On a generalization of QI-rings

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Abstract. In this paper rings for which every s-torsion quasi-injective module is weakly s-divisible for a hereditary preradical s are characterized in terms of the properties of the corresponding lattice of the (hereditary) preradicals. In case of a stable torsion theory these rings coincide with TQI-rings investigated by J. Ahsan and E. Enochs in [1]. Our aim was to generalize some results concerning QI-rings obtained by J.S. Golan and S.R. López-Permouth in [12]. A characterization of the QI-property in the category $\sigma[M]$ then follows as a consequence.

Keywords: s-QI-rings, s-stable preradicals, weakly s-divisible modules, s-tight modules

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In what follows, R stands for an associative ring with a unit element and R-Mod denotes the category of all unitary left R-modules.

First of all we recall some basic definitions from the theory of preradicals (for details see L. Bican, T. Kepka, P. Němec [5] and J.S. Golan [11]).

A preradical r for R-Mod is any subfunctor of the identity functor, i.e. r assigns to each module M its submodule $r(M)$ in such a way that every homomorphism of M into N induces a homomorphism of $r(M)$ into $r(N)$ by restriction. A preradical r is said to be

-idempotent if $r(r(M)) = r(M)$ for every module M,

-a radical if $r(M/r(M)) = 0$ for every module M,

-hereditary if $r(N) = N \cap r(M)$ for every submodule N of a module M,

-stable if $r(M)$ is a direct summand in M for every injective module M.

A module M is r-torsion if $r(M) = M$ and r-torsionfree if $r(M) = 0$. We shall denote by $\mathcal{T}_r$ ($\mathcal{F}_r$) the class of all r-torsion (r-torsionfree) modules. If r and s are preradicals then $r \circ s$ is the preradical defined by $(r \circ s)(M) = r(s(M))$, $M \in R$-Mod and $r \leq s$ means $r(M) \subseteq s(M)$ for every $M \in R$-Mod. The idempotent core $\overline{r}$ of a preradical r is defined by $\overline{r}(M) = \sum K$, where K runs through all r-torsion submodules K of M. The injective hull of a module M will be denoted by $E(M)$. The hereditary closure $h(r)$ of a preradical r is defined by $h(r)(M) = M \cap r(E(M))$ for every module M. If $\mathcal{A}$ is a non-empty class of modules then the idempotent preradical $p_{\mathcal{A}}$ is defined by $p_{\mathcal{A}}(M) = \sum \operatorname{Im} f$, $f \in \operatorname{Hom}_R(A, M)$, $A \in \mathcal{A}$ for every $M \in R$-Mod. The identity functor will be

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denoted by id. Finally, $\sigma[M]$ denotes the category of all modules subgenerated by a module M.

If M is a module then a module Q is called M-injective if every homomorphism from a submodule of M into Q can be extended to a homomorphism from M into Q. If s is a preradical then a module Q is called weakly s-divisible if it is M-injective for every $M \in \mathcal{T}_s$. Since $\mathcal{T}_s = \mathcal{T}_{\overline{s}}$ we will assume without loss of generality throughout the whole text that s is an idempotent preradical. It is easy to see that a module Q is weakly s-divisible if and only if $s(E(Q)) \subseteq Q$ (see [5]) and that a module Q is M-injective if and only if it is weakly $p_{\{M\}}$-divisible (see [4]). Further, $E_s(Q) = Q + s(E(Q))$ is the weakly s-divisible hull of a module Q.

If M is a module then a module Q is called M-tight if every homomorphic image of M which is embeddable in $E(Q)$ is also embeddable in Q (see [2]). Now we can define a tight module with respect to a preradical.

Definition 1. *A module Q is said to be s-tight if it is M-tight for every $M \in \mathcal{T}_s$.*

As it is easy to see a module Q is s-tight if and only if $s(E(Q))$ can be embedded in Q and consequently every weakly s-divisible module is s-tight. The converse is true in case of quasi-injective modules.

Lemma 1. *Let Q be a quasi-injective module. Then Q is s-tight if and only if it is weakly s-divisible.*

Proof: If Q is s-tight then there is a monomorphism $f\colon s(E(Q)) \to Q$. Let us consider the following diagram:

$$\begin{array}{ccc} s(E(Q)) & \xrightarrow{\ f\ } & Q \\ {\scriptstyle i}\big\downarrow & & \\ E(Q) & & \end{array},$$

where i is the inclusion map. Then there is a homomorphism $g\colon Q \to E(Q)$ such that $fg = i$. Now $Qg \subseteq Q$ since Q is quasi-injective and consequently $s(E(Q)) = \operatorname{Im} i \subseteq Qg \subseteq Q$. Hence Q is weakly s-divisible. □

The following notion of generalized stability of preradicals plays an important role in our characterization of generalized QI-rings.

Definition 2. *A preradical r is said to be s-stable if $r(M)$ is a direct summand in M for every weakly s-divisible s-torsion module M.*

Proposition 1. *Let r be a preradical. Then*

(i) *if r is s-stable then $\mathcal{T}_{r\circ s} = \mathcal{T}_r \cap \mathcal{T}_s$ is closed under weakly s-divisible hulls;*
(ii) *if $r \circ s$ is idempotent (e.g. if r is idempotent and s is hereditary) and $\mathcal{T}_{r\circ s}$ is closed under weakly s-divisible hulls then r is s-stable;*
(iii) *if $r \circ s$ is stable then r is s-stable. The converse is true if s is stable;*
(iv) *if r is hereditary and s-stable then $r(M/r(M)) = 0$ for every s-torsion module M.*

Proof: (i). If $T \in \mathcal{T}_{r\circ s}$ then $E_s(T) = s(E(T))$ and $r(s(E(T))) \oplus A = s(E(T))$ for some module A since r is s-stable. Now $T \subseteq r(s(E(T)))$ and $T \cap A = 0$ yields $A = 0$. Thus $E_s(T) = s(E(T)) \in \mathcal{T}_r$.

(ii). Let Q be a s-torsion weakly s-divisible module. Then $r(Q) \in \mathcal{T}_{r\circ s}$ and consequently $E_s(r(Q)) = s(E(r(Q))) \in \mathcal{T}_{r\circ s}$. Now $E_s(r(Q))$ is a direct summand in Q since Q is s-torsion and weakly s-divisible. But $E_s(r(Q)) \subseteq (r \circ s)(Q) = r(Q) \subseteq E_s(r(Q))$ which yields $E_s(r(Q)) = r(Q)$ and consequently r is s-stable.

(iii). If $r \circ s$ is stable and Q is a weakly s-divisible s-torsion module then $Q = s(E(Q))$. Now $r(Q) = (r \circ s)(E(Q))$ is a direct summand in $E(Q)$ and therefore $r(Q)$ is also a direct summand in Q.
On the other hand, if s is stable, r is s-stable and Q is injective then $s(Q)$ is weakly s-divisible s-torsion. Hence $(r \circ s)(Q)$ is a direct summand in $s(Q)$ and consequently also in Q since s is stable.

(iv). If M is s-torsion then we can consider the following short exact sequence:

$$0 \to (r \circ s)(E(M))/r(M) \to s(E(M))/r(M) \to s(E(M))/(r \circ s)(E(M)) \to 0.$$

Now $s(E(M))$ is weakly s-divisible s-torsion and therefore $(r \circ s)(E(M))$ is a direct summand in $s(E(M))$. From it follows that $s(E(M))/(r \circ s)(E(M))$ is r-torsionfree. Hence $r(s(E(M))/r(M)) \subseteq (r \circ s)(E(M))/r(M)$ and consequently $r(M/r(M)) = M/r(M) \cap r(s(E(M))/r(M)) \subseteq (M \cap (r \circ s)(E(M)))/r(M) = r(M)/r(M) = 0$, r being hereditary. □

Corollary 1. *Let M be a module. An idempotent preradical r is $h(p_{\{M\}})$-stable if and only if $\mathcal{T}_r \cap \sigma[M]$ is closed under M-injective hulls.*

Proposition 2. *Let r and s be hereditary preradicals. Then the following conditions are equivalent:*

- (i) *every $M \in \mathcal{T}_s \backslash \mathcal{T}_r$ contains a nonzero r-torsionfree submodule;*
- (ii) *$\mathcal{T}_r \cap \mathcal{T}_s$ is closed under weakly s-divisible hulls;*
- (iii) *if $A \subseteq B \subseteq C$ such that $C/A \in \mathcal{T}_s$ and $B/A \in \mathcal{T}_r$ then there is $D \subseteq C$ with $D \cap B = A$ and $C/D \in \mathcal{T}_r$;*
- (iv) *if $I \subseteq K$ are left ideals with $K/I = r(R/I)$, where $R/I \in \mathcal{T}_s$ then there is a left ideal L with $L \cap K = I$ and $R/L \in \mathcal{T}_r$;*
- (v) *if $I \subseteq K \neq R$ are left ideals with $K/I = r(R/I)$ and $R/I \in \mathcal{T}_s$ then there is a left ideal $L \neq I$ with $L \cap K = I$;*
- (vi) *r is s-stable;*
- (vii) *$\neg(\exists M)(M \in \mathcal{T}_s \;\&\; r(M) \subsetneqq {}' M)$;*
- (viii) *weakly s-divisible hulls of cyclic s-torsion modules split in r.*

Proof: (i) implies (ii). Let $N \in \mathcal{T}_r \cap \mathcal{T}_s$. If $E_s(N) = s(E(N)) \notin \mathcal{T}_r$ then there is $0 \neq K \subseteq s(E(N))$ with $r(K) = 0$. Hence $K \cap N \in \mathcal{T}_r \cap \mathcal{F}_r = 0$ and consequently $K = 0$ since $N \subseteq' s(E(N))$, a contradiction.

(ii) implies (iii). By Zorn's lemma there is a submodule D of C maximal with respect to the property $D \cap B = A$. Then $(B + D)/D \cong B/(B \cap D) = B/A \in \mathcal{T}_r \cap$

$\mathcal{T}_s$. From the maximality of D follows that $(B+D)/D \subseteq{}' C/D$. Now $C/D \in \mathcal{T}_s$ and therefore $C/D \subseteq s(E((B+D)/D))$. Further, $s(E((B+D)/D)) \in \mathcal{T}_r \cap \mathcal{T}_s$ by assumption and consequently $C/D \in \mathcal{T}_r$, r being hereditary.

(iii) implies (iv). Obvious.

(iv) implies (v). Obviously, if $L = I$ then $K = R$, a contradiction.

(vi) is equivalent to (ii). It follows immediately from Proposition 1(i) and (ii).

(v) implies (ii). Let $T \in \mathcal{T}_r \cap \mathcal{T}_s$. If $E_s(T) = s(E(T)) \notin \mathcal{T}_r$ then there is $x \in s(E(T)) \backslash r(s(E(T)))$. Put $I = (0 : x)_l$ and $K = \{t \in R; tx \in r(s(E(T)))\}$. Then $K \neq R$, $K/I = r(R/I)$ and $R/I \in \mathcal{T}_s$. Now by (v) there is a left ideal L of R such that $L \neq I$ and $L \cap K = I$. Let $a \in L \backslash I$. Then $0 \neq ax \in s(E(T))$ and consequently there is $b \in R$ such that $0 \neq bax \in r(s(E(T)))$ by the essentiality of $r(s(E(T)))$ in $s(E(T))$. Hence $ba \in K \cap L = I$, a contradiction.

(iii) implies (vii). Let us suppose on the contrary that there is a module $M \in \mathcal{T}_s$ such that $r(M) \subsetneqq{}' M$. Then by (iii) there is a module $D \subseteq M$ with $D \cap r(M) = 0$ and $M/D \in \mathcal{T}_r$, a contradiction.

(vii) implies (i). Let $M \in \mathcal{T}_s \backslash \mathcal{T}_r$. Then $\neg(r(M) \subsetneqq{}' M)$ by assumption. Thus there is $0 \neq N \subseteq M$ with $0 = r(M) \cap N = r(N)$.

(vi) implies (viii). Obvious.

(viii) implies (v). Let $I \subseteq K \neq R$ be left ideals of R with $K/I = r(R/I)$ and $R/I \in \mathcal{T}_s$. By assumption $s(E(R/I)) = r(s(E(R/I))) \oplus A$ for some module A. Put $L/I = A \cap (R/I)$. Obviously, $A \cap (K/I) \subseteq A \cap r(s(E(R/I))) = 0$ and therefore $L \cap K = I$. Now, if $L = I$ then $A \cap R/I = 0$. Hence $A = 0$ from the essentiality of R/I in $s(E(R/I))$. Thus $R/I \in \mathcal{T}_r$ since $s(E(R/I)) \in \mathcal{T}_r$, a contradiction. $\square$

Definition 3. *A ring R is said to be a left s-QI-ring if every s-torsion quasi-injective module is weakly s-divisible.*

Obviously, if $s \leq t$ are two preradicals then every left t-QI-ring is a left s-QI-ring.

Theorem 1. *Let r and s be hereditary preradicals. Then the following conditions are equivalent:*

(i) *every $r \circ s$-torsion quasi-injective module is weakly s-divisible;*
(ii) *every $r \circ s$-torsion quasi-injective module is s-tight;*
(iii) *every preradical $t \leq r$ is s-stable;*
(iv) *every hereditary preradical $t \leq r$ is s-stable.*

Proof: (i) is equivalent to (ii). It follows immediately from Lemma 1.

(i) implies (iii). Let $t \leq r$. If Q is a weakly s-divisible s-torsion module then $Q = s(E(Q))$ and $t(Q) = (t \circ s)(E(Q))$ is $r \circ s$-torsion and quasi-injective. Now $t(Q)$ is weakly s-divisible by assumption and consequently $t(Q)$ is a direct summand in Q.

(iii) implies (iv). Obvious.

(iv) implies (i). If Q is $r \circ s$-torsion and quasi-injective then $Q = p_{\{Q\}}(E(Q)) = h(p_{\{Q\}})(s(E(Q)))$ and $h(p_{\{Q\}}) \le r$. Hence $h(p_{\{Q\}})(s(E(Q)))$ is a direct summand in $s(E(Q))$ since $h(p_{\{Q\}})$ is s-stable. Thus $Q = h(p_{\{Q\}})(s(E(Q))) = s(E(Q))$ and Q is weakly s-divisible. □

Corollary 2. *Let s be a hereditary preradical. Then the following conditions are equivalent for a ring R:*

(i) *R is a left s-QI-ring;*
(ii) *every s-torsion quasi-injective module is s-tight;*
(iii) *every preradical is s-stable;*
(iv) *every hereditary preradical is s-stable.*

Proof: It follows immediately from the Theorem 1 if we put $r = id$. □

Corollary 3. *Let s be a stable hereditary radical. Then the following conditions are equivalent:*

(i) *every s-torsion quasi-injective module is injective;*
(ii) *$r \circ s$ is stable for every preradical r;*
(iii) *$r \circ s$ is stable for every hereditary preradical r.*

Proof: It follows immediately from the Corollary 2, Proposition 1(iii) and the fact that if s is stable and Q is s-torsion and weakly s-divisible then Q is injective. □

Corollary 4. *The following conditions are equivalent for a module M:*

(i) *every quasi-injective module from $\sigma[M]$ is M-injective;*
(ii) *every preradical is $h(p_{\{M\}})$-stable;*
(iii) *every hereditary preradical is $h(p_{\{M\}})$-stable.*

Proof: It follows immediately from the Corollary 2. □

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