# Nonuniqueness for some linear oblique derivative problems for elliptic equations

Gary M. Lieberman

*Abstract.* It is well-known that the "standard" oblique derivative problem, $\Delta u = 0$ in $\Omega$, $\partial u/\partial\nu - u = 0$ on $\partial\Omega$ ($\nu$ is the unit inner normal) has a unique solution even when the boundary condition is not assumed to hold on the entire boundary. When the boundary condition is modified to satisfy an obliqueness condition, the behavior at a single boundary point can change the uniqueness result. We give two simple examples to demonstrate what can happen.

*Keywords:* elliptic equations, uniqueness, a priori estimates, linear problems, boundary value problems

*Classification:* 35J25, 35A05, 35B65

## Introduction

For an elliptic operator $L$ defined in a domain $\Omega \subset \mathbb{R}^n$, the oblique derivative problem is

$$Lu = f \quad \text{in} \quad \Omega, \;\; \beta^i D_i u + \gamma u = g \quad \text{on } \partial\Omega, \tag{0.1}$$

where $\beta$ is a vector such that $\beta \cdot \nu > 0$ for $\nu$ the unit inner normal to $\partial\Omega$. When the data of the problem are sufficiently smooth and $L$ has the form $Lu = a^{ij}D_{ij}u + b^i D_i u + cu$ with $c \leq 0$, $\gamma \leq 0$ with at least one of these functions is not identically zero, it is a standard result in linear elliptic theory (see, e.g. [1, Section 6.7]) that (0.1) has a unique solution which is $C^1$ (actually $C^{2,\alpha}$) up to $\partial\Omega$. In a series of papers [2,3,4], the author looked at a generalization of this problem in which $\partial\Omega$ is only Lipschitz. Under suitable regularity hypotheses on $a^{ij}$, $b^i$, $c$, $\beta^i$, and $\gamma$, it was shown that (0.1) still has a unique solution provided the obliqueness condition $\beta \cdot \nu > 0$ is appropriately modified. A key assumption was that the boundary condition had to be satisfied at every point on the boundary. At a point of discontinuity of $\beta$, such a condition seems unnecessarily restrictive because it requires a specification of $\beta$ there. In this paper, we investigate the role of this assumption.

To understand the significance of our investigation, we note that the obvious choice of a model problem is (0.1) with $L = \Delta$, the Laplace operator, and $\beta = \nu$. If $\Omega$ is a (smoothly) truncated two-dimensional wedge, then (assuming that $\gamma \leq 0$ and $\gamma \not\equiv 0$) problem (0.1) with no specification of boundary condition at the vertex of the angle does not have a unique solution; however, the uniqueness is immediately recovered if we consider only bounded solutions. We shall see that the special nature of the boundary condition in this case is crucial to this uniqueness in a suitably restricted class even if we consider only simple domains and $L = \Delta$.

We recall here that a vector field $\beta$ is oblique at a point $x_0 \in \partial\Omega$ if there is a coordinate system $(x^1, \ldots, x^n)$ centered at $x_0$ with $\beta(x_0)$ parallel to the positive $x^n$-axis such that $\Omega$ can be written locally as

$$\{x : x^n > f(x^1, \ldots, x^{n-1})\}$$

for some Lipschitz function $f$. Moreover, the boundary condition need only hold in the following generalized sense:

$$\lim_{t\to 0^+} \frac{u(x + t\beta(x)) - u(x)}{t} = -\gamma(x)u(x) + g(x)$$

for any $x \in \partial\Omega$. The obliqueness of $\beta$ guarantees that $x + t\beta(x) \in \Omega$ for $x \in \partial\Omega$ and $t$ sufficiently small and positive. We also note that in some circumstances a discontinuous vector field defined in a deleted neighborhood of $x_0$ has a reasonable extension to $x_0$. Specifically, suppose $\Omega \subset \mathbb{R}^2$ and that $\sigma_1$ and $\sigma_2$ are two line segments in $\partial\Omega$ with a common endpoint $x_0$. Suppose also that there are two vectors $\beta_1$ and $\beta_2$ such that

$$\lim_{\substack{x\to x_0\\ x\in\sigma_1}} \beta(x) = \beta_1 \quad \text{and} \quad \lim_{\substack{x\to x_0\\ x\in\sigma_2}} \beta(x) = \beta_2,$$

with $\beta$ oblique at each interior point of $\sigma_1$ and $\sigma_2$. If $\beta \cdot Du$ is prescribed on $\sigma_1 \cup \sigma_2$ and if $Du$ is continuous at $x_0$, then we actually have prescribed $\beta_1 \cdot Du$ and $\beta_2 \cdot Du$ at $x_0$. Therefore it is reasonable to define $\beta(x_0)$ to be any convex combination of $\beta_1$ and $\beta_2$. (Taking a convex combination rather than an arbitrary linear combination is important because then inequalities of the form $\beta \cdot Du > 0$ on $\partial\Omega$ are true by extension if they are true on $\sigma_1 \cup \sigma_2$.) In this situation, it is convenient to say that $\beta$ has an *oblique* extension to $x_0$. (This idea was used significantly in [4].)

We also recall the following uniqueness result, which is essentially [2, Corollary 2.5].

**Proposition 1.** *Let $\partial\Omega$ be piecewise $C^1$. Suppose $L$ is a uniformly elliptic operator with bounded coefficients $a^{ij}$, $b^i$, and $c$. Suppose also that $\beta$ and $\gamma$ are bounded on $\partial\Omega$ with $\beta$ oblique at each point of $\partial\Omega$. If $c \le 0$ and $\gamma \le 0$, then any two $C^0(\overline{\Omega}) \cap C^2(\Omega)$ solutions of (0.1) differ by a constant, which must be zero unless $c \equiv 0$ and $\gamma \equiv 0$.*

In Section 1, we give an example of an oblique derivative problem in a smooth domain with discontinuous $\beta$ which has a unique solution provided the oblique derivative condition is satisfied at every point of the boundary but which has another solution if this condition fails at a single point. Moreover, the second solution that we find is always bounded and may be Hölder continuous with prescribed exponent less than one. In Section 2, we give an example of a vector field which is oblique in a deleted neighborhood of a nonsmooth point on the boundary but which does not have an oblique extension there; again, there will be multiple solutions if no boundary condition is specified at the nonsmooth point. For our second example, all solutions are analytic functions of the variables, so additional smoothness is not enough to guarantee uniqueness. Both examples are elementary and work when the operator is the two-dimensional Laplacian. Accordingly, we use the coordinates $(x, y)$ in $\mathbb{R}^2$.

## 1. Nonuniqueness in a smooth domain

Let $\Omega$ be an open connected set in $\mathbb{R}^2$ with $y > 0$ in $\Omega$ such that the segment $S = \{y = 0, -1 < x < 1\}$ is on $\partial\Omega$. We then define $\beta$ as follows: Choose $\lambda \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. For $(x, y) \in S$, we set

$$\beta(x,y) = \begin{cases} (0,1) & \text{if } x > 0 \\ (\tan(\lambda\pi), 1) & \text{if } x < 0. \end{cases}$$

On the remainder of $\partial\Omega$, $\beta$ is taken to be continuous such that $\beta \cdot \nu \geq 1$. We also take $\gamma$ to be a smooth, nonpositive function which vanishes on $S$ but which is not identically zero on $\partial\Omega$. Now we define $w$ via polar coordinates $(r, \theta)$ by

$$w(x, y) = r^\lambda \cos(\lambda\theta),$$

and then we define $g$ on $\partial\Omega$ by $g = 0$ on $S$, $g = \beta \cdot Dw + \gamma w$ on $\partial\Omega \setminus S$. It follows that

$$\Delta w = 0 \ \text{ in } \Omega, \ \beta^i D_i w + \gamma w = g \ \text{ on } \partial\Omega \setminus \{(0,0)\}. \tag{1.1}$$

On the other hand, from [3, Theorem 1.4], there is a unique solution $u$ of

$$\Delta u = 0 \ \text{ in } \Omega, \ \beta^i D_i u + \gamma u = g \ \text{ on } \partial\Omega, \tag{1.1$'$}$$

and this solution is $C^{1,\alpha}$ up to $\partial\Omega$ for some $\alpha > 0$. Since $w$ is only $C^\lambda$ near $(0, 0)$, it follows that $u$ and $w$ are different solutions of (1.1). On the other hand, $u$ is the unique solution of (1.1)$'$ even if we take $\beta(0, 0)$ to be any convex combination of $(0, 1)$ and $(\lambda \tan(\lambda\pi), 1)$. Moreover, as $\lambda \to 1$, $\lambda \tan(\lambda\pi) \to 0$, so even the combination of hypotheses that $u$ has Hölder exponent arbitrarily close to one and that $\beta$ is pointwise arbitrarily close to a constant vector in a neighborhood of a boundary point is not enough to guarantee uniqueness unless the boundary condition is in force at that point.

In fact, the regularity issue here is more subtle than in the normal derivative boundary condition case since Proposition 1 holds for continuous solutions which satisfy the boundary condition in the generalized sense indicated in the introduction, which means that the directional derivative of $u$ in the direction of $\beta$ exists. Hence (as is also clear from the explicit form of $w$), the directional derivative of $w$ in the direction of $\beta$ must not exist at $(0, 0)$ for any $\beta$ with $\beta^2 > 0$.

## 2. Nonuniqueness in a nonsmooth domain

Now we define the function $g$ by

$$g(y) = \begin{cases} (1-y^2)^{1/2} & \text{if } \frac{-1}{\sqrt{2}} < y < 1 \\ y+\sqrt{2} & \text{if } -\sqrt{2} < y \le \frac{-1}{\sqrt{2}} \end{cases}$$

and let

$$\Omega = \{(x,y) : |x| < g(y), -\sqrt{2} < y < 1\}.$$

Then we define

$$\beta(x,y) = \begin{cases} (-x,-y) & \text{if } y > 0 \\ (-x/|x|, 0) & \text{if } y \le 0 \end{cases}$$

and $\gamma(x,y) = \max\{y,0\}/(y-2)$. Finally we use $x_0$ to denote the point $(0,-\sqrt{2})$. Then the problem

$$\Delta u = 0 \;\text{ in } \Omega,\; \beta \cdot Du + \gamma u = 0 \;\text{ on } \partial\Omega \setminus \{x_0\} \tag{2.1}$$

has the solution $u = k[y-2]$ for any real constant $k$. The vector field $\beta$ is oblique in the classical sense at every point of $\partial\Omega \setminus \{x_0\}$ but it is not oblique at $x_0$. It is simple to see that we have a complete list of solutions of (2.1). The usual uniqueness theory tells us that a positive maximum or a negative minimum to this problem can only occur at $x_0$. If we prescribe $u(x_0) = 0$ (or $\beta \cdot Du(0,0) = 0$, for any vector other than $(\pm 1, 0)$), then the solution must be identically zero.

Note that the example in Section 1 relies, at least indirectly, on a regularity issue: $w$ is not smooth at the point of discontinuity of $\beta$ and the boundary condition is not in force there. Here, our explicit solutions are analytic but the boundary condition is not oblique at one point. In fact, the regularity is built into the local structure of (2.1).

In addition, this example can be modified to show that the relative size of the discontinuity set of $\beta$ is not important. (In many cases, sets of singularities are removable if they are small enough.) Returning to the general case of an $n$-dimensions, we write $x = (x^1, \dots, x^n)$ and $x' = (x^1, \dots, x^{n-1})$. We now take $\Omega$ to be the cone $\{|x'| < g(x^n), -\sqrt{2} < x^n < 1\}$, and we write

$$\beta(x) = \begin{cases} (-x', -x^n) & \text{if } x^n > 0 \\ (-x'/|x'|, 0) & \text{if } x^n \le 0 \end{cases}$$

and $\gamma(x) = \max\{x^n, 0\}/(x^n - 2)$. If we use $x_0$ to denote the point $(0, \dots, -\sqrt{2})$, then $u = k[x^n - 2]$ is a solution of (2.1) for any constant $k$. Now the singularity set has Lebesgue measure zero, Hausdorff dimension zero, and zero capacity, so the singular set for $\beta$ can be arbitrarily small relative to the underlying space $\mathbb{R}^n$.

## References

[1] Gilbarg D., Trudinger N.S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin-Heidelberg-New York, 1983.

[2] Lieberman G.M., *Local estimates for subsolutions and supersolutions of oblique derivative problems for general second-order elliptic equations*, Trans. Amer. Math. Soc. **304** (1987), 343–353.

[3] Lieberman G.M., *Oblique derivative problems in Lipschitz domains I. Continuous boundary values*, Boll. Un. Mat. Ital. **1-B** (1987), 1185–1210.

[4] Lieberman G.M., *Oblique derivative problems in Lipschitz domains II. Discontinuous boundary values*, J. Reine Angew. Math. **389** (1988), 1–21.

Department of Mathematics, Iowa State University, Ames, Iowa 50011, USA

(*Received* June 30, 1998)