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The Bordalo order on a commutative ring

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Abstract. If R is a commutative ring with identity and $\leq$ is defined by letting $a \leq b$ mean $ab = a$ or $a = b$, then $(R, \leq)$ is a partially ordered ring. Necessary and sufficient conditions on R are given for $(R, \leq)$ to be a lattice, and conditions are given for it to be modular or distributive. The results are applied to the rings Z_n of integers mod n for $n \geq 2$. In particular, if R is reduced, then $(R, \leq)$ is a lattice iff R is a weak Baer ring, and $(R, \leq)$ is a distributive lattice iff R is a Boolean ring, Z_3, Z_4, $Z_2[x]/x^2 Z_2[x]$, or a four element field.

Keywords: commutative ring, reduced ring, integral domain, field, connected ring, Boolean ring, weak Baer Ring, regular element, annihilator, nilpotents, idempotents, cover, partial order, incomparable elements, lattice, modular lattice, distributive lattice

Classification: 03G10, 06A06, 11A07, 13A99

1. Introduction

Throughout, R will denote a commutative ring with identity element 1. In 1986, in an unpublished paper [Bo], Gabriela Bordalo defined an order $\leq$ on R by letting $a \leq b$ mean $a = b$ or $ab = a$. In that paper, she observed that $\leq$ is a partial order for any commutative ring R, derived some of its elementary properties, gave some pertinent examples, and made this investigation possible. In her honor, we call $\leq$ the *Bordalo order on R.* (An ancestor $\leq'$ has long been used in the study of Boolean rings; $a \leq' b$ is defined to mean $ab = a$. This relation is reflexive if and only if R is Boolean, and if R is a Boolean ring with identity element, then $(R, \leq')$ is a complemented distributive lattice. See [J, Chapter 8].)

In Section 2, we characterize those rings R for which $(R, \leq)$ is a lattice and those for which it is a chain, and Section 3 is devoted to describing when it is a modular or a distributive lattice. In case R is reduced, $(R, \leq)$ is a lattice if and only if R is a weak Baer ring, $(R, \leq)$ is a chain if and only if R is a two or three element field, and if R has more than 2 idempotents, then $(R, \leq)$ is a modular lattice if and only if R is a Boolean ring (in which case $(R, \leq)$ is distributive). The results are more complicated in case R has nonzero nilpotent elements, but $(R, \leq)$ cannot be a lattice if R has a nilpotent of index 4 or more, or if it has more than one nilpotent of index 2.

We are indebted to P. Dwinger for valuable discussions in which interesting questions were raised.

2. When $(R, \leq)$ is a lattice and when it is a chain

If $a < b$ and $a \leq x \leq b$ imply $x = a$ or $x = b$, then b is said to *cover* a. If a and b are incomparable, we write $a \parallel b$. As usual, $a \vee b$ and $a \wedge b$ will denote the least upper bound and greatest lower bound of a and b when these latter are defined. If $ra = 0$ implies $a = 0$, then r is called a *regular* element of R.

Observe that $0 \leq x \leq 1$ for every $x \in R$.

The following technical lemma will be used in the sequel.

2.1 Lemma.

(a) *If* $(1-x) \vee (1-y)$ *is defined, then* $x \wedge y = 1 - (1-x) \vee (1-y)$. *Dually, if* $(1-x) \wedge (1-y)$ *is defined, then* $x \vee y = 1 - (1-x) \wedge (1-y)$. *Thus* $(R, \leq)$ *is a lattice if* $x \vee y$ *is defined for all* x, y.

(b) *If* $r \neq 1$ *is regular, then* 1 *covers* r *and* $1 - r$ *covers* 0. *In particular, every nonzero nilpotent covers* 0.

(c) *If* $xy = 0$, *then* $x = y$ *or* $x \wedge y = 0$.

(d) *If* $x^2 = 0$ *and* $yzx \neq 0$, *then* $1 + zx$ *covers* yx.

(e) x *and* $1 + x$ *are comparable if and only if* $x^2 = 0$ *or* $x^2 = 1$.

(f) *If* $x \wedge y$ *is defined and regular, then* x *and* y *are regular, and either one of* x, y *is* 1, *or* $x = y = x \wedge y$.

PROOF: (a) follows easily from the fact that $a \leq b$ if and only if $1 - b \leq 1 - a$.

(b) If $r \leq x \leq 1$, then $r = x$ or $rx = r$. If the latter holds, then $rx = r$, hence $r(1-x) = 0$. Because r is regular, this implies $x = 1$, so 1 covers r. The proof that $1 - r$ covers 0 is similar. The second assertion follows from the fact that if $0 \neq x$ is nilpotent, then $1 - x$ is a unit and hence is regular.

(c) If x or $y = 0$, then $x \wedge y = 0$. If neither x nor y is 0, $x \neq y$, and $t < x$, then $tx = t$. So $0 = txy = ty$, and $t < y$ implies $y = 0$. Thus, (c) holds.

(d) If $x^2 = 0$, then $yx(1 + zx) = yx$, so $yx \leq 1 + zx$. If $yx = 1 + zx$, then $0 = (yx)x = (1 + zx)x = x$, contrary to the assumption that $yzx \neq 0$. So $yx < 1 + zx$. If there is a $t \in R$ such that $yx < t < 1 + zx$, then (i) $yxt = yx$ and (ii) $t(1 + zx) = t$. Multiplying both sides of (ii) by y yields $tzxy = 0$, so by (i), $zyx = 0$, in violation of the hypothesis. So, (d) holds.

(e) Clearly $x(1 + x) = x$ if and only if $x^2 = 0$ and $x(1 + x) = 1 + x$ if and only if $x^2 = 1$.

(f) x and y must be comparable. For otherwise, $(x \wedge y)x = (x \wedge y)y = (x \wedge y)$, so $x = y$ since $x \wedge y$ is regular. If $x \leq y$, then $x = (x \wedge y)$ is regular and either $x = y$ or $x < y = 1$ by (b). If $y \leq x$, interchanging the role of x and y in the last sentence completes the proof. □

An immediate consequence of Lemma 2.1(b) follows. It shows that the order $\leq$ cannot distinguish between integral domains of the same cardinality. The next result may also be found in [Bo].

2.2 Proposition. *If R is an integral domain, then $(R, \leq)$ is a lattice in which $x \wedge y = 0$ and $x \vee y = 1$ if $x \neq y$ and neither is 0 or 1. See Figure 1.*

$$\begin{array}{c} 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \end{array}$$

Figure 1

2.3 Lemma. *If $(R, \leq)$ is a lattice and $x^2 = 0$, then $2x = 0$.*

PROOF: Suppose $2x \neq 0$. Applying Lemma 1(d) in case $y = 2$ and $z = 1$, and in case $y = 1$ and $z = 2$ yields that both $1 + 2x$ and $1 + x$ cover both x and $2x$. Because $x \parallel 2x$, it follows that x and $2x$ cannot have a least upper bound, contrary to assumption. So $2x = 0$. □

For $x \in R$, $A(x) = \{\, y \in R : xy = 0\}$ is called the *annihilator of x*.

2.4 Lemma. *If x and y are incomparable elements of R, then the following are equivalent:*

(a) *$z = x \vee y$ is defined in R;*
(b) *$(1 - z)R \subset A(x) \bigcap A(y) \subset A(z) \bigcup (1 - z)$;*
(c) *$z^2 = z$ and $A(x) \bigcap A(y) \subset A(z) = (1 - z)R$ or $(1 - z)^2 = 0$ and $A(x) \bigcap A(y) = (1 - z)R = \{0, 1 - z\}$.*

PROOF: Note first that since $x \parallel y$, w is an upper bound of x and y if and only if $1 - w \in A(x) \bigcap A(y)$; i.e., if and only if $xw = x$ and $yw = y$.

If (a) holds, then the first inclusion in (b) holds because annihilators of elements are ideals. If $t \in A(x) \bigcap A(y)$, then $x \leq 1 - t$ and $y \leq 1 - t$, so $z \leq 1 - t$ by definition of least upper bound. Thus $z(1 - t) = z$ and hence $t \in A(z)$, or $z = 1 - t$, in which case $t = 1 - z$. So the second inclusion of (b) holds as well.

Assume next that (b) holds. If $z^2 = z$, then $1 - z \in A(z) = (1 - z)R$, so (b) is equivalent to saying that $A(x) \bigcap A(y) = A(z) = (1 - z)R$.

Assume (b) holds and z is not an idempotent. By (b), $1 - z^2 = (1 - z)(1 + z) \in A(z)$ or $1 - z^2 = 1 - z$. This latter cannot hold because $z^2 \neq z$, so $z^3 = z$. Also, $(1 - z)z \in A(z)$ or $(1 - z)z = 1 - z$. If the former holds, then $z^2 = z^3 = z$, contrary to assumption. So the latter holds and $(1 - z)^2 = 0$. If $a \in A(z)$, then $0 = a(1 - z)z = a(1 - z) = a$, so $A(z) = \{0\}$. It follows from (b) that $(1 - z)R = \{0, 1 - z\}$ and (c) holds.

Assume finally that (c) holds. If $A(x) \bigcap A(y) = \{0, 1 - z\}$, then $x < z$ and $y < z$. If t is an upper bound for x and y, then $(1 - t) \in A(x) \bigcap A(y) = \{0, 1 - z\}$. If $1 - t = 0$, then $t = 1$, while if $1 - t = 1 - z$, then $t = z$. In either case, $z \leq t$, so $x \vee y = z$. The proof that $x \vee y = z$ if $A(z) = (1 - z)R$ is an exercise. □

The following consequence of the last lemma will be used later.

2.5 Lemma. *If $(R, \leq)$ is a lattice, $x \parallel y$ for some $x, y \in R$, and $z = x \vee y$, then:*
(a) $(z - z^2)^2 = 2(z - z^2) = 0$, *and*
(b) $z - z^2 = 0$ *or* $(1 - z)^2 = 2(1 - z) = 1 - z^2 = 0$.

Proof: (a) By Lemma 2.4(c), $z - z^2 = 0 = z(1 - z)$ or $(1 - z)^2 = 0$. In either case $(z - z^2)^2 = z^2(1 - z)^2 = 0$, so $2(z - z^2) = 0$ by Lemma 2.3. So (a) holds.

(b) By Lemmas 2.4(c) and 2.3, if $z - z^2 \neq 0$, then $0 = (1 - z)^2 = 2(1 - z)$. The last equation implies $z^2 = 1$, so (b) holds. □

The next lemma is well known. Its proof is an exercise.

2.6 Lemma. *If $A(x) = eR$ and $e^2 = e \neq 0$, then x is not nilpotent.*

We are now ready to characterize those rings for which the Bordalo order is a lattice order.

2.7 Theorem. *$(R, \leq)$ is a lattice if and only if:*
(a) *there is at most one $y \in R$ such that $yR = \{0, y\}$ and y is nilpotent of index 2, and*
(b) *if $0 \neq x$ and $xR \neq \{0, x\}$, then either*
  (1) *$A(x) = eR$ for some idempotent e, or*
  (2) *$A(x) = (1-z)R = \{0, 1-z\}$ for some z such that $(1-z)^2 = 2(1-z) = 1 - z^2 = 0$.*

Proof: Suppose first that $(R, \leq)$ is a lattice. If there are distinct nilpotents y, z of index 2 such that $yR = \{0, y\}$ and $zR = \{0, z\}$, then by Lemma 2.1(b), y and z cover 0, and hence $y \parallel z$. Then yz is neither y nor z, so $yz = 0 = y = z$, contrary to assumption. So (a) holds. If $x = 0$, then (1) holds, so we may assume $x \neq 0$. We consider three cases: (i) $A(x) = A(x^2)$, (ii) there is a $y \in A(x^2) \setminus A(x)$ such that $xy = x$, and (iii) for no $y \in A(x^2) \setminus A(x)$ is it true that $xy = x$.

Suppose first that $A(x) = A(x^2)$. If $x^3 = x$ or $x^3 = x^2$, then x^2 is an idempotent, and $A(x) = A(x^2) = (1 - x^2)R$, so (1) holds. If $x^3 \neq x$ and $x^3 \neq x^2$, then $x \parallel x^2$, so by Lemma 2.4, either there is $z \in R$ such that $z^2 = z$ and $A(x) = A(x) \cap A(x^2) = (1-z)R$, and (1) holds, or $(1-z)^2 = 0$ and $A(x) = A(x) \cap A(x^2) = (1 - z)R = \{0, 1 - z\}$, and by Lemma 2.5(b), $(1 - z)^2 = 2(1 - z) = (1 - z^2) = 0$, so (2) holds.

If there is a $y \in A(x^2) \setminus A(x)$, then $0 = x^2y = x(xy)$ and $y \neq xy \neq 0$. If $xy = x$ as in case (ii), then $0 = x(xy) = x^2$, so $2x = 0$ by Lemma 2.4. If $xR \neq \{0, x\}$, there is a $w \in R$ such that $xw \neq x$ and $xw \neq 0$, and $x(xw - x) = 0$. If $x = xw - x$, then $2x = xw = 0$, contrary to assumption. Hence $x \parallel (xw - x)$. Arguing as above using Lemmas 2.4 and 2.5, there is a z such that $A(x) \cap A(y) = A(z)$ and one of conditions (1) or (2) holds. So, in case (ii), one of (a) or (b) holds.

In case (iii), $x \parallel xy$, and by Lemmas 2.4 and 2.5, $A(x) = A(x) \cap A(xy)$ satisfies either condition (1) or (2).

Assume next that (a) or (b) holds and $x, y \in R$ with $x \parallel y$. We need to show that $x \vee y$ exists in R. To do this, we need to consider 5 cases: (i) (1) holds for both x and y, (ii) (2) holds for both x and y, (iii) (1) holds for one of x, y and (2) holds for the other; say (1) holds for x and (2) holds for y, (iv) (1) holds for one of x, y and (a) applies to the other; say (1) holds for x and $yR = \{0, y\}$, and (v) (2) holds for one of x, y and (a) applies to the other; say (2) holds for x and $yR = \{0, y\}$.

(i) If $A(x) = eR$ and $A(y) = fR$ for idempotents e, f, it is easy to verify that $A(x) \cap A(y) = efR$. So $x \vee y = (1 - ef)$ by Lemma 2.4.

(ii) Suppose $A(x) = \{0, 1 - z\} = (1 - z)R$ and $A(y) = \{0, 1 - z^{'}\} = (1 - z^{'})R$ as in (b). Then $z = z^{'}$ and $A(x) \cap A(y) = \{0, 1 - z\} = (1 - z)R$, so $x \vee y$ is defined by Lemma 2.4 or $z \neq z^{'}$, in which case $A(x) \cap A(y) = \{0\}$. If this latter holds, the only common upper bound for x and y is 1.

(iii) Suppose $A(x) = eR$ as in (1) and $A(y) = (1 - z)R = \{0, 1 - z\}$ as in (2), and $x \parallel y$. If also $A(x) \cap A(y) = \{0\}$, and t is an upper bound for x and y, then $x(1 - t) = y(1 - t) = 0$, so $x \vee y = 1$. Otherwise, $A(x) \cap A(y) = \{0, 1 - z\}$, and the existence of $x \vee y$ follows from Lemma 2.4.

(iv), (v) If $y^2 = 0$, $yR = \{0, y\}$ and $x \parallel y$, then $xy = y^2 = 0$. So $(1 + y)$ is an upper bound for both x and y. By Lemma 2.1(d), $(1 + y)$ covers y, so $x \vee y = (1 + y)$. □

Recall that a ring R is called *von Neumann regular* if for each $x \in R$, there is an $y \in R$ such that $xyx = x$, and is called a *weak Baer* ring if for each $x \in R$, there is an idempotent $e \in R$ such that $A(x) = eR$. Clearly every von Neumann regular ring is a weak Baer ring, and the ring of integers witnesses that the converse is false. (The terminology in this area is not standard. Most authors define a Baer ring to be one in which the annihilator of each ideal is generated by an idempotent, but in [ES], [K], [S1], and [S2], our weak Baer rings are called *Baer* rings. In [Be], weak Baer rings are called *weak Rickart* *-rings.)

Let $N_2(R) = \{x \in R : x^2 = 2x = 0\}$. An immediate consequence of Theorem 2.7 is:

2.8 Corollary. *If $N_2(R) = \{0\}$, then $(R, \leq)$ is a lattice if and only if R is a weak Baer ring.*

The next corollary will prove to be useful below.

2.9 Corollary. *If $(R, \leq)$ is a lattice, then:*

(a) *R does not contain a nilpotent element x such that both $A(x)$ and xR have at least three elements;*
(b) *R contains at most one nilpotent of index 2;*
(c) *each nilpotent in R has index ≤ 3;*
(d) *if R contains a nilpotent x of index 3, then $8 = 0$.*

Proof: (a) If $A(x)$ and xR have at least three elements, then, by Theorem 2.7, $A(x) = eR$ for some idempotent $e \neq 0$. By Lemma 2.6, this cannot happen if x is nilpotent.

(b) If there are distinct nilpotents x, y of index 2. If $xy = 0$, then $\{0, x, y\}$ is a three element subset of $A(x) \cap xR$, contrary to (a), while if $xy \neq 0$, then this set contains $\{0, x, xy\}$; again contrary to (a).

(c) If $x^{k+1} = 0 \neq x^k$ and $k \geq 3$, then x^k and x^{k-1} are distinct nilpotents of index 2, contrary to (b).

(d) By Lemma 2.3, $2x^2 = 0$, so $(2x)^2 = 0 = 2(2x)$. Because $\{0, x^2, 4\} \subset A(x)$ and $\{0, x, x^2\} \subset xR$, (a) implies $x^2 = 4$ or $0 = 4$. Multiplying both sides of either of these equations by 2 yields $8 = 0$. $\square$

The presence of a nilpotent of index 3 in a ring which is a lattice under the Bordalo order restricts its nature considerably.

2.10 Theorem. *If R has a nilpotent of index 3, $(R, \leq)$ is a lattice, and $4 \neq 0$, then R is isomorphic to Z_8.*

Proof: Using Corollary 2.9 and Lemma 2.3 yields $8 = 0, (x^2)^2 = 2x^2 = 0$, and $(2x)^2 = 2(2x) = 0$. By Corollary 2.9(b), $2x = 0$ or $2x = 4 = x^2$. If the former holds, then $A(2) \supset \{0, x, 4\}$ and $2R \supset \{0, 2, 4\}$, contrary to Corollary 2.9(a) and the fact that x is nilpotent of index 3. Thus $2x = 4$, $A(2) \supset \{0, x - 2, 4\}$, and Corollary 2.9(a) yields $x = 2$ or $x = 6$. Thus R contains an isomorphic copy S of Z_8. If there is a $y \in R \setminus S$, then $(4y)^2 = 0$ and hence by Corollary 2.9(b), (i) $4y = 0$ or (ii) $4y = 4$.

If (i) holds, then $(2y)^2 = 0$, so by Corollary 2.9(a), $2y = 0$ or $2y = 4$. In the first case, $A(2) \supset \{0, 4, y\}$ and $2R \supset \{0, 2, 4\}$, and Corollary 2.9(a) implies $y = 4 \in R$. In the second case, repeating this last argument with y replaced by $y - 2$ yields $y = 6 \in R$.

If (ii) holds, then note that $4(y - 1) = 0$. So, repeating this last argument yields $(y - 1) \in R$ and hence $y \in R$. This contradiction shows that $R = S$ and completes the proof. $\square$

Next, we determine when $(R, \leq)$ is a chain. The ring of integers will be denoted by Z, and for each positive integer n, the ring of integers mod n will be denoted by Z_n. First, we need to establish a rather technical lemma.

2.11 Lemma. *If $(R, \leq)$ is a lattice and there is a nonzero $x \in R$ such that $xR = \{0, x\} = A(x)$, then R is isomorphic to either Z_4 or $Z_2[x]/x^2 Z_2[x]$.*

Proof: Clearly $x^2 = 2x = 0$, so by Lemma 2.1(c), $2 = 0$ or $2 = x$ and $4 = x^2 = 0$. If $2 \neq 0$, then R contains an isomorphic copy S of Z_4 that contains 0 and 1. Suppose $y \in R$ and $a \in S$. By assumption, $2(a + y) = 0$ or $2(a + y) = 2$. If the former holds, then $a + y = 0$ or $a + y = 2$. In either case, $y \in S$. If $2(a + y) = 2$, then $2(a + y - 1) = 0$, and the same argument yields $y \in S$.

If $2 = 0$, then R contains a subring T isomorphic to $Z_2[x]/x^2 Z_2[x]$ that contains 0 and 1. Suppose $y \in R \setminus T$. Then $x(1 + x + y) = 0$ or $x(1 + x + y) = x$, so either $1 + x + y = x$ and $y = 1$, or $x(x + y) = 0$ and $y = 0$. It follows that $R = T$. □

2.12 Theorem. *$(R, \leq)$ is a chain if and only if R is isomorphic to Z_2, Z_3, Z_4, or $Z_2[x]/x^2Z_2[x]$.*

Proof: If $(R, \leq)$ is a chain and $0 \neq y \in R$, then y and $1 + y$ are comparable, so $y^2 = 0$ or $y^2 = 1$ by Lemma 2.1(e). If R is reduced, then $y = 1$ or -1, and R is isomorphic to Z_2 or Z_3 according as $-1 = 1$ or not. If R fails to be reduced, there is an $x \neq 0$ such that $x^2 = 0$. If $0 \neq y \in A(x)$, then $y = x$ by Lemma 2.1(c), so $A(x) = \{0, x\}$, and hence $2x = 0$. For any $z \in R$, $x(xz - x) = 0$, so $xz = x$ or $xz - x = xz + x = x$. Thus, $xz = x$ or $xz = 0$ and it follows that $xR = \{0, x\}$, and R is isomorphic to Z_4 or $Z_2[x]/x^2Z_2[x]$ by Lemma 2.10.

It is routine to verify that each of these rings is a chain under the Bordalo order. □

It is easy to see that a direct product of weak Baer rings is a weak Baer ring. So Corollary 2.8 implies that if each member of a family of reduced rings is a lattice under the Bordalo order, so is their direct product. It will be seen next that this situation changes drastically in the presence of nonzero nilpotent elements.

2.13 Lemma. *If S is not reduced, then the direct product $R \oplus S$ is a lattice under the Bordalo order if and only if R is an integral domain and S is a chain.*

Proof: Assume first that $(R \oplus S, \leq)$ is a lattice and observe that if $(r, s) \in R \oplus S$, then $A((r, s)) = A(r) \oplus A(s)$ and $(r, s)(R \oplus S) = rR \oplus sS$. If S is not reduced, it contains an element $s' \neq 0$ such that $(s')^2 = 0$, and if R is not an integral domain, it contains nonzero elements r', r'' such that $r'r'' = 0$. It follows that $A((r', s'))$ contains the three element set $\{(0, 0), (r'', 0), (0, s')\}$, and $(r', s')(R \oplus S)$ contains the three element set $\{(0, 0), (r', 0), (0, s')\}$. So, by Theorem 2.7, $A((r', s')) = (e, f)(R \oplus S) = eR \oplus fS$ for idempotents $e \in R$ and $f \in S$. It follows easily from Lemma 2.6 applied to the second summand that this cannot happen. So, R is an integral domain.

Because $A((1, s')) = \{0\} \oplus A(s')$ and $s' \neq 0$ is nilpotent, it cannot be generated by an idempotent. Also, $(1, s')(R \oplus S) = R \oplus s'S$ has more than the 2 elements, so $A(s') = \{0, s'\}$. Suppose there is a $w \in S$ such that $s'w \neq 0 \neq s'$. Then $s'(s'w - s') = 0$ and $A(s') = \{0, s'\}$, we must have $s'w - s' = s'$ or $s'w = 2s' = 0$ by Lemma 2.3. This contradiction yields $s'S = \{0, s'\}$, so by Lemma 2.11, S is isomorphic to Z_4 or $Z_2[x]/x^2Z_2[x]$.

Suppose conversely that R is an integral domain and S is isomorphic to Z_4 or $Z_2[x]/x^2Z_2[x]$. Then S contains a unique element s' such that $A(s') = \{0, s'\} = s'S$. Since R is an integral domain, if $(r, s) \in R \oplus S$, then $A((r, s))$ is generated by an idempotent unless $s = s'$. In this case, $A((1, s')) = \{(0, 0), (0, s')\} = (0, s')(R \oplus S)$. So $(R \oplus S, \leq)$ is a lattice by Theorem 2.7. □

We conclude this section by applying the above to determining for which integers $n > 1$, the ring Z_n is a lattice under the Bordalo order.

2.14 Theorem. *If $n > 1$ is an integer, then Z_n is a lattice under the Bordalo order if and only if either:*

(a) *n is square free,*
(b) *$n = 4p$ for some prime p, or*
(c) *$n = 4$.*

PROOF: Clearly $n = \prod_{i=1}^{t} p_i^{k_i}$ is a product of distinct prime powers, if and only if Z_n is the direct product of the rings $Z_{p_i^{k_i}}$ and n is square free if and only if Z_n is the direct product of the fields Z_{p_i} and hence is a weak Baer ring. So, if n is square free, then Z_n is a lattice under the Bordalo by Corollary 2.8.

Assume next that $n = p^2 m$ for some prime p, $(Z_n, \leq)$ is a lattice, and $n \neq 4$. If $2p = 0$ in Z_n, then $p = 2$ and $m = 1$, so $n = 4$, contrary to assumption. So $\{0, p, 2p\}$ is a three element subset of $A(pm)$. By Corollary 2.9(a), pmZ_n has cardinality 2. Thus, $2pm = p^2 m$, so $p = 2$. Because Z_4 is a chain, we may conclude that if Z_n is a lattice, then n is square free or $n = 4m$ for some $m \geq 1$.

If m is odd, then Z_{4m} is isomorphic to $Z_4 \oplus Z_m$, so if it is a lattice under the Bordalo order, the m is prime by Lemma 2.13. If m is even, then $n = 2^{k+1}$ for some $k \leq 2$. by Corollary 2.9. By Lemma 2.11, Z_4 is a chain. For $n = 8$, every nonzero element except for 2 and 4 has annihilator $\{0\}$, $4^2 = 0$, while $4Z_8 = \{0, 4\}$, and $A(2) = (1 - 5)Z_8 = \{0, -4\}$, so $(Z_8, \leq)$ is a lattice by Theorem 2.7. □

Some additional properties of a ring for which the Bordalo order is a lattice follow. We begin a lemma whose proof is an exercise:

2.15 Lemma. *If $x < y$, then $xz \leq y$ for any $z \in R$.*

If $0, 1$ are the only idempotents of a ring R, then it is said to be *connected.* Note that any integral domain and any ring with a unique maximal ideal (e.g., the rings Z_{p^k} for any prime p and integer k) are connected.

2.16 Lemma. *Suppose $(R, \leq)$ is a lattice and a, b are incomparable in R, then:*

(a) *$(a \wedge b)^3 = (a \wedge b)^2$, and hence $(a \wedge b)^2$ is idempotent, and*
(b) *if R is connected, then $(a \wedge b)^2 = 0$.*

PROOF: (a) Because $a \parallel b$, $(a \wedge b) < a$ and $(a \wedge b) < b$,so by Lemma 2.15, $(a \wedge b)^2 \leq (a \wedge b)$. Thus $(a \wedge b)^2 = (a \wedge b)$ or $(a \wedge b)^2 = (a \wedge b)^2 (a \wedge b) = (a \wedge b)^3$. In either case, the conclusion holds.

(b) Because R is connected, $(a \wedge b)^2 = 1$ or $(a \wedge b)^2 = 0$ by (a). The first case cannot hold by Lemma 2.1(f), so $(a \wedge b)^2 = 0$. □

Next, we give some more examples pertinent to the results of this section.

2.17 Examples. A. By Theorem 2.14, $(Z_8, \leq)$ is a lattice that witnesses the existence of a ring with nilpotents of index 3 in which $0 = 8 \neq 4$ that is a lattice under the Bordalo order.

B. By Theorem 2.12, if D is an integral domain of characteristic 0, then $(D \oplus Z_4, \leq)$ is a lattice. Note that $(0,2)$ is the only nonzero nilpotent in $D \oplus Z_4$ and that $n(1,1)$ is not $(0,0)$ if $n \geq 1$ is in Z. So the requirement in the hypothesis of Corollary 2.9(d) that there be a nilpotent of index 3 may not be omitted.

C. A connected ring S of characteristic 4 such that $(S, \leq)$ is a lattice. Moreover, S has cardinality 8 and has a nilpotent of index 3.

Let $R = Z_4[x]$, let $I = x^3R + 2xR + (x^2 - 2)R$, and let $S = R/I$. We may regard S as $\{0, 1, 2, 3, x, 1 + x, 2 + x, 3 + x\}$, where $x^3 = 2x = x^2 - 2 = 0$. The following assertions are easily verified: $A(0) = S, A(1) = A(3) = A(1 + x) = A(3 + x) = \{0\}$. Because $4 = 2x^2 = 0$, $2S = \{0, 2\}$ and $2^2 = 0$, while $A(x) = A(2+x) = \{0, 1-3\}$. By Theorem 2.7, $(S, \leq)$ is a lattice, and $x^3 = 2x^2 = 0 \neq x^2$. Clearly $(a + bx)^2 = a^2$, so S is connected. Thus, S has the indicated properties.

In the next section, we examine the properties of the lattice $(R, \leq)$ when R is a lattice under the Bordalo order.

3. What kind of lattice is $(R, \leq)$?

3.1 Definition. Suppose r, s, t are distinct elements of R different from 0 or 1.

(a) If $r \vee s = r \vee t, r \wedge s = r \wedge t$, and $s < t$, then $\{r, s, t\}$ is called a *nonmodular triple.*

(b) $r \vee s = r \vee t = s \vee t$ and $r \wedge s = r \wedge t = s \wedge t$, then $\{r, s, t\}$ is called a *nondistributive triple.*

The first part of the next lemma appears as Theorem 8.4 in [J].

3.2 Lemma. *Suppose $(R, \leq)$ is a lattice.*

(a) *$(R, \leq)$ is modular if and only if it fails to contain a nonmodular triple.*

(b) *$(R, \leq)$ is distributive if and only if it fails to contain either a nondistributive or a nonmodular triple.*

(c) *If $\{r, s, t\}$ is a triple of distinct regular elements none of which is 1, then R is not distributive. In particular if $\{r, s\}$ is a pair of distinct regular elements neither of which is 1 such that $rs \neq 1$, then R is not distributive.*

Proof: (a) and (b) are just restatements of the well-known fact that a lattice is modular if and only if it fails to contain the five element lattice illustrated on p. 435 of [J] and that a lattice is distributive if and only if it fails to contain either this lattice or the six element lattice in the same illustration.

(c) Not both rs and rt can be 1; say $rs \neq 1$, and let $w = r \wedge s$. Then $rw = w$ or $r = w$, in which case $r \leq s$. Because $r \neq s$, $rs = r$ and $s = 1$, contrary to assumption. Hence $wr = w$. The same argument shows that $ws = w$ and hence that $wrs = w$ or $w < rs$. Thus, if R is distributive, then $r = r \wedge 1 = r \wedge (s \vee rs) = (r \wedge s) \vee (r \wedge rs) = w \vee (r \wedge rs) = (r \wedge rs)$, so $r \leq rs$. This latter is impossible because $rs \notin \{1, r\}$. $\square$

3.3 Theorem. *Suppose $(R,\leq)$ is a lattice.*

(a) *If R is not connected, then $(R,\leq)$ is modular if and only if R is a Boolean ring. So, if R is not connected, then $(R,\leq)$ is modular if and only if it is distributive.*

(b) *If R is connected, then $(R,\leq)$ is distributive if and only if R is a chain or a four element field.*

Proof: (a) Suppose R is modular, $e^2 = e \notin \{0,1\}$, and there is a regular element x such that $ex \neq e$ (in which case $x \neq 1$). Then $(ex)e = ex$, so $ex < e$. A series of routine arguments using this and the fact that 1 covers x together imply $\{x \wedge e, ex, e\}$ is a nonmodular triple. Thus, $(R,\leq)$ is not modular by Theorem 3.2(a), contrary to the hypothesis. We conclude that $ex = e$ for every nontrivial idempotent e and regular element x. Because $(1-e) = (1-e)^2, (1-e) = (1-e)x = x - ex = 0$. This contradiction shows that 1 is the only regular element of R, and because -1 and 1 are regular, this implies $2 = 0$. Because $1 + y$ is invertible for every nilpotent element y, R is reduced.

Suppose $a \in R$ is not idempotent. By Theorem 2.7, there is an idempotent e such that $A(a) = eR$. Now $a < 1 + e$ since $a \neq 1 - e = 1 + e$. Let K denote $\{1 + a, a, 1 + e\}$. It is easy to see that $0 < a < 1 + e < 1$, and $0 < 1 + a < 1$. If $z = a \vee (1 + a)$, then $az = z$ and $(1 + a)z = 1 + a$, and hence $z = 1$. Similarly, $a \wedge (1 + a) = 0$. So $(K,\leq)$ is a nonmodular triple, and hence $(R,\leq)$ fails to be modular by Theorem 3.2(a). This contradiction shows that R is a Boolean ring. As noted in the introduction, $(R,\leq)$ is distributive and hence modular if R is a Boolean ring. So (a) holds.

(b) Suppose $(R,\leq)$ is distributive and R is connected. If R is reduced, then by Corollary 2.8, $A(x) = \{0\}$ whenever $x \neq 0$. So R is an integral domain, and must contain at least three regular elements other than 1 if R has at least 5 elements. So, by Theorem 3.2(c), R fails to be distributive unless R has no more than 4 elements and hence must be Z_2, Z_3, or a 4 element field. Hence R is a chain. If $xR \neq \{0, x\}$, there is a $w \in R$ such that $0 \neq xw \neq x$, so $\{1 + x, 1 + xw\}$ is a pair of regular elements satisfying the hypothesis of Theorem 3.2(c), contrary to the assumption that $(R,\leq)$ is distributive. If there is a $y \in R$ such that $y^2 = 0$ and $y \neq x$, then $\{1 + x, 1 + y, 1 + x + y\}$ would be a nondistributive triple, again contrary to the hypothesis. Thus, $xR = \{0, x\} = A(x)$ for a unique $x \in R$, and it is an exercise to show that R is isomorphic to Z_4 and hence a chain. □

Next, we describe a class of rings that are modular but not distributive lattices under the Bordalo order.

A ring in which every element is either regular or nilpotent will be called a *RON-ring*. Clearly every RON-ring is connected. Note that every integral domain and the rings Z_{p^k} (where p is a prime and $k \geq 1$ is an integer) are RON-rings. An example will be given below of a connected ring that is a lattice under the Bordalo order and fails to be an RON-ring. Let $r(R)$ denote the set of regular elements or R other than 1.

3.4 Lemma. *If R is an RON-ring and $(R, \leq)$ is a lattice, then:*

(a) *$(R, \leq)$ contains no chain with more than 4 elements;*

(b) *if R is not reduced, then there is a unique element z that is nilpotent of index 2 such that $r \wedge s = z$ whenever r, s are distinct elements of $r(R)$.*

PROOF: (a) If $0 < a < b < c < 1$ is a chain in $(R, \leq)$, then, contrary to Lemma 2.1, b is not regular since is not covered by 1, and is not nilpotent because it does not cover 0, so 4 is the maximal length of a chain in this lattice.

(b) By assumption and Corollary 2.9(b), R contains a unique nilpotent element z of index 2. If $1 - z \neq r \in r(R)$, then $A(zr) \supset \{0, z, zr\}$ and $zrR \supset \{0, zr, zr^2\}$. By Corollary 2.9(a), not both of these sets contain 3 distinct elements, so $zr - z = 0$ or $zr - zr^2 = (zr - z)r = 0$. Because r is regular, we conclude that $zr - z = 0$ in either case, and we know that $z < 1 - z$. So the conclusion follows. □

3.5 Theorem. *If R is an RON-ring and $(R, \leq)$ is a lattice, then $(R, \leq)$ is modular.*

PROOF: If R is reduced, then R is an integral domain and the conclusion follows immediately from Proposition 2.2. If R is not reduced, then it contains a unique nilpotent z of index two such that $r \wedge s = z$ whenever $r \neq s \in r(R)$. If $1 > t > s > 0$, then s cannot be regular by Lemma 2.1(b), and by the same lemma, t cannot be nilpotent. Thus, t is regular and s is nilpotent. If r, s, and t are distinct, then either r is regular and $r \wedge t = z \neq 0$, or r is nilpotent and $r \vee s = 1 - z \neq 1$. So, $\{r, s, t\}$ fails to be a nonmodular triple in R, and the conclusion follows. □

The next example shows that not every connected ring which is a lattice under the Bordalo order is a RON-ring.

3.6 Example. A connected ring S that fails to be an RON-ring such that $(S, \leq)$ is a lattice and S has a unique nonzero nilpotent element.

Let $R = Z_2[x, y]$, let $I = x^2 R + (xy - x)R$, and let $S = R/I$. Each element of S may be written in the form $a + bx + yp(y)$, where $a, b \in Z_2$, $p(y) \in Z_2[y]$, $x^2 = 0$, and $xyp(y) = p(1)x$. Thus, $(a + bx + yp(y))(c + dx + yq(y)) = ac + (ad + bc + bq(1) + dp(1)) + (aq(y) + cp(y) + yp(y)q(y))y$. Carrying out routine computations will enable the reader to verify the following assertions: If $p(y) \neq 0$, then (1) $A(yp(y)) = \{0\}$ or $\{0, x\}$ according as $p(1) = 0$ or 1, and $A(1 + yp(y)) = \{0\}$ or $\{0, x\}$ according as $p(1) = 1$ or 0. (2) $xS = \{0, x\}$. (3) $A(x + yp(y)) = \{0\}$ or $\{0, x\}$ according as $p(1) = 1$ or 0. (4) $A(1 + x + yp(y)) = \{0\}$ or $\{0, x\}$ according as $p(1) = 0$ or 1. So, by Theorem 2.7, $(S, \leq)$ is a lattice. Because $(a + bx + yp(y))^2 = a^2 + y^2[p(y)]^2$, S is connected, and because $(1 + x + y)^2 = 1 + y^2$ and $1 + x + y$ is not regular, S is not a RON-ring. It is clear from the above that x is the only nonzero nilpotent in S.

It is not difficult to verify that if S is the ring of Example 3.6, then $(S, \leq)$ is a modular lattice. Indeed, we wonder if it is true that whenever $(R, \leq)$ is a lattice, R is connected, and has a nonzero nilpotent, it follows that $(R, \leq)$ is modular?

We have neglected to include much about the geometry of the partially ordered sets $(R, \leq)$. [Bo] contains some results about the length of chains in such sets. This preprint together with the present paper should set the stage for further study.

References

[Be] Berberian S., *Baer*-rings*, Springer-Verlag, New York, 1972.

[Bo] Bordalo G., *Naturally ordered commutative rings*, preprint.

[ES] Speed T., Evans M., *A note on commutative Baer rings*, J. Austral. Math. Soc. **13** (1971), 1–6.

[J] Jacobson N., *Basic Algebra I*, W.H. Freeman and Co., San Francisco, 1974.

[K] Kist J., *Minimal prime ideals in commutative semigroups*, Proc. London Math. Soc. **13** (1963), 31–50.

[Sp1] Speed T., *A note on commutative Baer rings I*, J. Austral. Math. Soc. **14** (1972), 257–263.

[Sp2] Speed T., *A note on commutative Baer rings II*, ibid. **15** (1973), 15–21.

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