Linear programming duality and morphisms

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Abstract. In this paper we investigate a class of problems permitting a good characterisation from the point of view of morphisms of oriented matroids. We prove several morphism-duality theorems for oriented matroids. These generalize LP-duality (in form of Farkas' Lemma) and Minty's Painting Lemma. Moreover, we characterize all morphism duality theorems, thus proving the essential unicity of Farkas' Lemma. This research helped to isolate perhaps the most natural definition of strong maps for oriented matroids.

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1. Introduction

There is a great variety of combinatorial optimization results which are of min-max type or "good characterizations" ([6]) and from the point of view of theoretical computer science these results establish the membership in the class $NP \cap co\text{-}NP$. However, it seems to be convenient for a better understanding to have a more uniform description of certain classes of characterizations. The most important here is of course the linear programming approach and the duality of linear programming serves as a prototype and master theorem which implies many (but not all) min-max results. Yet, there are other approaches for example the lattice method of A. Hoffman [15], [16], or general methods related to the Ellipsoid Method [10]; see also [4]. Another possible approach of algebraic flavour was suggested in [21], [17], [12] and relies on the definition of classes by means of special morphisms. (Although this has a definitive category theory flavour one does not make use of any of the deep results of the theory of categories and thus we do not have to use the formal language of this theory.) One proceeds as follows:

Suppose that $\mathcal{K}$ is a class of objects together with a certain class of maps ("homomorphisms", "morphisms") between them. Given two objects A, B of $\mathcal{K}$ we denote by $A \to B$ the existence of a map from A to B. Given a fixed object A we denote by

$$\to (A)$$

the class of objects of $\mathcal{K}$ which map to A:

$$\to (A) \quad := \quad \{B \in \mathcal{K} \mid B \to A\}.$$

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Similarly, $(A)\rightarrow \;:=\; \{B \in \mathcal{K} \mid A \rightarrow B\}$.

The complementary classes will be denoted by $\not\rightarrow (A)$ and $(A) \not\rightarrow$. Explicitly, $\not\rightarrow (A)$ is the class of objects which do not map into A.

A *homomorphism duality theorem for $\mathcal{K}$* is the following equation of two classes

$$(*) \qquad (B)\rightarrow \;=\; \not\rightarrow (A).$$

The equation $(*)$ is the equation of two classes and thus, explicitly, it means:

For every object C of $\mathcal{K}$: $\quad B \rightarrow C$ iff $C \not\rightarrow A$.

Examples of homomorphism duality are numerous. The class of graphs and homomorphisms between them was studied in greater detail in [21], [17], [12], [13] (a homomorphism $f : G \rightarrow H$ is any edge preserving mapping of the vertices). For example the following is proved in [21] and [17]:

Theorem A. *The only homomorphism duality for the class of all undirected graphs and their homomorphisms is the following pair:*

$$(K_2)\rightarrow \;=\; \not\rightarrow (K_1).$$

Theorem B. *For every oriented tree (i.e. an orientation of a tree) T there exists a directed graph B_T such that*

$$(T)\rightarrow \;=\; \not\rightarrow (B_T).$$

These are the only homomorphism dualities for the class of all directed graphs and their homomorphisms.

For example (as the first non-trivial case) for T on Fig. 1a B_T is given on Fig. 1b.

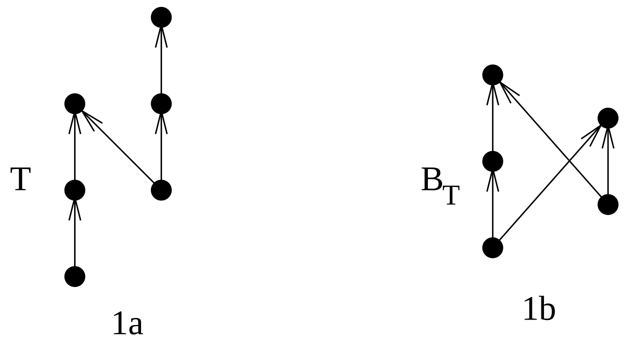

Figure 1: An example of Tree Duality in Digraphs

These results show that in the case of graphs and their homomorphism the examples of homomorphism dualities form a rather small group unable to describe deeper min-max theorems valid for graphs. There are two ways how to save this situation:

1. to consider more complicated objects,
2. to consider more special homomorphisms.

Approach 1 was taken in [11], [12], [13] where examples were found of path-dualities, tree-dualities and bounded tree-width dualities. In all these examples we deal with classes of graphs (instead of individual graphs). For example in [11] it has been shown

Theorem C. *For any directed path P the following holds for any directed graph G:*

$$G \not\to P \text{ iff there exists a path } P', P' \not\to P \text{ with } P' \to G.$$

Theorem C is an instance of a path-duality as it claims the following: There is an "easy" proof of $G \not\to P$. The reason is that G contains a homomorphic image of a path P', $P' \not\to P$ (i.e. a "bad path"). If the role of a path is played by trees (i.e. by bad trees) or by graphs with bounded tree-width then we speak about a tree-duality and a bounded tree-width duality. Let us state explicitely at least the latter notion:

The decision H-coloring problem

Instance: A graph G

Question: Does there exist a homomorphism $G \to H$?
admits a bounded tree-width duality if there exists $k \geq 1$ such that the following holds:

$G \not\to H$ if and only if there exists F with tree width $\leq k$ such that $F \not\to H$ and $F \to G$.

The importance of this lies in the fact that any bounded tree-width duality yields a polynomial algorithm for the corresponding H-coloring problem. This was independently (and by different means) shown in [12] and [8]. In fact, presently, this approach yields all polynomial instances of coloring problems.

Our approach here combines 1 and 2. We demonstrate the power of homomorphism duality scheme $(*)$ by showing that the duality theorem of linear programming, Farkas' Lemma, may be interpreted within this framework. We shall show that the standard class of morphisms for oriented matroids yields a particular instance of duality which is equivalent to Farkas' Lemma (Theorem 4.1). Moreover, we show that with properly defined morphisms this is the only existing duality (Theorem 5.5). Thus, in this setting, Farkas' Lemma is the only valid duality (i.e. it is the only valid duality for oriented matroids with a given class of maps).

The use of oriented matroids for linear programming in the context of homomorphism dualities was suggested by L. Lovász and A. Schrijver [19]. Our approach here is motivated by this and by their question whether there exists a homomorphism duality for Farkas' Lemma of such a form, that on both sides of the equation $(*)$ is the same type of morphisms (as is the case for homomorphisms of graphs; indeed our formulation of a homomorphism duality covers only such cases).

Thus our main results (Theorem 4.1, 5.2 and 5.4) have the simple form of a class-equation

$$A \not\to \quad = \quad \to B$$

for suitably defined objects and morphisms. However, there is a price to be paid for this apparent simplicity: the definitions of morphisms and objects are complex, yet we believe natural.

The paper is organized as follows: In Section 2 we review some basic facts about oriented matroids. In Section 3 we introduce our morphisms (Definition 3.2) and prove their basic properties. Section 4 contains a proof of the main result and finally in Section 5 we analyse the relations on the oriented matroids given by the morphism and extend them such that Farkas' Lemma is the only homomorphism duality theorem.

2. Oriented matroids

Oriented matroids ([5], [9]) are perhaps not as widely known and familiar as matroids but at least there are already a few books available ([1], [3]).

From the various cryptomorphic axiom systems for oriented matroids we have chosen the *circuit axioms*. The advantage is that they can be immediately observed from the circuit axioms of matroid theory by considering digraphs (see detailed comments after Definition 2.1). We will start with the definition of signed sets and basic notations for them. For reasons which we hope will become clear later on, we use a notation similar to that of Folkman and Lawrence [9].

Definition 2.1. *Let E be a set. For our purposes it will be convenient to consider two copies of E namely E^+, E^- with different signs. We will denote $E^+ \cup E^-$ by $\pm E$. The involution $- : \pm E \to \pm E$ is defined the obvious way mapping e.g. element e^+ to e^- and vice versa. The* support $\operatorname{supp}(X) \subseteq E$ *of a subset $X \subseteq \pm E$ is the set of elements of the underlying ground set. A* signed subset *$X \subseteq E^+ \cup E^-$ of E is a set with $\operatorname{supp}(X \cap E^+) \cap \operatorname{supp}(X \cap E^-) = \emptyset$. For short, we write $X^+ := X \cap E^+$ and $X^- := X \cap E^-$ and denote by $2^{\pm E}$ the set of all signed subsets of E. The* separator *of two signed subsets X, Y is defined as* $\operatorname{sep}(X,Y) := \operatorname{supp}(X \cap -Y)$.

A collection $\mathcal{C}$ of signed subsets of a set E is the set of signed circuits *of an* oriented matroid *on E if it satisfies the following axioms*

(C0) $\emptyset \notin \mathcal{C}$,

(C1) $\mathcal{C} = -\mathcal{C}$, (*symmetry*)

(C2) $\forall X, Y \in \mathcal{C} : \operatorname{supp}(X) \subseteq \operatorname{supp}(Y) \Rightarrow X \in \{Y, -Y\}$, (*incomparability*)

(C3) $\forall X \neq -Y \in \mathcal{C}\, \forall e \in \operatorname{sep}(X,Y)\, \exists Z \in \mathcal{C} : Z \subseteq X \cup Y \setminus \{e^+, e^-\}$ (*weak elimination*).

If $\mathcal{D}$ is a collection of signed subsets of a finite set E then $X \in \mathcal{D}$ is elementary *in E iff $\{Y \in \mathcal{D} \mid \emptyset \neq Y \subset X\} = \emptyset$.*

Consider a digraph $D = (V, E)$. For any circuit of the underlying graph we can derive two signed sets C and $-C$ on E as follows. Choose an orientation in which the circuit is traversed. Let

$$C^+ := \{e^+ \in E^+ \mid e \text{ is an edge which is traversed from tail to head}\} \text{ and}$$

$$C^- := \{e^- \in E^- \mid e \text{ is an edge which is traversed from head to tail}\}.$$

This uniquely defines C. It is immediate that the set of signed sets which can be derived from D satisfies the above axioms.

There is another classical example for oriented matroids. It stems from vector spaces over ordered fields $\mathbb{K}$. Let $E \subset \mathbb{K}^d$ be finite and $n = |E|$. Thus, we may consider E as a $(d \times n)$ matrix over the reals. Let $A \subseteq E$ be a circuit of E, i.e. a minimal dependent subset and $\lambda \in \mathbb{K}^{|E|}$ such that $A\lambda = 0$ and $\lambda_e \neq 0 \Leftrightarrow e \in A$. Hence λ is a *minimal non-trivial element in the kernel* of A. From λ we derive the signed set

$$\bigcup_{\substack{\lambda_i > 0 \\ e \in E}} e^+ \cup \bigcup_{\substack{\lambda_i < 0 \\ e \in E}} e^-.$$

Elementary computations show that the signed sets derived this way satisfy the circuit axioms of oriented matroid theory.

These signed sets correspond to the vectors of minimal support in a vector space. Arbitrary elements of that space correspond to compositions of circuits, which we are going to define now. A composition of any two circuits derived from say λ and μ yields the sign pattern of $\lambda + \varepsilon\mu$ for some small positive ε:

Definition 2.1. *Let C and D be two signed subsets of a finite set E. The composition $C \circ D$ is the signed set $C \cup (D \setminus -C)$ (where by $\setminus$ we mean set difference). Note, that this operation is associative. Let $\mathcal{C}$ be the set of circuits of an oriented matroid and let $C_1, \ldots, C_k \in \mathcal{C}$. Then we call the signed set $C_1 \circ \ldots \circ C_k$ a vector of $\mathcal{C}$.*

Thus the vectors of an oriented matroid which is given by a linear space over an ordered field correspond bijectively to the sign patterns of the vectors of that linear space.

It is useful to notice that each positive vector is the composition of positive circuits:

Proposition 2.1 (cf. [3, 3.7.2]). *If X is a vector of an oriented matroid $\mathcal{C}$ on a finite set E, and $e \in E$ such that $\{e^+, e^-\} \cap X \neq \emptyset$ then there is a $C \in \mathcal{C}$ such that $C \subseteq X$ and $\{e^+, e^-\} \cap C \neq \emptyset$.*

Both of the above examples show that oriented matroids provide a combinatorial framework to study linear programming duality. Actually, Minty's Painting

Lemma [20] for directed graphs is formulated in these terms. Thus note, that in the abstract setting of this paper we view linear programming duality as Farkas' Lemma in the homogeneous form (see [1] and [7]):

Lemma 2.1 (Farkas' Lemma). *Let $V \in \mathbb{R}^n$ be a subspace of the d-dimensional Euclidean space. Then*

$$\begin{aligned} &\exists\, x = (x_1, \dots, x_n) \in V : x \geq 0,\, x_1 > 0 \\ \Leftrightarrow\, &\nexists\, y = (y_1, \dots, y_n) \in V^{\perp} : y \geq 0,\, y_1 > 0. \end{aligned}$$

Before we can present Farkas' Lemma in oriented matroids we first have to define the *orthogonal complements* or *duals* of oriented matroids. Consider the polygonmatroid of a graph. The cocircuits of this matroid are the cuts of the graph. It is immediate how to derive signed subsets for the cuts of a directed graph. One observes that any circuit which traverses a cut has to traverse it in both directions. This and the definition of orthogonality in linear spaces motivate the following:

Definition 2.2. *Let X, Y be two signed sets. X and Y are* orthogonal $X \perp Y$ *if*

$$X \cap Y = \emptyset \Leftrightarrow -X \cap Y = \emptyset.$$

Proposition 2.2 (cf. [3, 3.7.6 and 3.7.12])**.** *Let $\mathcal{C}$ be an oriented matroid. Denote by $\mathcal{C}^*$ the set of signed subsets which are elementary in $\mathcal{C}^{\perp} := \{X \in 2^{\pm E} \mid \forall\, C \in \mathcal{C} : X \perp C\}$. Then $\mathcal{C}^*$ satisfies the circuit axioms of oriented matroid theory. We call $\mathcal{C}^*$ the* dual oriented matroid *of $\mathcal{C}$. Furthermore, we have $(\mathcal{C}^*)^* = \mathcal{C}$.*

Definition 2.3. *Let $\mathcal{C}$ be an oriented matroid. We call an element of $\mathcal{C}^*$ a* cocircuit *of $\mathcal{C}$ and an element from $\mathcal{C}^{\perp}$ a* covector *of $\mathcal{C}$.*

After these preparations Farkas' Lemma translates to oriented matroids.

Lemma 2.2 (Farkas' Lemma in oriented matroids cf. [3, 3.4.6])**.** *Let $\mathcal{C}$ be an oriented matroid on a finite set E and $e \in E$. Then exactly one of the following conditions holds:*

1. *there is a $C \in \mathcal{C}$ such that $e^+ \in C \subseteq E^+$,*
2. *there is a $C' \in \mathcal{C}^*$ such that $e^+ \in C' \subseteq E^+$.*

As mentioned above, Farkas' Lemma for digraphs is also known as Minty's Painting Lemma. It is possible to define oriented matroids as the unique combinatorial structure such that Farkas' Lemma holds for all possible sign patterns in all minors (cf. [3, 3.4.4] or [1]).

What are the minors of oriented matroids?

Proposition 2.3 (cf. [3, 3.3.1 and 3.3.2])**.** *Let E be finite and $\mathcal{C}$ an oriented matroid on E and $I \subseteq E$.*

1. $\mathcal{C} \setminus I := \{X \in 2^{\pm E \setminus I} \mid X \in \mathcal{C}\}$ *is an oriented matroid which we call the* submatroid induced by $E \setminus I$. *We as well say that $\mathcal{C} \setminus I$ arises from $\mathcal{C}$ by* deletion of I.
2. *Let $\mathcal{C}/I$ denote the set of vectors which are elementary in $\{X \in 2^{\pm E \setminus I} \mid \exists\, \tilde{X} \in \mathcal{C} : X = \tilde{X} \setminus (I^+ \cup I^-)\}$. Then $\mathcal{C}/I$ is an oriented matroid which we call the* contraction of $\mathcal{C}$ to $E \setminus I$. *We as well say that $\mathcal{C}/I$ arises from $\mathcal{C}$ by* contraction of I.

The reader will not be very surprised that contraction and deletion are dual operations.

Proposition 2.4 (cf. [3, 3.4.1])**.** *Let $\mathcal{C}$ be an oriented matroid on a finite set E and $I \subseteq E$. Then*

$$(M \setminus I)^* = M^* \,/\, I,$$
$$(M \,/\, I)^* = M^* \setminus I.$$

Definition 2.4. *Let $e, f \in E$. Let $X = \{e^+\}, Y = \{e^+, f^-\}$ and $Z = \{e^+, f^+\}$. Then e is called a* loop of $\mathcal{C}$ *if $X \in \mathcal{C}$. We say that e and f are* (anti)-parallel *if $Y \in \mathcal{C}\,(Z \in \mathcal{C})$.*

The following fact is immediate from the definition and well known from matroid theory: the duals of the loops are the bonds and the dual of a set of parallels is a series.

Proposition 2.5. *Let $e, f \in E$, then*

1. *e is a loop of $\mathcal{C}$ if and only if $\forall\, X \in \mathcal{C}^\perp : X_e = 0$,*
2. *e and f are (anti-)parallel in $\mathcal{C}$ if and only if $\forall\, X \in \mathcal{C}^\perp : X_e = X_f\ (X_e = -X_f)$.*

With these preparations we have objects in the right form to define our morphisms.

3. Strong maps

In matroid theory there are two reasonable notions of morphisms: "strong maps" (an abstraction of linear maps) and "weak maps" (which can move points from general to more special position). Linear algebra homomorphisms are related to strong maps and thus for our purposes weak maps do not play a role. Below we define strong maps for oriented matroids (Definition 3.2). We believe that this definition is most natural, as our strong maps are those maps where "the inverse image of a convex set is convex". The possibility to define strong maps in this way seems to have been overlooked so far. It is also the reason why we had to choose the slightly non-standard notation for oriented matroids due to Folkman and Lawrence. This notation enables us to define *convexity* not only for a special class of oriented matroids, namely the acyclic simple matroids, (see [3, Exercises 3.8–3.12]) and exploit the full structure that convexity may give.

Moreover, we show below (Proposition 3.1) that our definition coincides with the definition given in [3, 7.7.2.] for oriented matroids with the same ground set. In view of Lemmata 3.1 and 3.2 our definition is compatible with Exercise 7.27 in [3].

Definition 3.1. *Let $\mathcal{C}$ be the family of circuits of an oriented matroid $\mathcal{C}$ on the finite ground set E. Let $A \subseteq \pm E$. The* convex closure $\operatorname{conv}(A)$ of *A is defined as*

$$\operatorname{conv}(A) := A \cup \{f \in \pm E \mid \exists\, C \in \mathcal{C} : -f \in C \subseteq A \cup -f\}.$$

A set $A \subseteq \pm E$ is convex *if* $\operatorname{conv}(A) = A$.

Hence, by definition any loop is in the convex hull of the empty set.

Definition 3.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be oriented matroids on the ground sets E and F. A* strong map *σ from $\mathcal{C}$ to $\mathcal{D}$ is a function $\sigma : E \cup \{o\} \to F \cup \{o\}$ — where o is a loop on an additional element — mapping o to o and satisfying the following condition:*

The extension $\sigma^* : \pm E \cup \{o\} \to \pm F \cup \{o\}$ of σ mapping $e^+ \mapsto \sigma(e)^+$ and $e^- \mapsto \sigma(e)^-$ if $\sigma(e) \neq o$, and $\sigma^*(e) = o$ otherwise, has the property that the inverse image of any convex set of $\mathcal{D}$ is convex in $\mathcal{C}$.

Note, that a strong map between two oriented matroids is a strong map between the underlying matroids. However, the strong maps of oriented matroids are not as "nice" as in the unoriented case since they in general do not factor into an extension followed by a contraction (see [3, 7.7.4]). But we will see that, basically, it suffices to consider strong maps on fixed ground sets. First we check that for fixed ground sets we get the standard definition of strong maps.

Proposition 3.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two oriented matroids on the same ground set E. The identity map is a strong map from $\mathcal{C}$ to $\mathcal{D}$ iff any circuit of $\mathcal{C}$ is a vector of $\mathcal{D}$. In this case we say that the identity is a* quotient map *from $\mathcal{C}$ to $\mathcal{D}$.*

Proof: Let $A \subseteq \pm E$ be convex in $\mathcal{D}$. Assume $e \in \pm E \setminus A$ is in the convex closure of A in $\mathcal{C}$. Hence there exists a circuit $-e \in C \subseteq A \cup \{-e\}$ in $\mathcal{C}$. By assumption C is a vector of $\mathcal{D}$ contradicting the fact that $e \notin A$ but A is convex closed in $\mathcal{D}$. To prove the other implication let C be a circuit in $\mathcal{C}$ and $e \in C$. We have $\operatorname{conv}_{\mathcal{C}}(C \setminus e) = C \setminus \{e\} \cup \{-e\} \Rightarrow \{-e\} \in \operatorname{conv}_{\mathcal{D}}(C \setminus e)$. Hence C contains a circuit in $\mathcal{D}$ containing e. Since e was arbitrary C is a union of circuits in $\mathcal{D}$ and thus a vector. □

The following observation will be useful later on.

Corollary 3.1. *The identity map is a strong map from $\mathcal{C}$ to $\mathcal{D}$ if and only if it is a strong map from $\mathcal{D}^*$ to $\mathcal{C}^*$. We say that the identity map is a* dual strong map.

PROOF: Let X be a circuit in $\mathcal{D}^*$. Thus $\forall\, D \in \mathcal{D} : X \perp D \Rightarrow \forall\, C \in \mathcal{C} : X \perp C \Rightarrow X$ is a vector of $\mathcal{C}^*$. □

Here are some standard examples where the ground sets differ.

Example 3.1. *Let $\mathcal{C}$ be an oriented matroid on the ground set E and $U \subset E$. Let $\sigma : E \cup \{o\} \to E \setminus U \cup \{o\}$ be the identy map on $(E \setminus U)$ and constant o on U. Then σ is a strong map from $\mathcal{C}$ to $\mathcal{D} := \mathcal{C}/U$. We call these maps* contractions.

Example 3.2. *Let $\mathcal{C}, \mathcal{D}$ be oriented matroids on sets E and $F = E \dot{\cup} U$ such that $\mathcal{C} = \mathcal{D} \setminus U$ for some set U. Then the embedding map $\sigma : E \cup \{o\} \to F \cup \{o\}$ is a strong map which we will call an* embedding.

Example 3.3. *Let $\mathcal{C}$ be an oriented matroid on the ground set E, $e \in E$ be a loop of $\mathcal{C}$ and $f, g \in E$ be parallel in $\mathcal{C}$.*

Consider the following two modifications σ and σ' of the identity map $E \cup \{o\} \to E \cup \{o\}$,

$\sigma : E \cup \{o\} \to (E \setminus \{e\}) \cup \{o\}$ defined by $\sigma(e) = o$ and $\sigma(h) = h$ otherwise and

$\sigma' : E \cup \{o\} \to (E \setminus \{g\}) \cup \{o\}$ defined by $\sigma'(g) = f$ and $\sigma'(h) = h$ otherwise. Both σ and σ' are strong maps. A simplification *is any composition of such maps.*

Note, that a simplification only relates parallel elements f, g and does not relate antiparallel elements. Furthermore observe, that contractions are dual strong maps as were quotient maps (Corollary 3.1). Unfortunately, if σ is a strong map from $\mathcal{C} \to \mathcal{D}$ that simplifies a parallel we do not necessarily have a non-trivial strong map from $\mathcal{D}^* \to \mathcal{C}^*$ (see Remark 4.2 below).

Fixing this will lead to a definition where replacing a positive element by a positive series is a morphism. This is reasonable because the corresponding operation in a vector space (doubling a coordinate) in fact is even an isomorphism. We will have to pay for this with the fact that our morphisms will no longer be maps of the ground sets like strong maps. This will be done in Section 5 — the corresponding morphisms are denoted by $\rightsquigarrow$ and called generalized strong maps.

Lemma 3.1. *Let $\sigma : E \cup \{o\} \to F \cup \{o\}$ be a strong map from $\mathcal{C}$ to $\mathcal{D}$. Then σ can be factored into a surjective strong map followed by an embedding.*

PROOF: Let $F' \cup \{o\}$ be the image of σ in $F \cup \{o\}$. Let $\mathcal{D}' := \mathcal{D} \setminus (F \setminus F')$. Then σ induces a strong map from $\mathcal{C}$ to $\mathcal{D}'$ and the claim follows if we compose this with an embedding. □

Lemma 3.2. *Let $\sigma : E \cup \{o\} \to F \cup \{o\}$ be a surjective strong map from $\mathcal{C}$ to $\mathcal{D}$. Then σ can be factored into a bijective strong map — hence a quotient map followed by a relabeling of the elements — and a simplification.*

PROOF: Introduce a new loop for each element which is mapped to o and parallels for the elements of F which have more than one element in its inverse image. □

4. Affine strong maps

Strong maps per se cannot be used as homomorphisms for duality theorems (see Introduction) as we have strong maps between any pair of oriented matroids. This is clear for vector spaces and in general by the following:

Example 4.1. *7 Let $\mathcal{C}$ and $\mathcal{D}$ be two oriented matroids with ground sets E, F such that $E \cap F = \emptyset$. Then $\sigma : E \cup \{o\} \to F \cup \{o\}$ mapping all elements to o is a strong map.*

But it is not surprising that we cannot model Farkas' Lemma as homomorphism duality in combinatorial models for linear spaces. Farkas' Lemma needs a distinguished coordinate — or hyperplane — which may not disappear under the homomorphisms. Considering this hyperplane as the hyperplane at infinity we may view Farkas' Lemma as a theorem for affine spaces. Affine spaces are modelled by affine oriented matroids (cf. [14] and [3, 4.5]).

Definition 4.1. *Let $\mathcal{C}$ be an oriented matroid on a finite set E and $g \in E$ not a loop. Then we call the tuple $(\mathcal{C}, g)$ an* affine oriented matroid*. Let $\mathcal{D}$ be an oriented matroid on F and $h \in F$. An* affine strong map *from $(\mathcal{C}, g)$ to $(\mathcal{D}, h)$ is a strong map $\sigma : E \cup \{o\} \to F \cup \{o\}$ from $\mathcal{C}$ to $\mathcal{D}$ mapping g to h.*

The following observation encodes one key step in the proof that Farkas' Lemma is a homomorphism duality for strong maps.

Lemma 4.1. *Let $\sigma : E \cup \{o\} \to F \cup \{o\}$ be an affine strong map from $(\mathcal{C}, g) \to (\mathcal{D}, h)$. Then*

$$(\exists\, X \in \mathcal{D}^* : h^+ \in X \subseteq F^+) \Rightarrow (\exists\, Y \in \mathcal{C}^* : g^+ \in Y \subseteq E^+).$$

Proof: By the results of the last section it suffices to prove the assertion for quotient maps, embeddings and simplifications.

Quotient maps are clear by Corollary 3.1 and Proposition 3.1. So assume σ is an embedding. Thus $\mathcal{C} = \mathcal{D} \setminus I$ and hence $\mathcal{C}^* = \mathcal{D}^*/I$ and any positive cocircuit of $\mathcal{D}$ corresponds to a positive covector of $\mathcal{C}$. Finally, a deletion of a loop does not change the covectors at all and an elimination of a parallel only shortcuts a positive (or negative) series in the dual. $\square$

Now let $(\mathcal{L}, 1), (\mathcal{L}^*, 1)$ denote the affine oriented matroids defined on the set $\{1\}$ consisting of a loop resp. a coloop. We would like to have a homomorphism duality theorem with $\mathcal{L}$ and $\mathcal{L}^*$. This does not work with affine strong maps since in general we cannot invert simplifications, to be more precise:

Example 4.2. *Let* $(\mathcal{F}, 1) = (\{(+,+),(-,-)\}, 1)$ *denote the oriented matroid on* $\{1,2\}$ *consisting of a positive circuit of length* 2 *then* $(\mathcal{F}^*, 1) \to (\mathcal{L}^*, 1)$ *but* $(\mathcal{L}, 1) \not\to (\mathcal{F}, 1)$.

PROOF: $\mathcal{F}^*$ consists of two parallels which simplify to $\mathcal{L}^*$. The other assertion follows from the fact that $\{1^+, 2^-\}$ is convex in $\mathcal{F}$ but loops are in the convex hull of the empty set and therefore the inverse image of $\{1^+, 2^-\}$ can never be convex in $\mathcal{L}$. □

As we will see in the next section the only homomorphism duality for oriented matroids and affine strong maps with simple objects is the trivial equation

$$(\mathcal{L}, 1) \not\to \;=\; \to (\mathcal{F}, 1) \qquad \text{(see Theorem 5.2).}$$

To model Farkas' Lemma in terms of homomorphism duality with affine strong maps we say that an oriented matroid $\mathcal{C}$ on $\{1, \dots, n\}$ — for some positive natural number n — is from the *class* Γ if it consists of an all positive vector and an all negative vector.

We take a time out for a lemma.

Lemma 4.2. *Let* $(\mathcal{C}, g)$ *be an affine oriented matroid. Then*

$$(\mathcal{C}, g) \to (\mathcal{L}^*, 1) \Leftrightarrow \exists \mathcal{N} \in \Gamma : (\mathcal{N}, 1) \to (\mathcal{C}^*, g).$$

PROOF: "$\Rightarrow$" By Lemma 4.1 $\mathcal{C}$ has a positive cocircuit containing g.

"$\Leftarrow$" By Lemmata 3.1 and 3.2 any strong map from $(\mathcal{N}, 1)$ to $(\mathcal{C}^*, g)$ factors into a quotient map followed by a simplification and an embedding. By Propositions 3.1 and 2.1 we may assume that the quotient map is the identity. Since $\mathcal{N}$ has no loops and parallels we, furthermore, may assume that the strong map is an embedding. Hence $\mathcal{C}$ has a positive cocircuit containing g. Contracting the complement of its support leaves a bunch of parallels which simplify to $(\mathcal{L}^*, 1)$. □

Now Farkas' Lemma gives the homomorphism duality theorem:

Theorem 4.1.

$$\forall (\mathcal{C}, g) : (\exists \mathcal{N} \in \Gamma\, ((\mathcal{N}, 1) \to (\mathcal{C}, g))) \Leftrightarrow ((\mathcal{C}, g) \not\to (\mathcal{L}^*, 1))$$

or for short

$$(\Gamma, 1) \to \;=\; \not\to (\mathcal{L}^*, 1).$$

PROOF: Let $(\mathcal{C}, g)$ be an oriented matroid and assume there is a non-trivial non-negative cocircuit containing g in $\mathcal{C}$. As above we conclude that $(\mathcal{C}, g) \to (\mathcal{L}^*, 1)$.

On the other hand by Lemma 4.1 there is a non-negative covector which is positive on 1 in $\mathcal{L}^*$ only if there is such a covector in $\mathcal{C}$. Hence we have

$$\begin{aligned}
(\mathcal{C}, g) \to (\mathcal{L}^*, 1) \quad & \Leftrightarrow \quad \exists\, X \in \mathcal{C}^* : g^+ \in X \subseteq E^+ \\
& \overset{\text{Farkas}}{\Leftrightarrow} \quad \nexists Y \in \mathcal{C} : g^+ \in Y \subseteq E^+ \\
& \Leftrightarrow \quad (\mathcal{C}^*, g) \not\to (\mathcal{L}^*, 1) \\
& \overset{\text{Lemma 4.2}}{\Leftrightarrow} \quad (\Gamma, 1) \not\to (\mathcal{C}, g).
\end{aligned}$$

□

5. Unicity of Farkas' Lemma

In this section we will show that strong maps give only a trivial duality when considering simple objects (not a class of objects, such as the class $(\Gamma, 1)$ in Theorem 4.1).

However we shall find a proper generalization of affine strong maps — generalized strong maps — and we show not only that Farkas' Lemma fits into this framework (with simple objects) but also that it is unique.

First we examine the morphic structures of strong maps in more detail.

Definition 5.1. *We say that two affine oriented matroids $(\mathcal{C}, g), (\mathcal{D}, h)$ are* morphic *if $(\mathcal{C}, g) \to (\mathcal{D}, h)$ and $(\mathcal{D}, h) \to (\mathcal{C}, g)$. This will be denoted by $(\mathcal{C}, g) \leftrightarrow (\mathcal{D}, h)$.*

Proposition 5.1. *Let $(\mathcal{C}, g)$ be an affine oriented matroid.*

1. *There is a strong affine map $(\mathcal{L}^*, 1) \to (\mathcal{C}, g)$.*
2. *There is a strong affine map $(\mathcal{C}, g) \to (\mathcal{L}, 1)$.*
3. *The second alternative of Farkas' Lemma holds if and only if $(\mathcal{C}, g) \leftrightarrow (\mathcal{L}^*, 1)$.*

Proof: Note, that any subset of $\{o, 1^+, 1^-\}$ is convex in $\mathcal{L}^*$ and $\{o, 1^+, 1^-\}$ is the only convex set in $\mathcal{L}$.

The last claim is clear from the above and Theorem 4.1. □

Definition 5.2 (see [3, 7.6.1]). *Let $\mathcal{C}$ and $\mathcal{D}$ be two oriented matroids with ground sets E, F such that $E \cap F = \emptyset$. The* direct sum *of $\mathcal{C}$ and $\mathcal{D}$ is given as follows.*

$$\mathcal{C} \dot\cup \mathcal{D} \;:=\; \{X \in 2^{\pm E \dot\cup F} \mid X \subseteq \mathcal{C} \text{ or } X \subseteq \mathcal{D}\}.$$

An oriented matroids is connected *if it is not the direct sum of two non-trivial oriented matroids.*

In the next lemma we will see that only the connected component containing the special element is of interest for the existence of morphisms.

Lemma 5.1. *Let $(\mathcal{C}, g)$ and $(\mathcal{D}, h)$ be two affine oriented matroids, E, F their ground sets and $\tilde{\mathcal{C}}$ an oriented matroid on $\tilde{E}$ such that $E \cap \tilde{E} = \emptyset$. Then*

1. *there is a morphism from $(\mathcal{C}\dot{\cup}\tilde{\mathcal{C}}, g)$ to $(\mathcal{D}, h)$ if and only if there is a morphism from $(\mathcal{C}, g)$ to $(\mathcal{D}, h)$;*
2. *there is a morphism from $\mathcal{D}$ to $(\mathcal{C}\dot{\cup}\tilde{\mathcal{C}}, g)$ if and only if there is a morphism from $(\mathcal{D}, h)$ to $(\mathcal{C}, g)$.*

Proof: Observe that $(\mathcal{C}\dot{\cup}\tilde{\mathcal{C}})/\tilde{E} = (\mathcal{C}\dot{\cup}\tilde{\mathcal{C}}) \setminus \tilde{E}$. □

If we fix the special element in affine oriented matroids, say to 1, we get the following picture of the morphic structure of affine oriented matroids defined by existence of affine strong maps.

- The affine oriented matroids $(\mathcal{C}, 1)$ which have a positive cocircuit containing 1 are exactly those which are morphic to the coloop.
- If $(\mathcal{C}, 1)$ has a positive circuit containing 1 let n denote the cardinality of the support of the smallest of these circuits. Then $(\mathcal{C}, 1)$ has a strong map from all elements of Γ containing at least n elements. Furthermore, there is a largest element of Γ such that there is a strong affine map from $(\mathcal{C}, 1)$ to this element.
- The only affine oriented matroids morphic to $(\mathcal{L}, 1)$ are those where 1 is itself a loop.

Theorem 5.1. *Let $(\Xi, 1) \to \quad = \quad \not\to (\Delta, 1)$ be a homomorphism duality theorem, where Ξ and Δ are classes of oriented matroids and $\to$ are affine strong maps. Then each element of Ξ has a positive circuit containing* 1.

Proof: Assume a representative $\mathcal{C} \in \Xi$ had a positive cocircuit containing 1. Then $(\mathcal{C}, 1) \leftrightarrow (\mathcal{L}^*, 1)$ and there is no candidate left for Δ. □

In this direction we will not get any further. There even is a homomorphism duality theorem where on both sides we just have an object and not a class:

Theorem 5.2.

$$(\mathcal{L}, 1) \to \quad = \quad \not\to (\mathcal{F}, 1).$$

Proof: $(\mathcal{L}, 1) \to (\mathcal{C}, 1)$ iff 1 is a loop in $\mathcal{C}$. Otherwise we consider two cases. If $(\mathcal{C}, 1)$ has a positive cocircuit containing 1 it is morphic to $(\mathcal{L}^*, 1)$ which we can embed into $(\mathcal{F}, 1)$. Otherwise it has a positive circuit containing 1 and we prove by induction on the number of elements n that it embeds into $(\mathcal{F}, 1)$. The case $n = 2$ is clear. So assume that $n > 2$. If there is a parallel element in $\mathcal{C}$ we can simplify and apply inductive assumption. Otherwise fix 1 and a second element of the positive circuit containing 1 and contract an arbitrary third element. The resulting oriented matroid satisfies the inductive assumption. □

The profane truth of Theorem 5.2 is that 1 either is a loop or it is not. But this trivial duality is the only homomorphism duality theorem using strong maps with simple objects. (So we have a situation analogous to undirected graphs and their homomorphisms, cf. Theorem A):

Theorem 5.3. *Let $(B,1) \rightarrow \quad = \quad \not\rightarrow (A,1)$ be a homomorphism duality theorem for oriented matroids and affine strong maps. Then $(B,1) \leftrightarrow (\mathcal{L},1)$ and $(A,1) \leftrightarrow (\mathcal{F},1)$.*

Proof: Let $(B,1),(A,1)$ be a homomorphism duality pair. By Theorem 5.1 B has a positive circuit containing 1.

For $k \geq n \geq 3$ let S_k^n denote the oriented matroid induced by the following point configuration. Start with an $n-2$-dimensional simplex, e.g. spanned by the unit vectors and the zero vector in $\mathbb{R}^{n-2}$. Add $k-n+1$ points in the convex hull of these points, but in affinely general position. Embed these into the $\mathbb{R}^{n-1}$ by adding a 1 in the last coordinate. Finally, replace the points in the interior of the configuration by their negative inverse. Note, that the underlying matroid is given by linear dependency and isomorphic to $U_{n-1,k}$ the uniform matroid on k points of rank $n-1$.

Let 1 be one of the corner points of the simplex. Now, the corner points together with any — formerly — interior point form a positive circuit of length n. Note, that S_k^n does not contain any smaller circuit.

By Lemma 5.3 (see below) for any affine strong map any positive circuit containing 1 has to be mapped to a positive vector containing 1. Since B is finite, there exists some n such that $(B,1) \not\rightarrow (S_k^n,1)$ for all $k \geq n$. So, by assumption, for all k there is an affine strong map $\sigma : (S_k^n,1) \rightarrow (A,1)$.

Let m denote the number of elements in A and choose $k = m(n-1)$. Let x be an element of A such that $|\sigma^{-1}(x)| \geq n-1$ and consider the strong map induced on the underlying matroids. Since $\sigma^{-1}(x)$ must contain a basis of the corresponding matroid, we conclude that x has to span $\sigma(S_k^n)$ and therefore $\sigma(S_k^n)$ besides x may contain only loops and elements which are (anti-)parallel to x. Since any positive circuit containing 1 in S_k^n has to be mapped to a positive circuit containing 1, either 1 must be a loop in A or A has to contain an element antiparallel to 1. By Proposition 5.1 any oriented matroid has an affine strong map into the loop. Since $(B,1) \not\rightarrow (A,1)$, A has to contain a positive circuit of length two containing 1 and therefore is morphic to $(\mathcal{F},1)$. By Theorem 5.2 we get $(B,1) \rightarrow\subseteq (\mathcal{L},1) \rightarrow$. Since all members of $(\mathcal{L},1)$ are morphic to the loop the claim follows. □

Recall (see Introduction), that our ultimate goal was to express LP-duality (Farkas' Lemma) as the simple equation

$$A \not\rightarrow \quad = \quad \rightarrow B$$

defined by simple objects. By defining affine strong maps we lost a particular feature of linear maps: the existence of dual maps (see Example 4.2). This problem already arises in the case of (ordinary) strong maps in oriented matroids: Deleting a parallel element from a rank one matroid yields a strong map by Example 3.3. But, dually, there is no non-trivial strong map from a loop to a two circuit. This, because the two circuit has nontrivial convex sets consisting of one signed copy of each element.

The following definition of generalized strong maps will allow us to complete our project.

Definition 5.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be oriented matroids on the ground sets E and F. We say that there is a* generalized strong map *from $\mathcal{C}$ to $\mathcal{D}$ if there is a sequence $\mathcal{C} = \mathcal{C}_0, \dots, \mathcal{C}_k = \mathcal{D}$ of oriented matroids such that*

- *there is a strong map from $(\mathcal{C}_i, 1)$ to $(\mathcal{C}_{i+1}, 1)$ if i is even and $0 \leq i \leq k-1$ and*
- *there is a strong map from $(\mathcal{C}^*_{i+1}, 1)$ to $(\mathcal{C}^*_i, 1)$ if i is odd and $0 \leq i \leq k-1$.*

We denote the existence of a generalized strong map from $\mathcal{C}$ to $\mathcal{D}$ by $\mathcal{C} \rightsquigarrow \mathcal{D}$.

Observe that a composition of generalized strong maps is a generalized strong map and that the following two lemmata hold:

Lemma 5.2. $(\mathcal{C}, 1) \rightsquigarrow (\mathcal{D}, 1) \Leftrightarrow (\mathcal{D}^*, 1) \rightsquigarrow (\mathcal{C}^*, 1)$. □

Lemma 5.3. *Let $\sigma : E \cup \{o\} \to F \cup \{o\}$ be an affine strong map from $(\mathcal{C}, g) \to (\mathcal{D}, h)$. Then*

$$(\exists\, X \in \mathcal{C} : g^+ \in X \subseteq E^+) \Rightarrow (\exists\, Y \in \mathcal{D} : h^+ \in Y \subseteq F^+).$$

Proof: The claim is trivial for simplifications and embeddings and clear for quotient maps by Proposition 2.1. □

Theorem 5.4.

$$(\mathcal{L}, 1) \rightsquigarrow \quad = \quad \not\rightsquigarrow (\mathcal{L}^*, 1).$$

Proof: We proceed similarly to the above proof of Theorem 4.1. First we consider the following analogue of Lemma 4.1:

Let $\sigma : E \cup \{o\} \to F \cup \{o\}$ be a generalized affine strong map from $(\mathcal{C}, g) \rightsquigarrow (\mathcal{D}, h)$. Then

$$(\exists\, X \in \mathcal{D}^* : h^+ \in X \subseteq F^+) \Rightarrow (\exists\, Y \in \mathcal{C}^* : g^+ \in Y \subseteq E^+).$$

However this is a consequence of Lemmata 4.1 and 5.3. Now we have analogues as in Theorem 4.1

$$\begin{aligned}
(\mathcal{C}, g) \rightsquigarrow (\mathcal{L}^*, 1) \quad &\Leftrightarrow \quad \exists\, X \in \mathcal{C}^* : g^+ \in X \subseteq E^+ \\
&\overset{\text{Farkas}}{\Leftrightarrow} \quad \nexists\, Y \in \mathcal{C} : g^+ \in Y \subseteq E^+ \\
&\Leftrightarrow \quad (\mathcal{C}^*, g) \not\rightsquigarrow (\mathcal{L}^*, 1) \\
&\overset{\text{Lemma 5.2}}{\Leftrightarrow} \quad (\mathcal{L}, 1) \not\rightsquigarrow (\mathcal{C}, g).
\end{aligned}$$

□

Finally, Proposition 5.1, Theorem 5.1 and Lemma 5.2 now imply:

Theorem 5.5. *Let* $(B,1) \rightsquigarrow \quad = \quad \not\rightarrow (A,1)$ *be a homomorphism duality theorem for oriented matroids and generalized affine strong maps. Then* $(B,1)$ *is morphic to the loop and* $(A,1)$ *is morphic to the coloop.*

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