mappings, fixed points

Classification: 47H09, 47H10

1. Introduction

Many authors discussed the problem concerning the existence of fixed points for different class of mappings defined on nonempty closed convex subsets C of infinite dimensional Banach space E and satisfying some metric conditions. The main problem was connected with establishing some conditions of geometrical nature implying the fixed point property for *nonexpansive* mappings $T : C \to C$ (i.e. mappings satisfying $\|Tx - Ty\| \leq \|x - y\|$ for all x, y in C). The usual assumptions are those of uniform convexity and normal structure.

In 1981, Goebel and Koter [6] defined the conditions of *rotativeness* (see below) and proved the following

Theorem 1. *If C is a nonempty closed convex subset of a Banach space E, then any nonexpansive rotative mapping $T : C \to C$ has a fixed point.* □

Note that this result does not require weak compactness or even boundedness of C, or any special geometric structure on C.

Further on, the authors studied the existence of fixed points for some class of k-Lipschitzian ($k > 1$) and rotative mappings in Banach spaces ([7], [13]).

In this note we extend Goebel and Koter's results for a real p-uniformly convex Banach space and give an estimate for the function γ_3 in a Hilbert space.

2. Preliminaries

Let C be a nonempty closed convex subset of a Banach space E. A mapping $T : C \to C$ is called (n, a)-*rotative* if there exists an integer $n \geq 2$ and a real number $0 \leq a < n$ such that for any $x \in C$, $\|x - T^n x\| \leq a \cdot \|x - Tx\|$.

The simplest examples of rotative mappings are contractions and rotation of the Euclidean space $\mathbb{R}^n$ or any *periodic* nonexpansive mappings (i.e. $T^n = I$ for some $n \in \mathbb{N}$, where I means identity mapping) in any Banach space.

Definition 1. *Denote by $\Phi(n, a, k, C)$ the class of all mappings $T : C \to C$ which are (n, a)-rotative and satisfy the following condition*

$$\forall\, x, y \in C \quad \|Tx - Ty\| \leq k \cdot \|x - y\|.$$

A mapping $T \in \Phi(n, a, k, C)$ is said to be k-Lipschitzian (n, a)-rotative on C.

We shall now consider mappings of the family $\Phi(n, a, k, C)$ with $k > 1$. For fixed $n \in \mathbb{N}$ put

$$\gamma_n(a) = \inf \left\{ \begin{array}{l} k > 1 : \text{there exists a set } C \text{ (closed convex) and} \\ \qquad\quad \text{a mapping } T \text{ such that } T \in \Phi(n, a, k, C) \\ \qquad\quad \text{and } \ F(T) = \emptyset \end{array} \right\}$$

($F(T)$ denotes the set of all fixed points of T).

The definition of $\gamma_n(a)$ implies that for an arbitrary set C, if $T \in \Phi(n, a, k, C)$ and $k < \gamma_n(a)$, then T has at least one fixed point. It was proved in [7] that for an arbitrary Banach space E and for any $n \in \mathbb{N}$, we have $\gamma_n(a) > 1$ for all $a < n$. It is a qualitative result which raises a number of technical yet attractive questions concerning the precise values of $\gamma_n(a)$. Even the exact value of $\gamma_n(0)$ is of interest since it characterizes the fixed point behavior of mappings of period n (see [11], [16] and [4], [8], [9], [10] for *involutions*, i.e. mappings T for which $T^2 = I$).

3. About the function $\gamma_2(a)$

Now, we restrict our attention to the case $n = 2$. It was proved in [5] that for an arbitrary Banach space E

$$\gamma_2(a) \geq \gamma_B(a), \quad a \in [0, 2),$$

where

$$\gamma_B(a) = \max \left\{ \frac{1}{2} \cdot \left[2 - a + \sqrt{(2-a)^2 + a^2} \right], \right.$$
$$\left. \frac{1}{8} \cdot \left[a^2 + 4 + \sqrt{(a^2+4)^2 - 64 \cdot (a-1)} \right] \right\}.$$

Surprisingly, it is possible to show that the first term provides a better estimate if $a \leq 2(\sqrt{2}-1) \approx 0.828$, while the second is better for $a \in [2(\sqrt{2}-1), 2)$.

No upper bound for $\gamma_2(a)$ with $a \in [0,1]$ is known until now, while if $a \in (1,2)$ we have $\gamma_2(a) \leq \frac{k_R \cdot (a+1)}{a-1}$, where k_R is the minimal Lipschitz constant of the retraction of the unit ball onto the unit sphere in E (see Example 1 in [13]). In general, the value of k_R is unknown, so that the bound given above shows only that $\gamma_2(a) < +\infty$ for $a \in (1,2)$. It is however essential that this fact is true in an arbitrary Banach space. In $C[0,1]$ or $L^1[0,1]$, we have $\gamma_2(a) \leq \frac{1}{a-1}$, $a \in (1,2)$ (see Examples 1, 2 in [7] and Example 17.2 in [5]).

These results are illustrated in Figure 1.

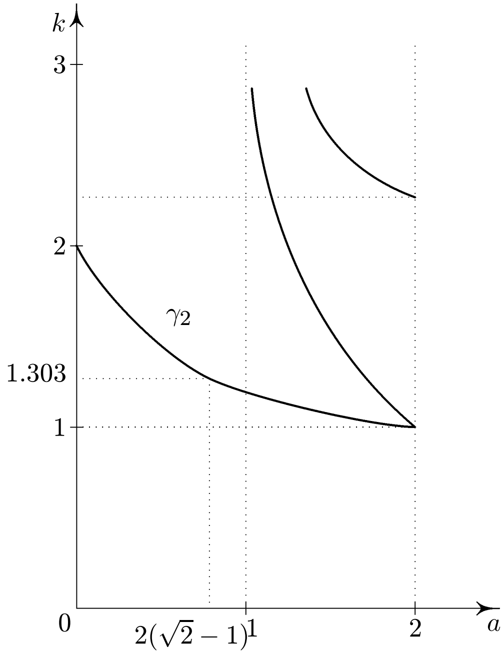

Figure 1

Denote

$D_1 = \{(a,k) \in [0,2) \times [0,+\infty) : k < \gamma_2(a)\}$;

$D_2 = \{(a,k) \in (1,2) \times (1,+\infty) : k \geq \frac{k_R \cdot (a+1)}{a-1}\}$;

$D_3 = \{(a,k) \in (1,2) \times (1,+\infty) : k \geq \frac{1}{a-1}\}$;

$D_4 = [0,2) \times [0,+\infty) \setminus (D_1 \cup D_3)$.

If T is k-Lipschitzian and $(2,a)$-rotative, where $(a,k) \in D_1$, then T has at least one fixed point. In other words: the graph of the function γ_2 for an arbitrary space E lies above the region D_1. On the other hand, it lies always below the curve which is the lower bound of the region D_2 (in some spaces even below the lower bound of D_3). The existence of fixed points for mappings $T \in \Phi(2, a, k, C)$, where $(a, k) \in D_4$, remains an open problem.

However, in some spaces one can sightly raise the lower bound of the region D_4. Koter [13] proved the following theorem (in spaces with known modulus of convexity, see [5]).

Theorem 2. *Let C be a nonempty closed convex subset of a Banach space E with the modulus of convexity δ_E. If $T \in \Phi(2, a, k, C)$ and*

$$1 - \delta_E(2/k) \leq \frac{2-a}{k},$$

then T has at least one fixed point. □

Since in the space L^p (or ℓ^p), $p \in (2, +\infty)$, we have $\delta_p(\varepsilon) = 1 - (1 - (\varepsilon/2)^p)^{1/p}$, routine calculations and the previous estimates (1) yield

Corollary 1. *Let C be a nonempty closed convex subset of the space L^p (or ℓ^p), $2 < p < +\infty$. If $T \in \Phi(2, a, k, C)$ and*

$$k < \max\left\{\gamma_B(a), [(2-a)^p + 1]^{1/p}\right\}, \quad a \in [0, 2),$$

then T has at least one fixed point. □

Hence, in the space L^p (or ℓ^p), $2 < p < +\infty$, we have

$$\gamma_2(a) \geq \max\left\{\gamma_B(a), [(2-a)^p + 1]^{1/p}\right\}, \quad a \in [0, 2).$$

Komorowski [12] shows that for a real Hilbert space $\mathcal{H}$ we have a better bound for γ_2, namely

$$\gamma_2(a) \geq \sqrt{\frac{5}{a^2+1}} = \gamma_{\mathcal{H}}(a), \quad a \in [0, 2)$$

(see Figure 2).

4. The function γ_2 in p-uniformly convex spaces

In this section we give some estimates of the function γ_2 by means of inequalities in Banach spaces.

Let $p > 1$ and denote by λ a number in $[0, 1]$ and by $W_p(\lambda)$ the function $\lambda \cdot (1-\lambda)^p + \lambda^p \cdot (1-\lambda)$.

The functional $\|\cdot\|^p$ is said to be *uniformly convex* ([22]) on the Banach space if

(∗) there exists a positive constant c_p such that for all $\lambda \in [0, 1]$ and $x, y \in E$ the following inequality holds:
$\|\lambda \cdot x + (1-\lambda) \cdot y\|^p \leq \lambda \cdot \|x\|^p + (1-\lambda) \cdot \|y\|^p - c_p \cdot W_p(\lambda) \cdot \|x - y\|^p$.

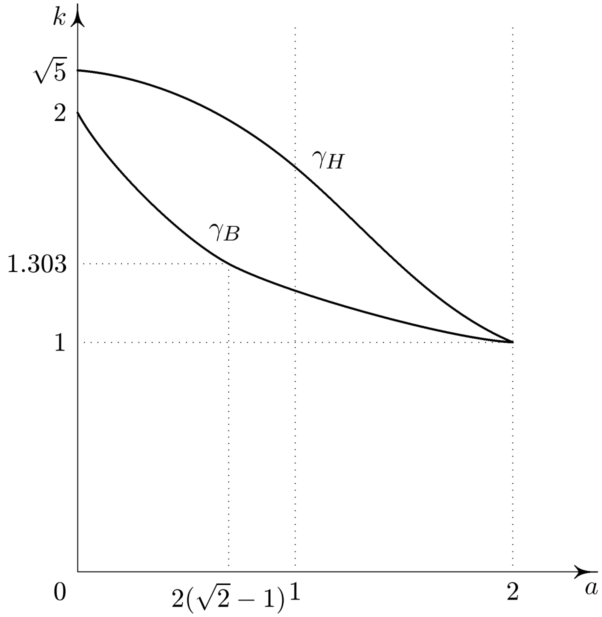

Figure 2

Xu [12] proved that the functional $\|\cdot\|^p$ is uniformly convex on the whole Banach space E if and only if E is *p-uniformly convex*, i.e. there exists constant $c > 0$ such that the modulus of convexity (see [5]) $\delta_E(\varepsilon) \geq c \cdot \varepsilon^p$ for all $0 \leq \varepsilon \leq 2$. We note that a Hilbert space $\mathcal{H}$ is 2-uniformly convex (indeed $\delta_{\mathcal{H}}(\varepsilon) = 1 - \sqrt{1-(\varepsilon/2)^2} \geq (1/8)\cdot\varepsilon^2$) and L^p (or ℓ^p) $(1 < p < +\infty)$ is $\max(2,p)$-uniformly convex.

Theorem 3. *Let E be a Banach space with the norm satisfying $(*)$ for some $p > 1$, let C be a nonempty closed convex subset of E. If $T \in \Phi(2,a,k,C)$ and*

$$k < \max\left\{1, \left[\frac{1+2^p}{2^{p-2}\cdot(1+a^p)}\right]^{1/p}\right\} \quad \text{if } c_p = 1,$$

or

$$k < \max\left\{1, \left[\frac{c_p+2^p}{2^{p-2}\cdot(2-c_p)(1+a^p)}\right]^{1/p}, \left[\frac{\sqrt{[2^{p-1}\cdot(1+a^p)]^2 + 8\cdot(1-c_p)\cdot(2^p+c_p)} - 2^{p-1}\cdot(1+a^p)}{2\cdot(1-c_p)}\right]^{1/p}\right\}$$
$$\text{if } 0 < c_p < 1 \text{ and } a \in [0,2),$$

then T has at least one fixed point.

Proof: If $k < 1$, then the Banach Contraction Principle implies that T has a fixed point. Thus we assume that $k \geq 1$. Let x be an arbitrary point in the set C and ε an arbitrary real positive number. Suppose that

$$\|T^2x - Tx\|^p > (1-\varepsilon) \cdot \|x - Tx\|^p$$

and put $z = (1/2)(Tx + T^2x)$. Then we have

$$\begin{aligned}
\|z - Tz\|^p &= \|(1/2)\cdot(Tx + T^2x) - Tz\|^p \\
&= \|(1/2)\cdot(Tx - Tz) + (1/2)\cdot(T^2x - Tz)\|^p \\
&\leq (1/2)\cdot\|Tx - Tz\|^p + (1/2)\cdot\|T^2x - Tz\|^p \\
&- c_p\cdot(1/2)^p\cdot\|T^2x - Tx\|^p \\
&\leq (1/2)\cdot k^p\|(1/2)\cdot(x - Tx) + (1/2)\cdot(x - T^2x)\|^p \\
&+ (1/2)\cdot k^p\cdot\|(1/2)\cdot(Tx - T^2x)\|^p - c_p\cdot(1/2)^p\cdot\|T^2x - Tx\|^p \\
&\leq \{(1/4)\cdot k^p + (1/4)\cdot k^p\cdot a^p\}\cdot\|x - Tx\|^p \\
&+ (1/2)^{p+1}\cdot k^p\cdot(1 - c_p)\cdot\|T^2x - Tx\|^p - c_p\cdot(1/2)^p\cdot\|T^2x - Tx\|^p.
\end{aligned}$$

If $c_p = 1$, then by last inequality we have

$$\begin{aligned}
\|z - Tz\|^p &\leq \Big\{(1/4)\cdot k^p + (1/4)\cdot k^p\cdot a^p\Big\}\cdot\|x - Tx\|^p \\
&- (1/2)^p\cdot\|T^2x - Tx\|^p \\
&\leq \Big\{(1/4)\cdot k^p + (1/4)\cdot k^p\cdot a^p - (1/2)^p\cdot(1-\varepsilon)\Big\}\cdot\|x - Tx\|^p \\
&= f(\varepsilon)\cdot\|x - Tx\|^p.
\end{aligned}$$

Now, assume $0 < c_p < 1$.

Case I. By the estimate

$$\begin{aligned}
\|T^2x - Tx\|^p &\leq \Big(\|T^2x - x\| + \|x - Tx\|\Big)^p \\
&\leq 2^{p-1}\cdot\Big(\|T^2x - x\|^p + \|x - Tx\|^p\Big) \\
&\leq 2^{p-1}\cdot(a^p + 1)\|x - Tx\|^p,
\end{aligned}$$

we have

$$\begin{aligned}
\|z - Tz\|^p \leq \Big\{&(1/4)\cdot k^p + (1/4)\cdot k^p\cdot a^p \\
&+ (1/2)^{p+1}\cdot k^p\cdot(1 - c_p)\cdot 2^{p-1}\cdot(a^p + 1) \\
&- (1/2)^p\cdot c_p(1-\varepsilon)\Big\}\cdot\|x - Tx\|^p \\
= g(\varepsilon)&\cdot\|x - Tx\|^p.
\end{aligned}$$

Case II. By the estimate

$$\|T^2x - Tx\|^p \le k^p \cdot \|Tx - x\|^p$$

we have

$$\begin{aligned}\|z - Tz\|^p &\le \Big\{(1/4) \cdot k^p + (1/4) \cdot k^p \cdot a^p + (1/2)^{p+1} \cdot k^{2p} \cdot (1 - c_p) \\ &\quad - (1/2)^p \cdot c_p \cdot (1 - \varepsilon)\Big\} \cdot \|x - Tx\|^p \\ &= h(\varepsilon) \cdot \|x - Tx\|^p.\end{aligned}$$

If the assumptions of the theorem are satisfied, then there exists $\varepsilon > 0$ such that $\max\{f(\varepsilon), g(\varepsilon), h(\varepsilon)\} < 1$, and we may consider the following sequence

$$\begin{aligned} x_1 &= x, \\ x_{n+1} &= Tx_n \ \text{ if } \ \|T^2x_n - Tx_n\|^p \le (1 - \varepsilon) \cdot \|Tx_n - x_n\|^p, \end{aligned}$$

or

$$x_{n+1} = \big(1/2\big)(Tx_n + T^2x_n\big) \ \text{ if } \ \|T^2x_n - Tx_n\|^p > (1 - \varepsilon) \cdot \|Tx_n - x_n\|^p$$

for $n = 1, 2, \dots$.

Now, we show the convergence of the sequence $\{x_n\}$. Indeed,

$$\|Tx_{n+1} - x_{n+1}\|^p \le A \cdot \|Tx_n - x_n\|^p, \quad \text{for} \quad n \in \mathbb{N},$$

where $A = \max\{f(\varepsilon), g(\varepsilon), h(\varepsilon), 1 - \varepsilon\} < 1$. Thus

$$\|Tx_{n+1} - x_{n+1}\|^p \le A^n \cdot \|Tx_1 - x_1\|^p \to 0,$$

as $n \to +\infty$, which shows that $\{x_n\}$ is a Cauchy sequence. Let $y = \lim_{n\to\infty} x_n$. Since $\|Tx_{n+1} - x_{n+1}\|^p \to 0$ as $n \to +\infty$, we have $Ty - y = 0$, and $Ty = y$. □

5. Applications

Note that in a Hilbert space $\mathcal{H}$ we have the identity

$$\|\lambda \cdot x + (1 - \lambda) \cdot y\|^2 = \lambda \cdot \|x\|^2 + (1 - \lambda) \cdot \|y\|^2 - \lambda \cdot (1 - \lambda) \cdot \|x - y\|^2$$

for all x, y in C and $0 \le \lambda \le 1$. In this case $p = 2$ and $c_2 = 1$. Thus by Theorem 3, we have the following corollary.

Corollary 2 ([12]). *Let $\mathcal{H}$ be a Hilbert space and let C be a nonempty closed convex subset of $\mathcal{H}$. If $T \in \Phi(2, a, k, C)$ and*

$$k < \sqrt{\frac{5}{a^2+1}}, \; a \in [0,2),$$

then T has at least one fixed point. □

If $1 < p < 2$, then we have for all x, y in L^p (or ℓ^p) and $\lambda \in [0,1]$,

$$\|\lambda \cdot x + (1-\lambda) \cdot y\|^2 \leq \lambda \cdot \|x\|^2 + (1-\lambda) \cdot \|y\|^2 - (p-1) \cdot \lambda \cdot (1-\lambda) \cdot \|x-y\|^2,$$

(see [20], [14]). Thus by Theorem 3 we have the following estimate for k in L^p (or ℓ^p) spaces ($1 < p < 2$):

$$k < \max\left\{1, \sqrt{\frac{3+2}{(1+a^2)(3-p)}}, \sqrt{\frac{\sqrt{4(1+a^2)^2+8(2-p)(3+p)}-2(1+a^2)}{2(2-p)}}\right\}$$
$$= f_p(a), \; a \in [0,2).$$

If $p \to 2_+$, then $f_p(a) \to f_2(a) = \gamma_{\mathcal{H}}(a)$. Moreover, $f_p(0) > 2$ for $2 > p > 9/5$. The case $p = 3/2$ is illustrated by means of computer graphic in Figure 3.

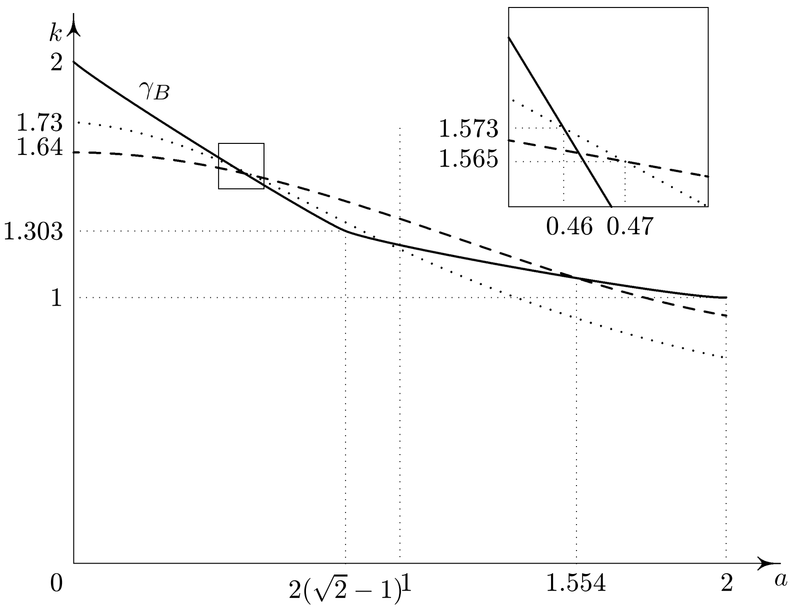

Figure 3

Thus in L^p (or ℓ^p), $1 < p < 2$, we have the following

Corollary 3. *Let C be a nonempty closed convex subset of L^p (or ℓ^p), $1 < p < 2$. If $T \in \Phi(2, a, k, C)$ and*

$$k < \max\left\{\gamma_B(a), \sqrt{\frac{3+2}{(1+a^2)(3-p)}}, \sqrt{\frac{\sqrt{4(1+a^2)^2 + 8(2-p)(3+p)} - 2(1+a^2)}{2(2-p)}}\right\}$$

for $a \in [0, 2)$, then T has at least one fixed point. □

For all x, y in L^p (or ℓ^p) spaces, $2 < p < +\infty$, and all $\lambda \in [0, 1]$, we have

$$\|\lambda \cdot x + (1-\lambda) \cdot y\|^p \leq \lambda \cdot \|x\|^p + (1-\lambda) \cdot \|y\|^p - c_p \cdot W_p(\lambda) \cdot \|x - y\|^p,$$

where $c_p = (p-1) \cdot (1-t_p)^{2-p}$, and t_p is the unique zero of the function $j(x) = -x^{p-1} + (p-1) \cdot x + (p-2)$ on the interval $(1, +\infty)$, see for example [18], [14].

By numerical approximation we obtain $c_{2.1} \approx 0.948917$ and the case $p = 2.1$ is illustrated in Figure 4.

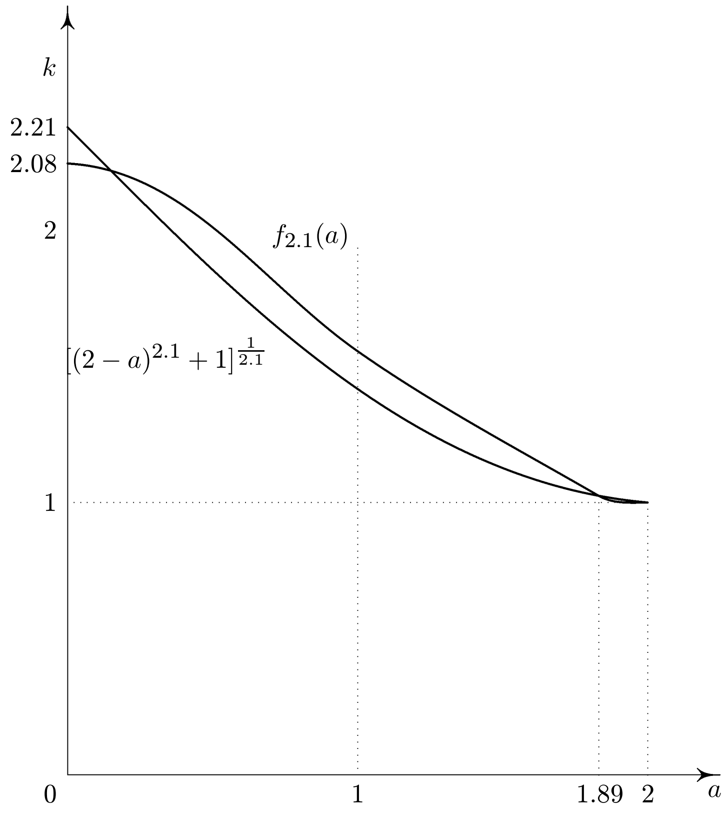

Figure 4

Thus by Corollary 1 and Theorem 3 we have

Corollary 4. *Let C be a nonempty closed convex subset of L^p (or ℓ^p), $2 < p < +\infty$. If $T \in \Phi(2, a, k, C)$ and*

$$k < \max\left\{\gamma_B(a), \left[(2-a)^p + 1\right]^{1/p}, \left[\frac{c_p + 2^p}{2^{p-2}\cdot(2-c_p)(1+a^p)}\right]^{1/p}, \left[\frac{\sqrt{[2^{p-1}\cdot(1+a^p) + 8\cdot(1-c_p)\cdot(2^p+c_p)} - 2^{p-1}\cdot(1+a^p)}{2\cdot(1-c_p)}\right]^{1/p}\right\}$$

for $a \in [0, 2)$, then T has at least one fixed point. □

Using the result of Prus, Smarzewski ([17], [19]) we obtain from Theorem 3 a fixed point theorem, for example, for Hardy and Sobolev spaces.

Let H^p, $1 < p < +\infty$, denote the *Hardy space* ([3]) of all functions x analytic in the unit disc $|z| < 1$ of the complex plane and such that

$$\|x\| = \lim_{r\to 1_-}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|x(re^{i\Theta})\right|^p d\Theta\right)^{1/p} < +\infty.$$

Now, let Ω be an open subset of $\mathbb{R}^n$. Denote by $W^{r,p}(\Omega)$, $r \geq 0$, $1 < p < +\infty$, the *Sobolev space* ([1, p. 149]) of distributions x such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq k$ equipped with the norm

$$\|x\| = \Big(\sum_{|\alpha|\leq k}\int_\Omega \left|D^\alpha x(\omega)\right|^p d\omega\Big)^{1/p}.$$

Let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, be a sequence of positive measure spaces, where Λ is finite or countable. Given a sequence of linear subspaces X_α in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < +\infty$, $q = \max(2, p)$ ([15]), the linear space of all sequences

$$x = \left\{x_\alpha \in X_\alpha : \alpha \in \Lambda\right\}$$

equipped with the norm

$$\|x\| = \Big[\sum_{\alpha\in\Lambda}\left(\left\|x_\alpha\right\|_{p,\alpha}\right)^q\Big]^{1/q},$$

where $\|\cdot\|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$.

Finally, let $L^p = L^p(S_1, \Sigma_1, \mu_1)$ and $L^q = L^q(S_2, \Sigma_2, \mu_2)$, where $1 < p < +\infty$, $q = \max(2, p)$ and (S_i, Σ_i, μ_i) are positive measure spaces. Denote by $L_q(L_p)$ the Banach space ([2, III.2.10]) of all measurable L^p-valued functions x on S_2 with the norm

$$\|x\| = \Big(\int_{S_2}\left(\|x(s)\|_p\right)^q \mu_2\,(ds)\Big)^{1/q}.$$

These spaces are q-uniform convex with $q = \max(2,p)$ ([17], [19]) and the norm in these spaces satisfies

$$\left\|\lambda \cdot x + (1-\lambda)\cdot y\right\|^q \le \lambda \cdot \left\|x\right\|^q + (1-\lambda)\cdot \left\|y\right\|^q - d \cdot W_q(\lambda)\cdot \left\|x-y\right\|^q$$

with a constant

$$d = d_p = \frac{p-1}{8} \quad \text{for} \quad 1 < p \le 2 \quad \text{and} \quad d = d_p = \frac{1}{p\cdot 2^p} \quad \text{for} \quad 2 < p < +\infty.$$

Hence it follows from Theorem 3 the following

Corollary 5. *Let C be a nonempty closed convex subset of the space X, where $X = H^p$ or $X = W^{r,p}(\Omega)$ or $X = L_{q,p}$ or $X = L_q(L_p)$ and $1 < p < +\infty$, $q = \max(2,p)$, $r \ge 0$. If $T \in \Phi(2,a,k,C)$ and*

$$k < \max\left\{\gamma_B(a), \left[\frac{d_p + 2^q}{2^{q-2}\cdot(2-d_p)(1+a^q)}\right]^{1/q}, \left[\frac{\sqrt{[2^{q-1}\cdot(1+a^q) + 8\cdot(1-d_p)\cdot(2^q+d_p)} - 2^{q-1}\cdot(1+a^q)}{2\cdot(1-d_p)}\right]^{1/q}\right\}$$

for $a \in [0,2)$, then T has at least one fixed point. □

6. γ_3 in a Hilbert space

We mentioned that the function γ_n may have different form in different spaces. Now we want to establish an evaluation of the function γ_3 in a Hilbert space.

Theorem 4. *Let $\mathcal{H}$ be a Hilbert space and let C be a nonempty closed convex subset of $\mathcal{H}$. If $T \in \Phi(3,a,k,C)$ and*

$$k < \max\left\{\sqrt{(1/2)\cdot\left[\sqrt{9a^4+2a^2+41} - 3\cdot a^2 + 1\right]}, \sqrt{(1/2)\cdot\left[\sqrt{(1+a^2)^2+40} - (1+a^2)\right]}\right\}, \ a \in [0,3),$$

then T has at least one fixed point.

(Note that it is possible to show that the second term provides a better estimate if $\sqrt{2} < a < \sqrt{(1/2)(\sqrt{29}+7)} \approx 2.48849$.)

Proof: Let x be an arbitrary point in the set C and ε an arbitrary real positive number. Suppose that

$$\left\|Tx - T^3x\right\|^2 + \left\|T^2x - T^3x\right\|^2 > (1-\varepsilon)\cdot\left\|x - Tx\right\|^2$$

and put

$$z = (1/3)(Tx + T^2x + T^3x) = (1/3) \cdot Tx + (2/3) \cdot [(1/2)(T^2x + T^3x)].$$

Then we have

$$\begin{aligned}
&\|z - Tz\|^2 = \|(1/3) \cdot Tx + (2/3) \cdot [(1/2)(T^2x + T^3x)] - Tz\|^2 \\
&= \|(1/3) \cdot (Tx - Tz) + (2/3) \cdot [(1/2)(T^2x + T^3x) - Tz]\|^2 \\
&= (1/3) \cdot \|Tx - Tz\|^2 + (2/3) \cdot \|(1/2)(T^2x + T^3x) - Tz\|^2 \\
&\quad - (2/9) \cdot \|Tx - (1/2)(T^2x + T^3x)\|^2 \\
&\le (1/3) \cdot k^2 \cdot \|x - z\|^2 + (2/3) \cdot \|(1/2) \cdot (T^2x - Tz) + (1/2) \cdot (T^3x - Tz)\|^2 \\
&\quad - (2/9) \cdot \|(1/2) \cdot (Tx - T^2x) + (1/2) \cdot (Tx - T^3x)\|^2 \\
&\le (1/3) \cdot k^2 \cdot \|x - (1/3) \cdot Tx - (2/3) \cdot [(1/2)(T^2x + T^3x)]\|^2 \\
&\quad + (2/3)\Big\{(1/2) \cdot k^2 \cdot \|Tx - z\|^2 + (1/2) \cdot k^2 \cdot \|T^2x - z\|^2 \\
&\quad - (1/4) \cdot \|T^2x - T^3x\|^2\Big\} \\
&\quad - (2/9) \cdot \Big\{(1/2) \cdot \|Tx - T^2x\|^2 + (1/2) \cdot \|Tx - T^3x\|^2 \\
&\quad - (1/4) \cdot \|T^2x - T^3x\|^2\Big\} \\
&= (1/3) \cdot k^2 \cdot \Big\{(1/3) \cdot \|x - Tx\|^2 + (2/3) \cdot \|x - (1/2)(T^2x - T^3x)\|^2 \\
&\quad - (2/9) \cdot \|Tx - (1/2)(T^2x - T^3x)\|^2\Big\} \\
&\quad + (2/3) \cdot \Big\{(1/2) \cdot k^2 \cdot \|(2/3)[Tx - (1/2)(T^2x + T^3x)]\|^2 \\
&\quad + (1/2) \cdot k^2 \cdot \|(1/3)(T^2x - Tx) + (2/3)[T^2x - (1/2)(T^2x + T^3x)]\|^2 \\
&\quad - (1/4) \cdot \|T^2x - T^3x\|^2\Big\} \\
&\quad - (2/9) \cdot \Big\{(1/2) \cdot \|Tx - T^2x\|^2 + (1/2) \cdot \|Tx - T^3x\|^2 \\
&\quad - (1/4) \cdot \|T^2x - T^3x\|^2\Big\} \\
&= (1/9) \cdot k^2 \cdot \|x - Tx\|^2 + (2/9) \cdot k^2 \cdot \Big\{(1/2) \cdot \|x - T^2x\|^2 \\
&\quad + (1/2) \cdot \|x - T^3x\|^2 - (1/4) \cdot \|T^2x - T^3x\|^2\Big\} \\
&\quad - (2/27) \cdot k^2 \cdot \|Tx - (1/2)(T^2x - T^3x)\|^2 \\
&\quad + (4/27) \cdot k^2 \cdot \|Tx - (1/2)(T^2x - T^3x)\|^2 \\
&\quad + (1/3) \cdot k^2 \cdot \Big\{(1/3) \cdot \|T^2x - Tx\|^2 + (2/3) \cdot \|T^2x - (1/2)(T^2x + T^3x)\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad - (2/9) \cdot \left\|Tx - (1/2)(T^2x - T^3x)\right\|^2 \Big\} - (1/6) \cdot \left\|T^2x - T^3x\right\|^2 \\
&\quad - (2/9) \cdot \Big\{ (1/2) \cdot \left\|Tx - T^2x\right\|^2 + (1/2) \cdot \left\|Tx - T^3x\right\|^2 \\
&\quad - (1/4) \cdot \left\|T^2x - T^3x\right\|^2 \Big\} \\
&\leq \ \text{(reduction)} \\
&\leq [(1/9) \cdot k^4 + (1/9) \cdot k^2] \cdot \left\|x - Tx\right\|^2 + (1/9) \cdot k^2 \cdot a^2 \cdot \left\|x - Tx\right\|^2 \\
&\quad + [(1/9) \cdot k^2 - (1/9)] \cdot \left\|x - T^2x\right\|^2 \\
&\quad - (1/9) \cdot \Big\{ \left\|Tx - T^3x\right\|^2 + \left\|T^2x - T^3x\right\|^2 \Big\}.
\end{aligned}
$$

Case I. By the estimate

$$
\begin{aligned}
\left\|x - T^2x\right\|^2 &\leq 2 \cdot \Big(\left\|x - T^3x\right\|^2 + \left\|T^3x - T^2x\right\|^2 \Big) \\
&\leq 2 \cdot (a^2 + k^2) \cdot \left\|x - Tx\right\|^2,
\end{aligned}
$$

we have

$$
\begin{aligned}
\left\|z - Tz\right\|^2 &\leq [(1/9) \cdot k^4 + (1/9) \cdot k^2] \cdot \left\|x - Tx\right\|^2 + (1/9) \cdot k^2 \cdot a^2 \cdot \left\|x - Tx\right\|^2 \\
&\quad + [(1/9) \cdot k^2 - (1/9)] \cdot 2 \cdot (a^2 + k^2) \cdot \left\|x - Tx\right\|^2 \\
&\quad - (1/9) \cdot \Big\{ \left\|Tx - T^3x\right\|^2 + \left\|T^2x - T^3x\right\|^2 \Big\} \\
&\leq \Big\{ (1/9) \cdot k^4 + [(3/9) \cdot a^2 - (1/9)] \cdot k^2 - (2/9) \cdot a^2 \\
&\quad - (1/9) \cdot (1 - \varepsilon) \Big\} \cdot \left\|x - Tx\right\|^2 \\
&= G(\varepsilon) \cdot \left\|x - Tx\right\|^2.
\end{aligned}
$$

Case II. By the estimate

$$
\begin{aligned}
\left\|x - T^2x\right\|^2 &\leq 2 \cdot \Big(\left\|x - Tx\right\|^2 + \left\|Tx - T^2x\right\|^2 \Big) \\
&\leq 2 \cdot (1 + k^2) \cdot \left\|x - Tx\right\|^2,
\end{aligned}
$$

we have

$$
\begin{aligned}
\left\|z - Tz\right\|^2 &\leq [(1/9) \cdot k^4 + (1/9) \cdot k^2] \cdot \left\|x - Tx\right\|^2 + (1/9) \cdot k^2 \cdot a^2 \cdot \left\|x - Tx\right\|^2 \\
&\quad + [(1/9) \cdot k^2 - (1/9)] \cdot 2 \cdot (1 + k^2) \cdot \left\|x - Tx\right\|^2 \\
&\quad - (1/9) \cdot \Big\{ \left\|Tx - T^3x\right\|^2 + \left\|T^2x - T^3x\right\|^2 \Big\} \\
&\leq \Big\{ (1/9) \cdot k^4 + (1/9)(1 + a^2) \cdot k^2 - (1/9) \cdot (1 - \varepsilon) \Big\} \cdot \left\|x - Tx\right\|^2 \\
&= H(\varepsilon) \cdot \left\|x - Tx\right\|^2.
\end{aligned}
$$

If the assumptions of the theorem are satisfied, then there exists $\varepsilon > 0$ such that $\max\{G(\varepsilon), H(\varepsilon)\} < 1$, and we may consider the following sequence

$$\begin{aligned} &x_1 = x, \\ &x_{n+1} = T^2 x_n \ \text{ if} \\ &\qquad \left\|Tx_n - T^3x_n\right\|^2 + \left\|T^2x_n - T^3x_n\right\|^2 \le (1-\varepsilon)\cdot\left\|x_n - Tx_n\right\|^2, \end{aligned}$$

or

$$\begin{aligned} &x_{n+1} = (1/3)(Tx_n + T^2x_n + T^3x_n) \ \text{ if} \\ &\qquad \left\|Tx_n - T^3x_n\right\|^2 + \left\|T^2x_n - T^3x_n\cdot\right\|^2 > (1-\varepsilon)\cdot\left\|x_n - Tx_n\right\|^2, \end{aligned}$$

$n = 1, 2, \ldots$.

It is easy to see that this sequence is convergent. Indeed,

$$\left\|Tx_{n+1} - x_{n+1}\right\|^2 \le A\cdot\left\|Tx_n - x_n\right\|^2, \quad \text{for} \quad n \in \mathbb{N},$$

where $A = \max\{G(\varepsilon), H(\varepsilon), 1-\varepsilon\} < 1$. Thus

$$\left\|Tx_{n+1} - x_{n+1}\right\|^2 \le A^n\cdot\left\|Tx_1 - x_1\right\|^2 \to 0$$

as $n \to +\infty$, which proves that $\{x_n\}$ is a Cauchy sequence. Let $y = \lim_{n\to\infty} x_n$. Since $\|Tx_{n+1} - x_{n+1}\|^2 \to 0$ as $n \to +\infty$, we have $\|Ty - y\| = 0$ and $Ty = y$. $\square$

Kirk [11] showed that a mapping $T : C \to C$ (C is a nonempty closed convex bounded subset of a reflexive Banach space with the normal structure) for which $T^n = I$ $(n > 1)$ has a fixed point if $\|T^ix - T^iy\| \le k\cdot\|x - y\|$, $x, y \in C$, $i = 1, 2, \ldots, n-1$, where k satisfies

$$(n-1)(n-2)\cdot k^2 + 2(n-1)\cdot k < n^2.$$

Thus a k-Lipschitzian mapping satisfying $T^n = I$ $(n > 1)$ has fixed point if

$$(n-1)(n-2)\cdot k^{2(n-1)} + 2(n-1)\cdot k^{n-1} < n^2.$$

For $n = 3$, we have the estimate $k < (1/2)\cdot\sqrt{\sqrt{88} - 4} \approx 1.1598$. Linhart [16] showed (in an arbitrary Banach space) that this mapping has a fixed point if

$$\frac{1}{n}\cdot\sum_{i=n-1}^{2n-3} k^i < 1.$$

Hence, for $n = 3$ we have the estimate for $k < k_0 \approx 1.174$.

By Theorem 4 a k-Lipschitzian involution T of order $n = 3$ in a Hilbert space (i.e. $T \in \Phi(3, 0, k, C)$) has fixed points if $k < \sqrt{(1/2)(\sqrt{41} + 1)} \approx 1.92394$.

Theorem 5. *Let C be a nonempty closed convex bounded subset of a Hilbert space $\mathcal{H}$. If $T : C \to C$ is k-Lipschitzian with $k < \sqrt{(1/2)(\sqrt{41}+1)}$ and $\|T^3x - T^3y\| \leq \|x-y\|$ for x, y in C, then there exists a fixed point of T.*

Proof: According to Browder-Göhde-Kirk's fixed point theorem [5] the set $C^* = \{x \in C : x = T^3x\}$ is nonempty. The strict convexity of $\mathcal{H}$ implies that C^* is convex. Obviously, we have $T(C^*) = C^*$ and $T^3 = I$ on C^*. Hence, by Theorem 4, we obtain our result. □

7. Open problems

The main problem of rather technical nature is whether γ_n is continuous. Other questions concern the evaluation of $\gamma_n(a)$. The evaluation given in Theorem 3 seem, in my opinion, to be not exact (for example, k-Lipschitzian involutions defined on a nonempty closed convex subset of a Hilbert space have a fixed point if $k < (1/2)(\pi + \sqrt{\pi^2 - 4}) \approx 2.78215$, see [13]). We do not even know whether there exist $a \in [0,1]$ such that $\gamma_2(a) < +\infty$ (in any Banach space), i.e. whether there exist $T \in \Phi(2, a, k, C)$, $0 \leq a \leq 1$, without fixed points. The same question can be stated for the whole interval $[0, 2)$ in the case of a Hilbert space. Analogous questions can be formulated for the function γ_3.

References

[1] Barros-Neto J., *An Introduction to the Theory of Distributions*, M. Dekker, New York, 1973.

[2] Dunford N., *Linear Operators*, vol. I, Interscience, New York, 1958.

[3] Duren W.L., *Theory of H^p Spaces*, Academic Press, New York, 1970.

[4] Goebel K., *Convexity of balls and fixed point theorems for mappings with nonexpansive square*, Compositio Math. **22** (1970), 269–274.

[5] Goebel K., Kirk W.A., *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics 28, Cambridge University Press, London, 1990.

[6] Goebel K., Koter M., *A remark on nonexpansive mappings*, Canad. Math. Bull. **24** (1981), 113–115.

[7] Goebel K., Koter M., *Fixed points of rotative Lipschitzian mappings*, Rend. Sem. Mat. Fis. di Milano **51** (1981), 145–156.

[8] Goebel K., Złotkiewicz E., *Some fixed point theorems in Banach spaces*, Colloquium Math. **23** (1971), 103–106.

[9] Górnicki J., *Fixed points of involutions*, Math. Japonica **43** (1996), 151–155.

[10] Górnicki J., Rhoades B.E., *A general fixed point theorem for involutions*, Indian J. Pure Appl. Math. **27** (1) (1996), 13–23.

[11] Kirk W.A., *A fixed point theorem for mappings with a nonexpansive iterate*, Proc. Amer. Math. Soc. **29** (1971), 294–298.

[12] Komorowski T., *Selected topics on Lipschitzian mappings* (in Polish), Thesis, Univ. M. Curie-Sklodowskiej, Lublin, 1987.

[13] Koter M., *Fixed points of Lipschitzian 2-rotative mappings*, Bolletino U.M.I., Ser. VI, **5** (1986), 321–339.

[14] Lim T.C., Xu H.K., Xu Z.B., *An L^p inequality and its applications to fixed point theory and approximation theory*, in: Progress in Approximation Theory (P. Nevai and A. Pinkus, eds.), Academic Press, New York, 1991, pp. 609–624.

[15] Lindenstrauss J., Tzafriri L., *Classical Banach Spaces, II – Function Spaces*, Springer-Verlag, Berlin, 1979.

[16] Linhart J., *Fixpunkte von Involutionen n-ter Ordnung*, Österich. Acad. Wiss. Math.-Natur. kl. S-B **180** (1973), 89–93.

[17] Prus B., Smarzewski R., *Strongly unique best approximations and centers in uniformly convex spaces*, J. Math. Anal. Appl. **121** (1987), 10–21.

[18] Smarzewski R., *Strongly unique minimization of functionals in Banach spaces with applications to theory of approximation and fixed points*, J. Math. Anal. Appl. **115** (1986), 155–172.

[19] Smarzewski R., *Strongly unique best approximation in Banach spaces, II*, J. Approx. Theory **51** (1987), 202–217.

[20] Smarzewski R., *On the inequality of Bynum and Drew*, J. Math. Anal. Appl. **150** (1990), 146–150.

[21] Xu H.K., *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), 1127–1138.

[22] Zălinescu C., *On uniformly convex function*, J. Math. Anal. Appl. **95** (1983), 344–374.

Department of Mathematics, Rzeszów Institute of Technology, P.O. Box 85,
35-959 Rzeszów, Poland

E-mail: gornicki@prz.rzeszow.pl

(*Received* August 26, 1996)