Comment.Math.Univ.Carolin. 40,3 (1999)561–570

Spaces with σ-n-linked topologies as special subspaces of separable spaces

Ronnie Levy, Mikhail Matveev

Abstract. We characterize spaces with σ-n-linked bases as specially embedded subspaces of separable spaces, and derive some corollaries, such as the **c**-productivity of the property of having a σ-linked base.

Keywords: separable, c.c.c., σ-centered base, σ-n-linked base, I_n-embedding, $I_{<\omega}$-embedding, product, Martin's Axiom, C_p-spaces

Classification: Primary 54D65, 54D70; Secondary 54B10, 54C25

A collection $\mathcal{C}$ of sets is *centered* if every finite subcollection of $\mathcal{C}$ has non-empty intersection. A collection $\mathcal{L}$ of sets is *linked* if every two elements of $\mathcal{L}$ have non-empty intersection. Thus, every centered collection is linked. More generally, a collection is *n-linked* (where $n \geq 2$) if every $\leq n$-element subfamily has non-empty intersection; "linked" is a synonym for "2-linked". A collection which is the union of countably many centered (respectively, n-linked, linked) families is said to be *σ-centered* (respectively, *σ-n-linked*, *σ-linked*). The following chain of implications is obvious:

$$\begin{array}{c}
\text{separable} \\
\downarrow \\
\sigma\text{-centered base} \\
\downarrow \\
\cdots \\
\downarrow \\
\sigma\text{-}(n+1)\text{-linked base} \\
\downarrow \\
\sigma\text{-}n\text{-linked base} \\
\downarrow \\
\cdots \\
\downarrow \\
\sigma\text{-linked base} \\
\downarrow \\
\text{c.c.c.}
\end{array}$$

It is known (see [vD1]; implicitly in [LMcD]) that a compact Hausdorff space has a σ-centered base if and only if it is separable, and that a Tychonoff space has a σ-centered base if and only if it has a separable Hausdorff compactification (or, which is equivalent, if every Hausdorff compactification is separable). This characterization of spaces with σ-centered base as dense subspaces of separable spaces makes reasonable the question whether there is similar characterization (i.e. as certain special subspaces of separable spaces) for spaces with σ-n-linked bases. We give such a characterization below. It has as a corollary the result that a regular space with a σ-linked base has weight at most $\mathbf{c}$ where $\mathbf{c} = 2^\omega$ and that there is a bound on the cardinalities of Hausdorff spaces with σ-linked bases. The result for Tychonoff spaces is a consequence of Theorem 5.8 in [BvD] where it is shown that if $\mathcal{B}$ is a σ-linked Boolean algebra, then $|\mathcal{B}| \leq \mathbf{c}$. Another corollary of our characterization is the $\mathbf{c}$-productivity of the classes of spaces with σ-n-linked bases.

It is interesting to note that, unlike in the case of spaces with σ-centered bases, a Hausdorff compact space with a σ-linked base need not be separable. This follows from a series of examples of Steprans and Watson [SW] of Tychonoff spaces with a σ-n-linked bases but without σ-$(n+1)$-linked bases, and a simple remark that a compactification of a space with a σ-linked base also has a σ-linked base.

1. Preliminaries

In this section, we gather some results which will be used in what follows. We make some conventions. All spaces are assumed to be at least Hausdorff. Also, whenever we talk about a collection of sets, such as a base for a topology, we always assume that each of the sets is non-empty.

The following theorem is proven in [vD1].

Theorem 1.1. *Suppose that X is a Tychonoff space. Then the following are equivalent.*

(i) *X has a σ-centered base.*
(ii) *X has a separable compactification.*
(iii) *Every compactification of X is separable.*

A particular case of Theorem 1.1 arises when X is compact.

Corollary 1.2. *A compact space has a σ-centered base if and only if it is separable.*

The question arises of whether the result in Corollary 1.2 holds if σ-centeredness is replaced with σ-linkedness. To answer this, we need some lemmas.

A π-network is a family of sets, not necessarily open, such that every open set contains an element of this family.

Lemma 1.3. *The following are equivalent for a space X.*

(i) *X has a σ-linked π-network.*

(ii) *X has a σ-linked base.*
(iii) *Every base for X is σ-linked.*
(iv) *The collection of non-empty open subsets of X is σ-linked.*

PROOF: Suppose that $\bigcup_{n\in\omega}\mathcal{B}_n$ is a π-network for X where each of the collections $\mathcal{B}_n$ is linked. If $\mathcal{C}$ denotes the topology of X with the emptyset removed, let $\mathcal{C}_n = \{C \in \mathcal{C} : \text{there exists } B \in \mathcal{B}_n \text{ such that } B \subseteq C\}$. Then $\mathcal{C}_n$ is linked for each $n \in \omega$ and $\mathcal{C} = \bigcup_{n\in\omega}\mathcal{C}_n$.

The other implications are trivial. □

Definition 1.4. *A space having a σ-n-linked (respectively, σ-linked) base is said to be n-SLINKY (respectively, SLINKY).*

Thus, "slinky" is a synonym for "2-slinky".

We note that since a linked family cannot have disjoint elements, a slinky space must satisfy the countable chain condition (see the diagram above). In particular, a slinky space can have only countably many isolated points.

Lemma 1.5. *The following are equivalent for a space X.*
(i) *X is n-slinky.*
(ii) *Every dense subset of X is n-slinky.*
(iii) *X has a n-slinky dense subset.*
(iv) *The collection of all non-empty open subsets of X is σ-n-linked.*

PROOF: (i) ⇒ (ii). Let $\bigcup_{k\in\omega}\mathcal{B}_k$ be a σ-n-linked base for X where each collection $\mathcal{B}_k$ is n-linked and suppose that D is dense in X. Let $\mathcal{C}_k = \{B \cap D : B \in \mathcal{B}_k\}$. Then it follows easily from the fact that D is dense in X that $\mathcal{C}_k$ is n-linked for each $k \in \omega$, and $\bigcup_{k\in\omega}\mathcal{C}_k$ is a base for D.

(iii) ⇒ (iv). Suppose D is a dense subset of X having the σ-n-linked base $\bigcup_{k\in\omega}\mathcal{C}_k$ where each collection $\mathcal{C}_k$ is n-linked. Let $\mathcal{U}_k = \{U \subseteq X : U \text{ is non-empty and open in } X \text{ and there exists } C \in \mathcal{C}_k \text{ such that } C \subseteq U\}$. Then $\mathcal{U}_k$ is n-linked for each $k \in \omega$ and if U is any non-empty open subset of X, then $U \cap D$ is non-empty and open in D so there exists an $k \in \omega$ and $C \in \mathcal{C}_k$ such that $C \subseteq U$. Therefore, $\bigcup_{k\in\omega}\mathcal{U}_k$ is the collection of all non-empty open subsets of X.

The other implications are obvious. □

Corollary 1.6. *The following are equivalent for a Tychonoff space X.*
(i) *X is n-slinky.*
(ii) *X has a n-slinky Hausdorff compactification.*
(iii) *Every Hausdorff compactification of X is n-slinky.*

The following observation will be used later.

Proposition 1.7. *If X is n-slinky and $f : X \to Y$ is a continuous surjection, then Y is n-slinky.*

PROOF: Suppose X is slinky and write the collection of non-empty open subsets of X as $\bigcup_{k\in\omega}\mathcal{B}_k$ where each collection $\mathcal{B}_k$ is n-linked. For $k \in \omega$, let $\mathcal{C}_k = \{V \subseteq$

$Y : U$ is open in Y, $f^{\leftarrow}(V) \in \mathcal{B}_k\}$. Then each non-empty open subset of Y is in $\mathcal{C}_k$ for some $k \in \omega$, because f is continuous, and since each collection $\mathcal{C}_k$ is n-linked, so is $\mathcal{B}_k$ for each $k \in \omega$. □

We close this section with a series of examples due to Steprans and Watson.

Example 1.8 (Steprans, Watson [SW])**.** *For every $n \geq 2$ there is a slinky Tychonoff space X_n which is not $(n+1)$-slinky. Furthermore, there is a Tychonoff space X_∞ which is n-slinky for each $n \geq 2$, but which does not have a σ-centered base.*

We note that the spaces in this examples are subspaces of the Pixley-Roy exponent of the irrationals, and, therefore, have additional nice properties: they are metacompact Moore spaces (see [vD2] or [T]). Metacompact plus Moore implies having a point countable base; this shows how far slinky is from separable: a separable space with a point countable base is second-countable.

2. Separability and σ-linked bases

By a result of Baumgartner and van Douwen (Theorem 5.8 in [BvD]), every Tychonoff slinky space has weight at most $\mathbf{c}$. Therefore, since every Tychonoff space of weight $\mathbf{c}$ is a subspace of a separable space, every slinky Tychonoff space embeds in a separable space. In this section, we discuss the relationship between separability and slinkiness. We first point out that even for compact spaces, having a σ-linked base is different from having a σ-centered base, that is, slinkiness is different from separability.

Example 2.1. *There exists a nonseparable compact slinky space.*

Proof: Let X_∞ be the space in Example 1.8 and let Y be a Hausdorff compactification of X_∞. By Corollary 1.6, Y has a σ-linked base, but by Theorem 1.1, Y is not separable. □

We now turn to the task of showing that slinky Tychonoff spaces not only embed in separable spaces, as follows from the Baumgartner-van Douwen result, but they embed in a nice way. Furthermore, we show that even non-Tychonoff slinky spaces embed in separable spaces in a nice way.

Notation 2.2. *Suppose X is a space having a base $\bigcup_{n\in\omega} \mathcal{B}_n$ where the collection $\mathcal{B}_n$ is linked for each $n \in \omega$. We consider ω as a copy of a countable discrete space disjoint from X. If U is an open subset of X and $F \subseteq \omega$, we denote by $\widehat{U_F}$ the subset $U \cup \{n \in \omega : U \in \mathcal{B}_n\} \setminus F$ of $X \cup \omega$. We write $\widehat{U}$ for $\widehat{U}_\emptyset$.*

The notation in 2.2 should include reference to the particular σ-linked base being used, and to the representation of that base as the union of countably many linked collections. However, in our construction, there will be only one such base and representation under consideration.

Lemma 2.3. *Suppose $n \geq 2$ is an integer, X is a space having a base $\bigcup_{k\in\omega} \mathcal{B}_k$ where the collection $\mathcal{B}_k$ is n-linked for each $k \in \omega$ and $U_1 \cdots U_n$ are open subsets of X. Then $U_1 \cap \cdots \cap U_n = \emptyset$ if and only if $\widehat{U_1} \cap \cdots \cap \widehat{U_n} = \emptyset$.*

Proof: We prove the case where $n = 2$, the other cases being essentially the same. One direction is trivial—since $U \subseteq \widehat{U}$ and $V \subseteq \widehat{V}$, if $\widehat{U} \cap \widehat{V} = \emptyset$, then $U \cap V = \emptyset$. For the converse, we note that if $k \in \widehat{U} \cap \widehat{V}$, where $k \in \omega$, then $U \in \mathcal{B}_k$ and $V \in \mathcal{B}_k$, and since $\mathcal{B}_k$ is linked, $U \cap V \neq \emptyset$. $\square$

Definition 2.4. *Let $n \geq 2$. Suppose $\langle Y, \mathcal{T}_Y\rangle$ is a space and $\langle X, \mathcal{T}_X\rangle$ is a subspace of $\langle Y, \mathcal{T}_Y\rangle$. Then X is I_n-*EMBEDDED* in Y if there exists a function $\widehat{\ }\colon \mathcal{T}_X \to \mathcal{T}_Y$ such that for each $U \in \mathcal{T}_X$, $\widehat{U} \cap X = U$ and such that $U_1 \cap \cdots \cap U_n = \emptyset$ implies $\widehat{U_1} \cap \cdots \cap \widehat{U_n} = \emptyset$, where $\widehat{U} = \widehat{\ }(U)$. The function $\widehat{\ }$ is called an I_n *OPERATOR*. If the operator $\widehat{\ }$ is I_n for each $n \geq 2$, then we will call it an $I_{<\omega}$-operator, and we will say that X is $I_{<\omega}$-embedded into Y.*

We note that the property of being I_n embedded is transitive:

Proposition 2.5. *If X is I_n-embedded in Z and Z is I_n-embedded in Y, then X is I_n-embedded in Y.*

The obvious proof works.

Lemma 2.6. *If X is dense in Z, then X is $I_{<\omega}$-embedded in Z.*

Proof: For an open subset U of X, let $K_U = Cl_X U \setminus U$, and let $\widehat{U} = Int_Z(Cl_Z U) \setminus Cl_Z K_U$. Since U is open in X and X is dense in Z, $U \subseteq Int_Z(Cl_Z U)$; since K_U is closed in X and disjoint from U, $U \subseteq \widehat{U}$. Therefore, $U \subseteq \widehat{U} \cap X$. For the reverse inclusion, note that the only elements of X which are in $Int_Z(Cl_Z U)$ are elements of $Cl_X U$, so $\widehat{U} \cap X \subseteq U$ and, hence, $\widehat{U} \cap X = U$. Next, it is easy to see, that for open sets U and V in X, $Int_Z Cl_Z U \cap Int_Z Cl_Z V = Int_Z Cl_Z(U \cap V)$. Therefore $Int_Z Cl_Z U_1 \cap \cdots \cap Int_Z Cl_Z U_n = \emptyset$ whenever $U_1 \cap \cdots \cap U_n = \emptyset$. $\square$

We are now ready to show that slinky spaces are I_2-embedded in separable spaces. This construction will lead to a characterization of slinky spaces.

Proposition 2.7. *Suppose X is a slinky space. Then there exists a separable space Y such that X is an I_2-embedded subspace of Y. Furthermore, if X is Tychonoff, Y may be taken to be Tychonoff.*

Proof: We give the proof for the case of Tychonoff spaces. It is not difficult to see that the same construction, but without the additional step of embedding X into a compact space, provides the preservation of lower axioms of separation than $T_{3\frac{1}{2}}$.

If X is Tychonoff, then by Corollary 1.6, X has a Hausdorff compactification having a σ-linked base; if we can prove the result for this compactification, then we have also proved the result for X. Therefore, if X is Tychonoff, we can and do assume that it is compact Hausdorff.

By Lemma 1.5, the collection of all non-empty open subsets of X is σ-linked, so we can write this collection as $\bigcup_{k\in\omega}\mathcal{B}_k$ where $\mathcal{B}_k$ is linked for each $k \in \omega$. Moreover, if X is n-slinky, we can assume that each $\mathcal{B}_k$ is n-linked (this remark is important for the use in Proposition 2.10 below). Furthermore, we may assume that if $U \in \mathcal{B}_k$ and $U \subseteq V$, then $V \in \mathcal{B}_k$. As a set, let Y be the disjoint union of X and ω. Let each element of ω be isolated, and for each open subset U of X and each finite subset F of ω, declare $\widehat{U_F}$ to be open in Y. Observe that for open subsets U and V of X, if $U\cap V \in \mathcal{B}_k$, then $U \in \mathcal{B}_k$ and $V \in \mathcal{B}_k$, so $\widehat{U\cap V} \subseteq \widehat{U}\cap\widehat{V}$. It follows that we have defined a base for a topology on Y.

To show that Y is Hausdorff, suppose that p and q are distinct elements of Y. Since each element of ω is isolated, and each finite subset of ω is closed in Y, we may assume that $p,\ q \in X$. Let U and V be disjoint neighborhoods of p and q in X. Then by Lemma 2.3, $\widehat{U}$ and $\widehat{V}$ are disjoint neighborhoods of p and q in Y.

We must also show that Y is Tychonoff. We will do more: using compactness, hence normality of X (recall that we have replaced X by its compactification) we prove that Y is normal. So, let F_1 and F_2 be disjoint closed sets in Y, and let $H_1 = F_1 \cap X$, $H_2 = F_2 \cap X$. Then H_1 and H_2 are disjoint closed sets in X. By normality of X, there are disjoint open sets in X, $U_1 \supset H_1$ and $U_2 \supset H_2$. By Lemma 2.3, the sets $\widehat{U_1}$ and $\widehat{U_2}$ are disjoint open neighbourhoods of H_1 and H_2 in Y. We put $V_1 = (\widehat{U_1} \setminus F_2) \cup (F_1 \cap \omega)$ and $V_2 = (\widehat{U_2} \setminus F_1) \cup (F_2 \cap \omega)$. Then V_1 and V_2 are disjoint open neighbourhoods of F_1 and F_2 in Y.

It is immediate from Lemma 2.3 that X is I_2-embedded in Y. Therefore, the only thing left to show is that Y is separable. Let $D = \omega \cup \{p \in X : p \text{ is isolated in } X\}$. Since X is slinky, it has only countably many isolated points, and, therefore, D is countable. Every non-empty set of isolated points of Y intersects D, so if we can show that every set of the form $\widehat{U_F}$, where U is a non-empty open subset of X and F is a finite subset of ω, intersects D, then we will have shown that D is dense in X. If U is finite, then U contains an isolated point p of X and $p \in D \cap \widehat{U_F}$. Therefore, we assume that U is infinite. If we can show that $\{k \in \omega : U \in \mathcal{B}_k\}$ is infinite, then there exists $m \in \omega \setminus F$ such that $U \in \mathcal{B}_m$ and $m \in D \cap \widehat{U_F}$. Suppose $|\{k \in \omega : U \in \mathcal{B}_k\}| = r < \omega$. Choose $r+1$ points $p_0, \cdots, p_r \in U$ and let $V_0, \cdots, V_r$ be disjoint open subsets of X such that $p_i \in V_i \subseteq U$ for $i = 0, \cdots r$. For $i = 0, \cdots, r$, let $k_i \in \omega$ be such that $V_i \in \mathcal{B}_{k_i}$. Then, since the V_i's are pairwise disjoint, if $i \neq j$, $\mathcal{B}_{k_i} \neq \mathcal{B}_{k_j}$. Since $V_i \subseteq U$ for each i, this means that U is an element of the $r+1$ families $\mathcal{B}_{k_0}, \cdots, \mathcal{B}_{k_r}$, contradicting the assumption that $|\{k \in \omega : U \in \mathcal{B}_k\}| = r$. □

Since every separable Tychonoff space has weight at most $\mathbf{c}$ and every separable Hausdorff space has cardinality at most $2^{\mathbf{c}}$, the following is an immediate corollary.

Corollary 2.8.

(1) *Every Hausdorff slinky space has cardinality at most* $2^{\mathbf{c}}$.
(2) *Every slinky Tychonoff space has weight at most* $\mathbf{c}$.

Saying that the weight of a slinky space is $\leq \mathbf{c}$, it is natural to ask what happens if the weight is strictly less than $\mathbf{c}$? It turns out that assuming MA, Martin's Axiom, every slinky space of weight $< \mathbf{c}$ has a σ-centered base. Indeed, it follows from ([W, Theorem 4.5]) that assuming MA, every partial order of cardinality $< \mathbf{c}$ with c.c.c. is σ-centered. Therefore, assuming MA, every c.c.c. space of weight $< \mathbf{c}$ has a σ-centered base, and in particular a σ-linked base becomes σ-centered.

It is easy to see that the property of having a σ-centered or a σ-n-linked base is not preserved by arbitrary, even by closed, subspaces (since one can embed an uncountable discrete space into a separable space). However, these properties are preserved by $I_{<\omega}$- and by I_n-embeddings, respectively.

Proposition 2.9. *An $I_{<\omega}$-embedded subspace of a space with a σ-centered base has a σ-centered base.*

An I_n-embedded subspace of an n-slinky space is an n-slinky space.

The proof is straightforward. We are now ready to characterize slinky spaces.

Proposition 2.10. *A (Tychonoff) space has a σ-centered base iff it can be $I_{<\omega}$-embedded in a separable (Tychonoff) space.*

A (Tychonoff) space is n-slinky iff it can be I_n-embedded into a separable (Tychonoff) space.

Proof: The "if" part follows from Lemma 2.3. The "only if" part follows from Proposition 2.9. □

Proposition 2.10 can be generalized to larger cardinals. To state the generalization, we need the following definition.

Definition 2.11. *Suppose κ is a cardinal. A collection which is the union of κ centered (respectively, n-linked, linked) families of sets is said to be κ-CENTERED (respectively, κ-n-LINKED, κ-LINKED).*

Proposition 2.12. *Suppose that κ is an infinite cardinal.*

A (Tychonoff) space X has a κ-centered base if and only if there exists a (Tychonoff) space Y of density at most κ such that X is $I_{<\omega}$-embedded in Y.

A (Tychonoff) space X has a κ-n-linked base if and only if there exists a (Tychonoff) space Y of density at most κ such that X is I_n-embedded in Y.

The proof is the trivial generalization of the proof of Proposition 2.7 and Proposition 2.10.

3. Products

In this section we deal with questions related to products and n-slinky spaces. In particular, we show that if X is Tychonoff, then the space $C_p(X)$ of continuous real-valued functions defined on X is slinky if and only if $C_p(X)$ has a σ-centered base, and these conditions hold if and only if $|X| \le \mathbf{c}$. We also show that the property of being n-slinky is $\mathbf{c}$-productive. We note here that all of the results carry over in straightforward ways to higher cardinal analogues of slinky spaces.

Proposition 3.1. *Suppose that X is a Tychonoff space. Then the following are equivalent:*

(1) *$C_p(X)$ has a σ-centered base;*
(2) *$C_p(X)$ is slinky;*
(3) $|X| \le \mathbf{c}$.

Proof: Since $C_p(X)$ is dense in $\mathbf{R}^X$, which is (homeomorphic to) a dense subset of $[0,1]^X$, it follows from Lemma that $C_p(X)$ is slinky if and only if $[0,1]^X$ is slinky. If $|X| \le \mathbf{c}$, then $[0,1]^X$ is separable and, therefore, slinky. If $|X| > \mathbf{c}$, then the weight of $[0,1]^X$ is greater than $\mathbf{c}$ and, hence, not slinky by Proposition 2.10. It follows from Theorem 1.1 that $C_p(X)$ has a σ-centered base if and only if $|X| \le \mathbf{c}$. □

Note that the equivalent conditions in the previous proposition do not imply the separability of $C_p(X)$: it is known that $d(C_p(X)) = iw(X)$ ([A, Theorem I.1.5]).
(Recall that $iw(X) = \inf\{w(Y) :$ there exists a continuous bijection of X onto $Y\}$, see [A].)

We now turn to the question of productivity of n-slinky spaces. We note that since a product of more than $\mathbf{c}$ spaces each of which has at least two points has weight greater than $\mathbf{c}$, the best that can be hoped for is that the product of at most $\mathbf{c}$ n-slinky spaces is n-slinky. We prove this result in three steps. We first show that in order to prove that a subspace is I_n-embedded in a space, it suffices to define an I_n operator on a base for the subspace topology.

Lemma 3.2. *Suppose $\langle Y, \mathcal{T}_Y\rangle$ is a space, $\langle X, \mathcal{T}_X\rangle$ is a subspace of $\langle Y, \mathcal{T}_Y\rangle$, and $\mathcal{B}$ is a base for $\langle X, \mathcal{T}_X\rangle$. If there exists a function $h\colon \mathcal{B} \to \mathcal{T}_Y$ such that $h(B) \cap Y = B$ for all $B \in \mathcal{B}$, and $B_0 \cap \cdots \cap B_n = \emptyset$ implies $h(B_0) \cap \cdots \cap h(B_n) = \emptyset$ for all $B_0, \cdots, B_n \in \mathcal{B}$, then X is I_n-embedded in Y.*

Proof: An I_n operator is defined by letting $\widehat{U} = \bigcup\{h(B) : B \in \mathcal{B},\ B \subset U\}$ for each $U \in \mathcal{T}_X$. □

Next we show that the property of being I_n-embedded is productive.

Lemma 3.3. *Suppose that Λ is a non-empty set and for each $\lambda \in \Lambda$, the space X_λ is an I_n-embedded subspace of the space Y_λ. Then $\prod_{\lambda\in\Lambda} X_\lambda$ is I_n-embedded in $\prod_{\lambda\in\Lambda} Y_\lambda$.*

Proof: Denote the products $\prod_{\lambda\in\Lambda} X_\lambda$ and $\prod_{\lambda\in\Lambda} Y_\lambda$ by X and Y respectively, and denote their topologies by $\mathcal{T}_X$ and $\mathcal{T}_Y$ respectively. Let $\mathcal{B}$ be the collection $\{\bigcap_{k\in F} \pi_k^{\leftarrow} U_k : F$ is a finite subset of Λ, U_k is open in $X_k\}$ of canonical open subsets of the product $\prod_{\lambda\in\Lambda} X_\lambda$. By Lemma 3.2, it suffices to show that there exists a function $h : \mathcal{B} \to \mathcal{T}_Y$ such that $h(B) \cap Y = B$ for all $B \in \mathcal{B}$, and $B_0 \cap \cdots \cap B_n = \emptyset$ implies $h(B_0) \cap \cdots \cap h(B_n) = \emptyset$ for all $B_0, \cdots, B_n \in \mathcal{B}$. For $\lambda \in \Lambda$, let $U \to \widehat{U}$ be an I_n operator from X_λ to Y_λ. (The fact that we use the same notation for all of these I_n operators should not cause confusion.) Define $h : \mathcal{B} \to \mathcal{T}_Y$ by $h(\bigcap_{k\in F} \pi_k^{\leftarrow} U_k) = \bigcap_{k\in F} \pi_k^{\leftarrow} \widehat{U_k}$, where the projections on the left side of the equal sign are taken in X and the projections on the right side of the equal sign are taken in Y. Then it is clear that h has the required properties. □

Proposition 3.4. *Suppose Λ is a non-empty set of cardinality at most* $\mathbf{c}$, *and suppose that for each $\lambda \in \Lambda$, X_λ is a space. Then the product $\prod_{\lambda\in\Lambda} X_\lambda$ is n-slinky if and only if X_λ is n-slinky for each $\lambda \in \Lambda$.*

Proof: Suppose first that each space X_λ is n-slinky. By Proposition 2.10, each space X_λ is I_n-embedded in a separable space Y_λ. By Lemma 3.3, $\prod_{\lambda\in\Lambda} X_\lambda$ is I_n-embedded in $\prod_{\lambda\in\Lambda} Y_\lambda$, which is separable since it is a product of at most $\mathbf{c}$ separable spaces. Therefore, by Proposition 2.10, $\prod_{\lambda\in\Lambda} X_\lambda$ is n-slinky.

Conversely, each factor of a product is a continuous image of the product.

Therefore, if is $\prod_{\lambda\in\Lambda} X_\lambda$ is n-slinky, then for each $\lambda \in \Lambda$, the space X_λ is n-slinky by Proposition 1.7. □

Acknowledgment. The authors express their gratitude to Alan Dow for drawing their attention to the paper [BvD].

Added in proof. Properties of embedding similar to I_n, called K_n and J_n, have been known in the literature for a considerable period, see, e.g. [vDTh], [vD0], [B1], [B2]. We decided to introduce a new notion because I_n is just what works in our constructions. Another application of I_n, in the study of star covering properties, can be found in [BM].

Murray Bell has informed the authors that he constructed examples of n-slinky, not $n+1$-slinky spaces before [SW], in [M. Bell, *Two Boolean algebras with extreme cellular and compactness properties*, Canad. J. Math. **35** (1983), no. 5, 824–838]. Also Bell noted that in a slinky space there are at most $\mathbf{c}$ regular open sets. The authors express their gratitude to Murray Bell for these and other remarks.

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Department of Mathematics, George Mason University, Fairfax, VA 22030, USA

E-mail: RLevy@vms1.gmu.edu

Proezd Odoevskogo, 3, korp. 5, kv. 601, Moscow 117574, Russia

E-mail: misha@matveev.mccme.ru

(*Received* July 23, 1998)