Topological dual of non-locally convex Orlicz-Bochner spaces

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Abstract. Let $L^\varphi(X)$ be an Orlicz-Bochner space defined by an Orlicz function φ taking only finite values (not necessarily convex) over a σ-finite atomless measure space. It is proved that the topological dual $L^\varphi(X)^*$ of $L^\varphi(X)$ can be represented in the form: $L^\varphi(X)^* = L^\varphi(X)^\sim_n \oplus L^\varphi(X)^\sim_s$, where $L^\varphi(X)^\sim_n$ and $L^\varphi(X)^\sim_s$ denote the order continuous dual and the singular dual of $L^\varphi(X)$ respectively. The spaces $L^\varphi(X)^*$, $L^\varphi(X)^\sim_n$ and $L^\varphi(X)^\sim_s$ are examined by means of the H. Nakano's theory of conjugate modulars. (Studia Mathematica 31 (1968), 439–449). The well known results of the duality theory of Orlicz spaces are extended to the vector-valued setting.

Keywords: vector-valued function spaces, Orlicz functions, Orlicz spaces, Orlicz-Bochner spaces, topological dual, order dual, order continuous linear functionals, singular linear functionals, modulars, conjugate modulars

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0. Introduction and preliminaries

For a given real Banach space $(X, \|\cdot\|_X)$ and an ideal E of L^0 one can consider X-valued function spaces $E(X)$ defined as subspaces of the space $L^0(X)$ of strongly measurable functions and consisting of all those $f \in L^0(X)$ for which the scalar function $\widetilde{f} = \|f(\cdot)\|_X$ belongs to E. In case when $(E, \|\cdot\|_E)$ is a complete F-normed function space, the space $E(X)$ provided with the F-norm $\|f\|_{E(X)} = \|\widetilde{f}\|_E$ is usually called a Köthe-Bochner space. The most important class of Köthe-Bochner spaces are Lebesgue-Bochner spaces $L^p(X)$ $(0 < p < \infty)$ and their generalization, Orlicz-Bochner spaces $L^\varphi(X)$ (see [9], [16]). In 1938 S. Bochner and A.E. Taylor [4] showed that the topological dual of a Lebesgue-Bochner space $L^p(X)$ $(1 < p < \infty)$ is identifiable with $L^q(X^*)$ $(p^{-1} + q^{-1} = 1)$ if and only if X^* satisfies the Radon-Nikodym property. A. Ionescu Tulcea and C. Ionescu Tulcea [15] showed that the dual space of $L^p(X)$ is identifiable with the space $L^q(X^*, X)$ consisting of weak*-measurable functions. Next, A.V. Bukhvalov [5], [6] extended this result to the class of Köthe-Bochner spaces $(E(X), \|\cdot\|_{E(X)})$, when $(E, \|\cdot\|_E)$ is a Banach function space with an order continuous norm $\|\cdot\|_E$. The integral representation of order continuous linear functionals on $E(X)$ in terms of the space $E'(X^*, X)$ of weak*-measurable functions (E' = the Köthe dual of E) was found by A.V. Bukhvalov [6].

Let us recall that a linear functional F on $E(X)$ is order continuous, whenever for a sequence (f_n) in $E(X)$, $\widetilde{f}_n \xrightarrow{(o)} 0$ in E implies $F(f_n) \to 0$. A.V. Bukhvalov and G. Lozanowskii [8] showed that if $(E, \|\cdot\|_E)$ is a Banach function space, then the topological dual $E(X)^*$ of a Köthe-Bochner space $(E(X), \|\cdot\|_{E(X)})$ admits a direct sum decomposition: $E(X)^* = E(X)_n^\sim \oplus E(X)_s^\sim$, where $E(X)_n^\sim$ and $E(X)_s^\sim$ denote the order continuous dual and the singular dual of $E(X)$ respectively.

As far as we know the first results concerning the topological dual of non-locally convex Orlicz-Bochner spaces $L^\varphi(X)$ are due to F. Hernandez [14], who studied the spaces $L^\varphi(X)$, whenever $\lim\limits_{t\to\infty} \frac{\varphi(t)}{t} = 0$ and a measure space is atomic. Duals of Orlicz spaces of functions valued in locally convex spaces are studied in [12].

In this paper we examine the topological dual of Orlicz-Bochner spaces $L^\varphi(X)$ defined by a finite valued Orlicz function φ (not necessarily convex) over a σ-finite atomless measure space and provided with its complete F-norm topology $\mathcal{T}_\varphi(X)$. In [29] we showed that the Mackey topology $\tau_{L^\varphi(X)}$ of $(L^\varphi(X), \mathcal{T}_\varphi(X))$ coincides with the supremum of the topology $\mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$ ($\overline{\varphi}$ = the convex minorant of φ) and the topology $\pi_\varphi(X)$ of the Minkowski functional of the Orlicz class $L_0^\varphi(X)$. This result allows us to use the methods of the theory of locally convex spaces to examine the topological dual $L^\varphi(X)^*$ of $(L^\varphi(X), \mathcal{T}_\varphi(X))$. In particular, it is shown that $L^\varphi(X)^* = L^\varphi(X)_n^\sim \oplus L^\varphi(X)_s^\sim$. Moreover, we make use of the Nakano's theory of conjugate modulars [23] to study the spaces $L^\varphi(X)^*, L^\varphi(X)_n^\sim$ and $L^\varphi(X)_s^\sim$. We extend to the "vector valued setting" the well known results concerning the dual of scalar Orlicz spaces (cf. [2], [10], [13], [19], [25], [32], [33], [34]).

For terminology concerning Riesz spaces we refer to [1], [17]. Throughout the paper let (Ω, Σ, μ) be a σ-finite atomless measure space and let L^0 stand for the corresponding space of equivalence classes of all Σ-measurable real valued functions defined and finite μ-a.e. Then L^0 is a super Dedekind complete Riesz space under the ordering $u_1 \le u_2$ whenever $u_1(\omega) \le u_2(\omega)$ μ-a.e. For a subset A of Ω let χ_A stand for its characteristic function. As usual, let $\mathbb{N}$ be the set of all natural numbers. We will write $A_n \searrow_\mu \emptyset$, whenever (A_n) is a decreasing sequence in Σ such that $\mu(A_n \cap A) \to 0$ for every $A \in \Sigma$ with $\mu(A) < \infty$.

Let $(X, \|\cdot\|_X)$ be a real Banach space, and let S_X and B_X denote the unit sphere and the unit ball in X respectively. Let X^* stand for the topological dual of X. By $L^0(X)$ we will denote the linear space of equivalence classes of all strongly Σ-measurable functions $f\colon \Omega \to X$. For a function $f \in L^0(X)$ let us put $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$.

Now we recall some terminology concerning Orlicz spaces and Orlicz-Bochner spaces (see [2], [9], [16], [22], [26], [27], [29], [33], [35]).

By an *Orlicz function* we mean here a function $\varphi\colon [0,\infty) \to [0,\infty)$ that is non-decreasing, left continuous, continuous at 0 with $\varphi(0) = 0$. An Orlicz function φ is said to be *strict* if φ is not identically equal to 0.

For an Orlicz function φ by $\overline{\varphi}$ we will denote its *convex minorant*, i.e., $\overline{\varphi}$ is the largest convex Orlicz function that is smaller than φ on $[0,\infty)$. Clearly $\overline{\varphi}$ is strict iff $\liminf\limits_{t\to\infty} \frac{\varphi(t)}{t} > 0$.

Let φ be an Orlicz function. For each $u \in L^0$ let

$$m_\varphi(u) = \int_\Omega \varphi(|u(\omega)|)\, d\mu.$$

The Orlicz space defined by φ is an ideal of L^0 defined by

$$L^\varphi = \{u \in L^0 : m_\varphi(\lambda u) < \infty \ \text{ for some } \ \lambda > 0\}$$

and endowed with the complete semi-metrizable topology $\mathcal{T}_\varphi$ of the Riesz pseudonorm $|u|_\varphi = \inf\{\lambda > 0\colon m_\varphi(u/\lambda) \le \lambda\}$. $\mathcal{T}_\varphi$ is a Hausdorff topology iff φ is strict. Moreover, if φ is a convex Orlicz function then $\mathcal{T}_\varphi$ can be generated by two Riesz seminorms: $\|u\|_\varphi = \inf_{\lambda>0}\{\lambda^{-1}(1 + m_\varphi(\lambda u))\}$ and $|||u|||_\varphi = \inf\{\lambda > 0\colon m_\varphi(u/\lambda) \le 1\}$.

Let $E^\varphi = \{u \in L^0\colon m_\varphi(\lambda u) < \infty$ for all $\lambda > 0\}$. Then E^φ is $|\cdot|_\varphi$-closed ideal of L^φ with $\operatorname{supp} E^\varphi = \Omega$ and $L^\varphi = E^\varphi$ iff φ satisfies the suitable Δ_2-condition (in symbols $\varphi \in \Delta_2$) i.e., $\limsup \frac{\varphi(2t)}{\varphi(t)} < \infty$ as $t \to 0$ and $t \to \infty$, whenever $\mu(\Omega) = \infty$; resp. as $t \to \infty$, whenever $\mu(\Omega) < \infty$.

For each $f \in L^0(X)$ let

$$M_\varphi(f) = m_\varphi(\widetilde{f}\,).$$

The space

$$L^\varphi(X) = \{f \in L^0(X) : \widetilde{f} \in L^\varphi\}$$

is called *an Orlicz-Bochner space* and can be endowed with a complete semimetrizable topology $\mathcal{T}_\varphi(X)$ of the F-pseudonorm $|f|_{L^\varphi(X)} = |\widetilde{f}|_\varphi$ for $f \in L^\varphi(X)$. If φ is a convex Orlicz function then $\mathcal{T}_\varphi(X)$ can be generated by two norms $\|f\|_{L^\varphi(X)} = \|\widetilde{f}\|_\varphi$ and $|||f|||_{L^\varphi(X)} = |||\widetilde{f}|||_\varphi$.

Now we recall some terminology concerning the solid structure and the duality theory of vector valued function spaces (see [27]).

A subset H of $L^\varphi(X)$ is said to be *solid* if $\widetilde{f_1} \le \widetilde{f_2}$ with $f_1 \in L^\varphi(X)$ and $f_2 \in H$ imply $f_1 \in H$. A linear subspace B of $L^\varphi(X)$ is called *an ideal* if B is a solid subset of $L^\varphi(X)$. In particular, $E^\varphi(X)$ is an ideal of $L^\varphi(X)$.

A pseudonorm ρ on $L^\varphi(X)$ is said to be *solid* if $\rho(f_1) \le \rho(f_2)$ whenever $f_1, f_2 \in L^\varphi(X)$ with $\widetilde{f_1} \le \widetilde{f_2}$. Clearly $|\cdot|_{L^\varphi(X)}$ is a solid F-norm on $L^\varphi(X)$.

For a linear functional F on $L^\varphi(X)$ let us set

$$|F|(f) = \sup\{|F(h)| : h \in E(X), \widetilde{h} \le \widetilde{f}\} \ \text{ for all } \ f \in E(X).$$

The linear space

$$L^\varphi(X)^\sim = \{F \in L^\varphi(X)^\# : |F|(f) < \infty \ \text{ for all } \ f \in E(X)\}$$

is called *the order dual* of $L^\varphi(X)$. (Here $L^\varphi(X)^\#$ denotes the algebraic dual of $L^\varphi(X)$.) One can show that

$$L^\varphi(X)^* = L^\varphi(X)^\sim, \tag{0.1}$$

where $L^\varphi(X)^*$ stands for the topological dual of $(L^\varphi(X), \mathcal{T}_\varphi(X))$ (see [8]).

For $F_1, F_2 \in L^\varphi(X)^\sim$ we will write $|F_1| \le |F_2|$ whenever $|F_1|(f) \le |F_2|(f)$ for all $f \in L^\varphi(X)$.

A linear subspace I of $L^\varphi(X)^\sim$ is said to be an *ideal* of $L^\varphi(X)^\sim$ whenever $|F_1| < |F_2|$, $F_1 \in L^\varphi(X)^\sim$, $F_2 \in I$ imply $F_1 \in I$.

1. Conjugate modulars

From now on in this paper we will assume that φ is a strict Orlicz function. The functional M_φ restricted to $L^\varphi(X)$ is a semimodular (see [21], [22]).

Due to H. Nakano [23] *the conjugate* $\overline{M}_\varphi$ of the semimodular M_φ can be defined on $L^\varphi(X)^\#$ by

$$\overline{M}_\varphi(F) = \sup\{|F(f)| - M_\varphi(f) : f \in L^\varphi(X)\}.$$

A functional $F \in L^\varphi(X)^\#$ is said to be *bounded* for M_φ if there exists a number $\gamma > 0$ such that $|F(f)| \le \gamma(M_\varphi(f) + 1)$ for all $f \in L^\varphi(X)$. The collection of all $F \in L^\varphi(X)^\#$ that are bounded for M_φ will be denoted by $\overline{L^\varphi(X)}$. Following [23, §5] one can define *the polar* P_{M_φ} of M_φ by

$$P_{M_\varphi}(F) = \sup\{|F(f)| : f \in L^\varphi(X), M_\varphi(f) \le 1\}$$

for $F \in L^\varphi(X)^\#$. It is known that P_{M_φ} is a norm on $\overline{L^\varphi(X)}$ (see [23, §5]).

Theorem 1.1. *For each $F \in L^\varphi(X)^*$ we have $P_{M_\varphi}(F) < \infty$ and $|F(f)| \le P_{M_\varphi}(F)(M_\varphi(f) + 1)$ for all $f \in L^\varphi(X)$.*

Proof: Following [31] we can easily show that $P_{M_\varphi}(F) < \infty$ for all $F \in L^\varphi(X)^*$. Now let $M_\varphi(f) < \infty$ and let $M_\varphi(f) = n + r$, where $n \in \mathbb{N}$, $0 \le r < 1$. Since the measure space (Ω, Σ, μ) is assumed to be atomless we can choose a finite partition $\{A_1, \dots, A_n, A\}$ of Ω such that $M_\varphi(\chi_{A_i} f) = 1$ for $i = 1, 2, \dots, n$ and $M_\varphi(\chi_A f) = r$. Hence for $f \in L^\varphi(X)$ we have $f = (\sum_{i=1}^n \chi_{A_i} f) + \chi_A f$ and $|F(f)| \le \sum_{i=1}^n |F(\chi_{A_i} f)| + |F(\chi_A f)| \le P_{M_\varphi}(F)(n+1) \le P_{M_\varphi}(F)(M_\varphi(f) + 1)$. $\square$

Theorem 1.2. *The following identities hold:*

$$L^\varphi(X)^* = \overline{L^\varphi(X)} = \{F \in L^\varphi(X)^\# : \overline{M}_\varphi(\lambda F) < \infty \text{ for some } \lambda > 0\}. \tag{$*$}$$

Moreover, $\overline{M}_\varphi$ restricted to $L^\varphi(X)^*$ is a convex semimodular and

$$\overline{M}_\varphi(F) = \sup\{|F|(f) - M_\varphi(f) : f \in L^\varphi(X),\ M_\varphi(f) < \infty\}. \tag{$**$}$$

Hence $\overline{M}_\varphi(F_1) \le \overline{M}_\varphi(F_2)$ *if* $|F_1| \le |F_2|$.

Proof: Arguing as in the proof of [26, Theorem 3.1] we obtain the identities $(*)$. Clearly $\overline{M}_\varphi$ is a convex semimodular on $L^\varphi(X)^*$. Now, let $f \in L^\varphi(X)$. Then for each $h \in L^\varphi(X)$ with $\widetilde{h} \le \widetilde{f}$ we have $|F(h)| \le \overline{M}_\varphi(F) + M_\varphi(f)$, so $|F|(f) \le \overline{M}_\varphi(F)+M_\varphi(f)$. It follows that the identity $(**)$ holds, and the proof is complete. □

By means of the convex semimodular $\overline{M}_\varphi$ one can define on $L^\varphi(X)^*$ two norms according to the general definitions (see [23, §6]):

$$\|F\|_{\overline{M}_\varphi} = \inf_{\lambda>0}\{\lambda^{-1}(1+\overline{M}_\varphi(\lambda f))\} \quad \text{and} \quad |||F|||_{\overline{M}_\varphi} = \inf\{\lambda > 0 : \overline{M}_\varphi(F/\lambda) \le 1\}$$

and in view of [30, 1.51] we have:

$$|||F|||_{\overline{M}_\varphi} \le \|F\|_{\overline{M}_\varphi} \le 2|||F|||_{\overline{M}_\varphi} \quad \text{and} \quad |||F|||_{\overline{M}_\varphi} \le 1 \;\text{ iff }\; \overline{M}_\varphi(F) \le 1.$$

Theorem 1.3. *Let* $F \in L^\varphi(X)^*$. *Then*

$$(*) \qquad P_{M_\varphi}(F) = \sup\{|F|(f) : f \in L^\varphi(X), M_\varphi(f) \le 1\},$$

$$(**) \qquad |||F|||_{\overline{M}_\varphi} \le P_{M_\varphi}(F) \le \|F\|_{\overline{M}_\varphi}.$$

Moreover, $L^\varphi(X)^*$ *provided with* $P_{\overline{M}_\varphi}$ *(resp.* $\|\cdot\|_{\overline{M}_\varphi}$, $|||\cdot|||_{\overline{M}_\varphi}$*) is a Banach space.*

Proof: Let $M_\varphi(f) \le 1$. Then for each $h \in L^\varphi(X)$ with $\widetilde{h} \le \widetilde{f}$ we have $|F(h)| \le P_{M_\varphi}(F)$. Hence $|F|(f) \le P_{M_\varphi}(F)$, and it follows that $(*)$ holds. To prove that $(**)$ holds it is enough to repeat the argument from the proof of [26, Theorem 3.2]. To show that $(L^\varphi(X)^*, P_{M_\varphi})$ is a Banach space assume that (F_n) is P_{M_φ}-Cauchy sequence in $L^\varphi(X)^*$, and let $\varepsilon > 0$ be given. Then $P_{M_\varphi}(F_n - F_m) \le \varepsilon$ for $n, m \ge n_0$ for some $n_0 \in \mathbb{N}$. For each $f \in L^\varphi(X)$ take $\lambda > 0$ such that $M_\varphi(\lambda f) < \infty$. By Theorem 1.1 $|F_n(\lambda f) - F_m(\lambda f)| \le P_{M_\varphi}(F_n - F_m)(M_\varphi(\lambda f) + 1)$, so $|F_n(f)-F_m(f)| \le \frac{\varepsilon}{\lambda}(M_\varphi(\lambda f)+1)$. Putting $F(f) = \lim F_n(f)$ for each $f \in L^\varphi(X)$, one can easily observe that F is a bounded for M_φ linear functional on $L^\varphi(X)$, so by Theorem 1.2, $F \in L^\varphi(X)^*$. Moreover, let $M_\varphi(f) \le 1$ and let $n \ge n_0$ be given. Then for each $m \ge n_0$ we have $|F_n(f) - F(f)| \le |F_n(f) - F_m(f)| + |F_m(f) - F(f)| \le \varepsilon(M_\varphi(f) + 1) + |F_m(f) - F(f)|$, so $|F_n(f) - F(f)| \le 2\varepsilon$. Hence $P_{M_\varphi}(F_n - F) \le 2\varepsilon$. It follows that $P_{M_\varphi}(F_n - F) \to 0$, as desired. □

2. Order continuous linear functionals on Orlicz-Bochner spaces

Throughout this section we will assume that $\liminf\limits_{t\to\infty} \frac{\varphi(t)}{t} > 0$.

Let us recall that the Köthe dual $(L^\varphi)'$ of L^φ is equal to L^{φ^*} (see [20], [25, Theorem 3.2]), where φ^* denotes the Young function conjugate to φ in the sense of Young, i.e., $\varphi^*(s) = \sup\{ts - \varphi(t)\colon t \ge 0\}$ for $s \ge 0$.

It is known that if $\liminf\limits_{t\to\infty} \frac{\varphi(t)}{t} = \infty$ (resp. $\liminf\limits_{t\to\infty} \frac{\varphi(t)}{t} = a,\ 0 < a < \infty$) then φ^* takes only finite values (resp. $\varphi^*(s) < \infty$ for $0 \le s < a,\ \varphi^*(s) = \infty$ for $s > a$) (see [25, Lemmas 2.2 and 2.3], [28]).

Definition 2.1 (see [6]). A linear functional F on $L^\varphi(X)$ is said to be *order continuous* whenever $\widetilde{f_\sigma} \xrightarrow{(o)} 0$ in L^φ implies $F(f_\sigma) \to 0$. In view of the super Dedekind completeness of $L^\varphi(X)$ one can restrict ourselves to usual sequences (f_n) (see [27, Theorem 2.2.]). The set consisting of all order continuous linear functionals on $L^\varphi(X)$ will be denoted by $L^\varphi(X)_n^\sim$. It is known that $L^\varphi(X)_n^\sim$ is an ideal of $L^\varphi(X)^\sim$ (see [27, Theorem 2.2]).

To describe the space $L^\varphi(X)_n^\sim$ we recall terminology concerning spaces of weak*-measurable functions (see [6], [27]).

Let $L^0(X^*, X)$ stand for the linear space of weak*-equivalence classes of all weak*-measurable functions $g \colon \Omega \to X^*$. One can define the so-called abstract norm $\vartheta : L^0(X^*, X) \to L^0$ by

$$\vartheta(g) = \sup\{|g_x| : x \in B_X\}$$

where $g_x(\omega) = g(\omega)(x)$ for $\omega \in \Omega$. Let

$$L^{\varphi^*}(X^*, X) = \{g \in L^0(X^*, X) : \vartheta(g) \in L^{\varphi^*}\}.$$

Then $L^{\varphi^*}(X^*) = L^{\varphi^*}(X^*, X) \cap L^0(X^*)$ and $\vartheta(g) = \widetilde{g}$ for $g \in L^{\varphi^*}(X^*)$. It is known that $L^{\varphi^*}(X^*, X) = L^{\varphi^*}(X^*)$ whenever X^* has the Radon-Nikodym property with respect to μ (see [7, Theorem 3.5]). $L^{\varphi^*}(X^*, X)$ can be provided with two norms:

$$\|g\|_{L^{\varphi^*}(X^*,X)} = \|\vartheta(g)\|_{\varphi^*}, \quad |||g|||_{L^{\varphi^*}(X^*,X)} = |||\vartheta(g)|||_{\varphi^*}.$$

We shall need the following technical lemma.

Lemma 2.1 (cf. [6, Theorem 1.1]). *Let $f \in L^\varphi(X)$ and $g \in L^{\varphi^*}(X^*, X)$. Then*

$$\sup\left\{\Big|\int_\Omega \langle h(\omega), g(\omega)\rangle \, d\mu\Big| : h \in L^\varphi(X), \widetilde{h} \le \widetilde{f}\right\} = \int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega)\, d\mu.$$

The following important result describes order continuous linear functionals on $L^\varphi(X)$ in terms of the space $L^{\varphi^*}(X^*, X)$ (see [6, Theorem 4.1]).

Theorem 2.2. *For a linear functional F on $L^\varphi(X)$ the following statements are equivalent:*

(i) *F is order continuous;*
(ii) *F is modular continuous (i.e. $M_\varphi(f_n) \to 0$ implies $F(f_n) \to 0$);*
(iii) *there exists a unique $g \in L^{\varphi^*}(X^*, X)$ such that*

$$F(f) = F_g(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu \quad \text{for all } \ f \in L^\varphi(X).$$

Moreover, for $g \in L^{\varphi^*}(X^*, X)$

$$(*) \qquad |F_g|(f) = \int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega)\, d\mu \quad \textit{for all} \quad f \in L^\varphi(X).$$

PROOF: (i) $\Rightarrow$ (ii) Let $m_\varphi(\widetilde{f_n}) \to 0$. In view of [24, Theorem 2.3] it follows that $\widetilde{f_n} \xrightarrow{(o)^*} 0$ in L^φ. Hence $F(f_n) \to 0$, as desired.

(ii) $\Rightarrow$ (i) Let $\widetilde{f_n} \xrightarrow{(o)} 0$ in L^φ. Then $m_\varphi(\lambda \widetilde{f_n}) \to 0$ for some $\lambda > 0$. Hence $F(f_n) \to 0$ as desired.

(i) $\Leftrightarrow$ (iii) It follows from [6, Theorem 4.1].

The identity $(*)$ follows from Lemma 2.1. $\square$

Lemma 2.3 (cf. [25], [28]). *Assume that* $\liminf_{t\to\infty} \frac{\varphi(t)}{t} = \infty$ (*resp.* $\liminf_{t\to\infty} \frac{\varphi(t)}{t} = a$, *where* $0 < a < \infty$). *Then for a measurable bounded function* $v \geq 0$ (*resp.* v *with* $0 \leq v(\omega) < a$ μ-a.e.) *there exists a measurable bounded function* $u \geq 0$ *such that* $\varphi(u(\omega)) + \varphi^*(v(\omega)) = u(\omega)v(\omega)$ μ-a.e.

Now we are ready to describe $\overline{M}_\varphi$, $\|\cdot\|_{\overline{M}_\varphi}$ and $|||\cdot|||_{\overline{M}_\varphi}$ on $L^\varphi(X)^\sim_n$ (cf. [19], [25, Theorem 4.2]).

Theorem 2.4. *For each* $g \in L^{\varphi^*}(X^*, X)$ *the following identities hold:*

(i) $\overline{M}_\varphi(F_g) = m_{\varphi^*}(\vartheta(g))$;

(ii) $\|F_g\|_{\overline{M}_\varphi} = \|\vartheta(g)\|_{\varphi^*} = \|g\|_{L^{\varphi^*}(X^*,X)}$;

(iii) $|||F_g|||_{\overline{M}_\varphi} = |||\vartheta(g)|||_{\varphi^*} = |||g|||_{L^{\varphi^*}(X^*,X)}$;

(iv) $P_{M_\varphi}(F_g) = \sup\{|\int_\Omega u(\omega)\vartheta(g)(\omega)\, d\mu| : u \in E^\varphi,\ m_\varphi(u) \leq 1\}$
$= \sup\{|\int_\Omega \langle h(\omega),\ g(\omega)\rangle\, d\mu| : h \in E^\varphi(X),\ M_\varphi(h) \leq 1\}$.

PROOF: (i) From the definition of φ^* it follows that

$$\overline{M}_\varphi(F_g) \leq m_{\varphi^*}(\vartheta(g)).$$

To prove that $\overline{M}_\varphi(F_g) \geq m_{\varphi^*}(\vartheta(g))$ we will distinguish two cases:

A. Assume that $\liminf_{t\to\infty} \frac{\varphi(t)}{t} = a < \infty$. Then $\varphi^*(s) < \infty$ for $0 \leq v \leq a$, $\varphi^*(s) = \infty$ for $s > a$. Thus the inclusion $L^{\varphi^*} \subset L^\infty$ holds and we can consider two subcases:

1^0. Assume that $\|\vartheta(g)\|_\infty \leq a$ (here $\|\cdot\|_\infty$ stands for the norm in L^∞). Since $\operatorname{supp} E^\varphi = \Omega$ there exists a sequence (Ω_n) in Σ such that $\Omega_n \uparrow \Omega$, $\mu(\Omega_n) < \infty$ and $\chi_{\Omega_n} \in E^\varphi$ (see [37, Theorem 86.2]). For $n \in \mathbb{N}$ let us set

$$v_n(\omega) = \begin{cases} \vartheta(g)(\omega) & \text{if } \vartheta(g)(\omega) \leq n \ \text{ and } \ \omega \in \Omega_n, \\ 0 & \text{elsewhere.} \end{cases}$$

Hence $v_n \in E^\varphi$, because $v_n \le n\chi_{\Omega_n}$ for $n \in \mathbb{N}$. In view of Lemma 2.3 for $n \in \mathbb{N}$ there exists $0 \le w_n \in L^0$ with $\operatorname{supp} w_n \subset \Omega_n$ and such that $\varphi(w_n(\omega)) + \varphi^*(v_n(\omega)) = w_n(\omega)v_n(\omega)$ for $\omega \in \Omega$. Since $\varphi^*(v_n(\omega)) \uparrow_n \varphi^*(\vartheta(g)(\omega))$ for $\omega \in \Omega$, by the Fatou lemma we get

$$\begin{aligned} m_{\varphi^*}(\vartheta(g)) &= \int_\Omega \varphi^*(\vartheta(g)(\omega))\,d\mu \le \sup_n \int_\Omega \varphi^*(v_n(\omega))\,d\mu \\ &= \sup_n \Big(\int_\Omega w_n(\omega)v_n(\omega)\,d\mu - \int_\Omega \varphi(w_n(\omega))\,d\mu \Big) \\ &\le \sup_n \Big(\int_\Omega w_n(\omega)\vartheta(g)(\omega)\,d\mu - \int_\Omega \varphi(w_n(\omega))\,d\mu \Big). \end{aligned} \tag{1}$$

For $n \in \mathbb{N}$ let $f_n(\omega) = w_n(\omega)x_0$ for $\omega \in \Omega$ and some $x_0 \in S_X$. Let $\varepsilon > 0$ be given. In view of Lemma 2.1 for $n \in \mathbb{N}$ there exists $h_n \in L^\varphi(X)$ with $\widetilde{h}_n \le \widetilde{f}_n$ and such that

$$\int_\Omega \widetilde{f}_n(\omega)\vartheta(g)(\omega)\,d\mu \le \Big| \int_\Omega \langle h_n(\omega), g(\omega)\rangle\,d\mu \Big| + \varepsilon. \tag{2}$$

Hence by (1) and (2)

$$\begin{aligned} m_{\varphi^*}(\vartheta(g)) &\le \sup_n \Big\{ \Big| \int_\Omega \langle h_n(\omega), g(\omega)\rangle\,d\mu \Big| - \int_\Omega \varphi(\widetilde{h}_n(\omega))\,d\mu \Big\} + \varepsilon \\ &\le \sup_n \Big\{ \Big| \int_\Omega \langle h(\omega), g(\omega)\rangle\,d\mu \Big| - \int_\Omega \varphi(\widetilde{h}(\omega))\,d\mu : h \in L^\varphi(X) \Big\} + \varepsilon \\ &\le \overline{M}_\varphi(F_g) + \varepsilon. \end{aligned}$$

It follows that $m_{\varphi^*}(\vartheta(g)) = \overline{M}_\varphi(F_g)$, as desired.

2^0. Assume that $\|\vartheta(g)\|_\infty > a$. Then $m_{\varphi^*}(\vartheta(g)) = \infty$. Since $\liminf_{t\to\infty} \frac{\varphi(t)}{t} = a$, there exists a sequence (t_n) such that $0 < t_n \uparrow \infty$ and $\varphi(t_n) < (a+\delta)t_n$. Choose $0 < \lambda < 1$ and $0 < \delta < a$ such that $\|\lambda\vartheta(g)\|_\infty = a$ and $\frac{\lambda(a+\delta)}{a-\delta} < 1$. Let $A = \{\omega \in \Omega\colon \lambda\vartheta(g)(\omega) > a - \delta\}$ and choose $C \in \Sigma$ with $C \subset A$ such that $0 < \mu(C) < \infty$. Let $u_n = t_n\chi_C$ for $n \in \mathbb{N}$ and $f_n = u_n x_0$ for some $x_0 \in S_X$.

Let $\varepsilon > 0$ be given. Then by Lemma 2.1 for $n \in \mathbb{N}$ there exists $h_n \in L^\varphi(X)$ with $\widetilde{h}_n \le \widetilde{f}_n = u_n$ and such that

$$\int_\Omega u_n(\omega)\vartheta(g)(\omega)\,d\mu \le \Big| \int_\Omega \langle h_n(\omega), g(\omega)\rangle\,d\mu \Big| + \varepsilon.$$

Thus

$$\overline{M}_\varphi(F_g) \ge \Big|\int_\Omega \langle h_n(\omega), g(\omega)\rangle\, d\mu\Big| - \int_\Omega \varphi(\widetilde{h}_n(\omega))\, d\mu$$
$$\ge \int_\Omega u_n(\omega)\vartheta(g)(\omega)\, d\mu - \int_\Omega \varphi(u_n(\omega))\, d\mu - \varepsilon.$$

But

$$\frac{\lambda(a+\delta)}{a-\delta}\int_\Omega u_n(\omega)\vartheta(g)(\omega)\, d\mu \ge \frac{\lambda(a+\delta)}{a-\delta} t_n \cdot \frac{a-\delta}{\lambda}\mu(C)$$
$$= (a+\delta)t_n\mu(C) \ge \int_\Omega \varphi(u_n(\omega))\, d\mu.$$

Hence for $n \in \mathbb{N}$

$$\overline{M}_\varphi(F_g) \ge \Big(1 - \frac{\lambda(a+\delta)}{a-\delta}\Big)\int_\Omega u_n(\omega)\vartheta(g)(\omega)\, d\mu - \varepsilon$$
$$\ge \Big(1 - \frac{\lambda(a+\delta)}{a-\delta}\Big)\frac{a-\delta}{\lambda} t_n\mu(C) - \varepsilon.$$

It follows that $\overline{M}_\varphi(F_g) = \infty$, because $t_n \uparrow \infty$.

B. Assume that $\liminf_{t\to\infty} \frac{\varphi(t)}{t} = \infty$. Then $\varphi^*(s) < \infty$ for all $s \ge 0$ and one can repeat the argument of the subcase 1^0 of A.

(ii) Since $\lambda F_g = F_{\lambda g}$ and $\vartheta(\lambda g) = \lambda\vartheta(g)$ for $\lambda > 0$, by (i)

$$\|F_g\|_{\overline{M}_\varphi} = \inf\Big\{\lambda^{-1}(1 + \overline{M}_\varphi(\lambda F_g))\Big\}$$
$$= \inf\Big\{\lambda^{-1}(1 + m_{\varphi^*}(\lambda\vartheta(g)))\Big\} = \|\vartheta(g)\|_{\varphi^*}.$$

(iii) Similarly by (ii) we get

$$|||F_g|||_{\overline{M}_\varphi} = \inf\{\lambda > 0 : \overline{M}_\varphi(F_g/\lambda) \le 1\}$$
$$= \inf\{\lambda > 0 : m_{\varphi^*}(\vartheta(g)/\lambda) \le 1\} = |||\vartheta(g)|||_{\varphi^*}.$$

(iv) Combining Theorem 1.3 and Lemma 2.1 and using the Fatou lemma we get

$$P_{M_\varphi}(F_g) = \sup\{|\int_\Omega u(\omega)\vartheta(g)(\omega)\, d\mu| : u \in E^\varphi,\ m_\varphi(u) \le 1\}.$$

Thus for $\varepsilon > 0$ there exists $u_0 \in E^\varphi$ with $m_\varphi(u_0) \le 1$ such that

$$P_{M_\varphi}(F_g) \le \int_\Omega |u_0(\omega)|\vartheta(g)(\omega)\, d\mu + \frac{\varepsilon}{2}. \tag{3}$$

Next, by Lemma 2.1 there exists $h_0 \in L^\varphi(X)$ with $\widetilde{h}_0 \le |u_0|$ such that

$$\int_\Omega |u_0(\omega)\vartheta(g)(\omega)\, d\mu \le |\int_\Omega \langle h_0(\omega), g(\omega)\rangle\, d\mu| + \frac{\varepsilon}{2}. \tag{4}$$

Thus by (3) and (4), $P_{M_\varphi}(F_g) \le \int_\Omega \langle h_0(\omega), g(\omega)\rangle| + \varepsilon$ and since $h_0 \in E^\varphi(X)$ the proof is complete. □

3. Singular linear functionals on Orlicz-Bochner spaces

Definition 3.1 (see [27, Definition 2.3])**.** A functional $F \in L^\varphi(X)^\sim$ is said to be *singular* if there exists an ideal B of $L^\varphi(X)$ with $\operatorname{supp} B = \Omega$ and such that $F(f) = 0$ for all $f \in B$. The set consisting of all singular functionals on $L^\varphi(X)$ will be denoted by $L^\varphi(X)^\sim_s$ and called *the singular dual* of $L^\varphi(X)$. $L^\varphi(X)^\sim_s$ is an ideal of $L^\varphi(X)^\sim$ (see [27, Theorem 2.8]).

The set $L^\varphi_0(X) = \{f \in L^\varphi(X) : M_\varphi(f) < \infty\}$ is an absolutely convex absorbing subset of $L^\varphi(X)$. Let K_φ stand for its Minkowski functional, i.e., for $f \in L^\varphi(X)$

$$K_\varphi(f) = \inf\{\lambda > 0 : M_\varphi(f/\lambda) < \infty\}.$$

Clearly $K_\varphi(f) \leq \|f\|_{L^\varphi(X)}$ for $f \in L^\varphi(X)$ and $E^\varphi(X) = \ker K_\varphi$.

Lemma 3.1. *Let B be an ideal of $L^\varphi(X)$ with* $\operatorname{supp} B = \Omega$ *and let $M_\varphi(f) < \infty$. Then for each $\varepsilon > 0$ there exists $h \in B$ such that $M_\varphi(f - h) \leq \varepsilon$.*

Proof: Let $\widetilde{B} = \{u \in L^\varphi \colon |u| \leq \widetilde{h}$ for some $h \in B\}$. Then $\widetilde{B}$ is an ideal of L^φ with $\operatorname{supp} \widetilde{B} = \Omega$ and $B = \widetilde{B}(X) = \{h \in L^\varphi(X) \colon \widetilde{h} \in \widetilde{B}\}$ (see [27, Lemma 1.1]). Since $\operatorname{supp} \widetilde{B} = \Omega$ there exists a sequence (Ω_n) in Σ such that $\Omega_n \uparrow \Omega$, $\mu(\Omega_n) < \infty$ with $\chi_{\Omega_n} \in \widetilde{B}$ for $n \in \mathbb{N}$ (see [37, Theorem 86.2]). For $n \in \mathbb{N}$ let

$$f_n(\omega) = \begin{cases} f(\omega) & \text{if } \ \widetilde{f}(\omega) \leq n \ \text{ and } \ \omega \in \Omega_n, \\ 0 & \text{elsewhere.} \end{cases}$$

Since $\widetilde{f_n} \leq n\chi_{\Omega_n}$ we get $\widetilde{f_n} \in \widetilde{B}$, so $f_n \in B$ for $n \in \mathbb{N}$. By the Lebesgue dominated convergence theorem $M_\varphi(f - f_n) \to 0$, so $M_\varphi(f - f_{n_0}) \leq \varepsilon$ for some $n_0 \in \mathbb{N}$. $\square$

Theorem 3.2. *Let $F \in L^\varphi(X)^\sim_s$. Then*

$$\begin{aligned} P_{M_\varphi}(F) &= \sup\{|F(f)| : f \in L^\varphi(X), M_\varphi(f) < \infty\} \\ &= \sup\{|F(f)| : f \in L^\varphi(X), K_\varphi(f) \leq 1\}. \end{aligned}$$

Proof: Since $F \in L^\varphi(X)^\sim_s$, $F(h) = 0$ for all $h \in B$, where B is an ideal of $L^\varphi(X)$ with $\operatorname{supp} B = \Omega$. Let $f \in L^\varphi(X)$ with $K_\varphi(f) \leq 1$, and let $\varepsilon > 0$ be given. Then $\frac{f}{K_\varphi(f)+\varepsilon} \in L^\varphi_0(X)$, i.e., $M_\varphi(\frac{f}{K_\varphi(f)+\varepsilon}) < \infty$. In view of Lemma 3.1 there exists $h \in B$ such that $M_\varphi(\frac{f}{K_\varphi(f)+\varepsilon} - h) \leq 1$. Hence

$$P_{M_\varphi}(F) \geq \Big|F\Big(\frac{f}{K_\varphi(f)+\varepsilon} - h\Big)\Big| = \Big|F\Big(\frac{f}{K_\varphi(f)+\varepsilon}\Big)\Big| = \frac{1}{K_\varphi(f)+\varepsilon}|F(f)|.$$

It follows that $|F(f)| \leq P_{M_\varphi}(F) \cdot K_\varphi(f) \leq P_{M_\varphi}(F)$. Hence $\sup\{|F(f)| : f \in L^\varphi(X), K_\varphi(f) \leq 1\} \leq P_{M_\varphi}(F)$, and the proof is complete. $\square$

For a non-empty subset A of $L^\varphi(X)$ let $A^\perp$ stand for its annihilator in $L^\varphi(X)^*$ i.e., $A^\perp = \{F \in L^\varphi(X)^* \colon F(f) = 0$ for all $f \in A\}$.

Corollary 3.3. $L^\varphi(X)^\sim_s = (L^\varphi(X), K_\varphi)^* = E^\varphi(X)^\perp$.

Proof: Let $F \in L^\varphi(X)^\sim_s$. Then for $f \in L^\varphi(X)$ and $\varepsilon > 0$ we have $M_\varphi(\frac{f}{K_\varphi(f)+\varepsilon}) < \infty$, so by Theorem 3.2 $|F(f)| \le P_{M_\varphi}(F) \cdot K_\varphi(f)$. Thus $F \in (L^\varphi(X), K_\varphi)^*$, so the inclusion $L^\varphi(X)^\sim_s \subset (L^\varphi(X), K_\varphi)^*$ holds.

To show that $(L^\varphi(X), K_\varphi)^* \subset E^\varphi(X)^\perp$ holds let $F \in (L^\varphi(X), K_\varphi)^*$. Then $|F(f)| \le M \cdot K_\varphi(f)$ for some $M > 0$ and all $f \in L^\varphi(X)$. Hence for $f \in E^\varphi(X) = \ker K_\varphi$ we have $F(f) = 0$, so $F \in E^\varphi(X)^\perp$.

Since $E^\varphi(X)^\perp \subset L^\varphi(X)^\sim_s$, the proof is complete. □

Theorem 3.4. *For $F \in L^\varphi(X)^\sim_s$ we have*

$$\overline{M}_\varphi(F) = P_{M_\varphi}(F) = \|F\|_{\overline{M}_\varphi} = |||F|||_{\overline{M}_\varphi}.$$

Proof: To prove that $\overline{M}_\varphi(F) \ge P_{M_\varphi}(F)$ let $\varepsilon > 0$ be given. Then by Theorem 3.2 there exists $f \in L^\varphi(X)$ with $M_\varphi(f) < \infty$ and such that $P_{M_\varphi}(F) \le |F(f)| + \frac{\varepsilon}{2}$. By Lemma 3.1 there exists $h \in E^\varphi(X)$ such that $M_\varphi(f-h) \le \frac{\varepsilon}{2}$. Since $L^\varphi(X)^\sim_s = E^\varphi(X)^\perp$ (see Corollary 3.3) we get $P_{M_\varphi}(F) \le |F(f-h)| + \frac{\varepsilon}{2}$. Hence $\overline{M}_\varphi(F) \ge |F(f-h)| - M_\varphi(f-h) \ge P_{M_\varphi}(F) - \varepsilon$. In view of Theorem 3.2 the inequality $\overline{M}_\varphi(F) \le P_{M_\varphi}(F)$ holds, so $\overline{M}_\varphi(F) = P_{M_\varphi}(F)$.

Hence $\overline{M}_\varphi(\lambda F) = \lambda \overline{M}_\varphi(F)$ for $\lambda > 0$, because P_{M_φ} is a norm on $L^\varphi(X)^*$. Thus we get

$$\|F\|_{\overline{M}_\varphi} = \inf_{\lambda>0}\Big\{\frac{1}{\lambda}(1+\lambda\overline{M}_\varphi(F))\Big\} = \overline{M}_\varphi(F)$$

and

$$|||F|||_{\overline{M}_\varphi} = \inf\{\lambda > 0 : \overline{M}_\varphi(F) \le \lambda\} = \overline{M}_\varphi(F).$$

□

4. Topological dual of Orlicz-Bochner spaces

We start with some results concerning Mackey topologies of Orlicz-Bochner spaces. Let us recall that the Mackey topology τ_L of a topological vector space (L, ξ) is the finest locally convex topology on L that produces the same continuous linear functionals as the original topology ξ.

The next theorem will be of importance (see [29, Theorem 2.4, Theorem 3.4], [11, Theorem 5.1, Theorem 5.3]).

Theorem 4.1 (cf. [11]).

(i) *The Mackey topology $\tau_{E^\varphi(X)}$ of $(E^\varphi(X), \mathcal{T}_\varphi(X)|_{E^\varphi(X)})$ coincides with the topology $\mathcal{T}_{\overline{\varphi}}(X)|_{E^\varphi(X)}$ induced from $(L^{\overline{\varphi}}(X), \mathcal{T}_{\overline{\varphi}}(X))$, i.e., $\tau_{E^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{E^\varphi(X)}$. Hence $\tau_{E^\varphi(X)}$ is normable iff $\liminf\limits_{t\to\infty} \frac{\varphi(t)}{t} > 0$.*

(ii) *The Mackey topology* $\tau_{L^\varphi(X)}$ *of* $(L^\varphi(X), \mathcal{T}_\varphi(X))$ *coincides with the supremum of* $\mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$ *and the topology* $\pi_\varphi(X)$ *of the seminorm* K_φ, *i.e.,* $\tau_{L^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)} \vee \pi_\varphi(X)$.

Theorem 4.2 (see [29, Corollary 2.5]). *The following statements are equivalent:*

(i) $(L^\varphi(X), \mathcal{T}_\varphi(X))$ *is locally convex;*

(ii) $(E^\varphi(X), \mathcal{T}_\varphi(X)|_{E^\varphi(X)})$ *is locally convex;*

(iii) φ *is equivalent to* $\overline{\varphi}$.

Now, we are ready to state our main result that extends the well known results concerning the dual of scalar Orlicz spaces (cf. [2], [14], [26], [32], [33]).

Theorem 4.3. (i) *Let* $\liminf_{t\to\infty} \frac{\varphi(t)}{t} > 0$. *Then*

$$L^\varphi(X)^* = L^\varphi(X)^\sim_n \oplus L^\varphi(X)^\sim_s.$$

(ii) *Let* $\liminf_{t\to\infty} \frac{\varphi(t)}{t} = 0$. *Then* $L^\varphi(X)^* = L^\varphi(X)^\sim_s$.

PROOF: (i) Let $F \in L^\varphi(X)^*$. Then by Theorem 4.1(i) the functional $F_0 = F|_{E^\varphi(X)}$ restricted to $E^\varphi(X)$ is $\mathcal{T}_{\overline{\varphi}}(X)|_{E^\varphi(X)}$-continuous. Since $E^\varphi(X) \hookrightarrow E^{\overline{\varphi}}(X)$, by the Hahn-Banach extension theorem there exists a $|||\cdot|||_{L^{\overline{\varphi}}(X)}$-continuous linear functional $\overline{F}_0$ on $E^{\overline{\varphi}}(X)$ such that $\overline{F}_0(h) = F_0(h)$ for all $h \in E^\varphi(X)$.

It is known that $E^{\overline{\varphi}} = (L^{\overline{\varphi}})_a$ (= the $|||\cdot|||_{\overline{\varphi}}$ -closed ideal of absolutely continuous elements of $L^{\overline{\varphi}}$) and the identity $(E^{\overline{\varphi}})' = L^{\varphi^*}$ holds (see [18, Theorem 2.3.2]), where $(E^{\overline{\varphi}})'$ stands for the Köthe dual of $E^{\overline{\varphi}}$. Hence by [6, Corollary 4.1], [5, Theorem 7] there exists a unique $g_0 \in L^{\varphi^*}(X^*, X)$ such that $\overline{F}_0(h) = \int_\Omega \langle h(\omega), g_0(\omega)\rangle\, d\mu$ for all $h \in E^{\overline{\varphi}}(X)$. Hence $F_0(h) = \int_\Omega \langle h(\omega), g_0(\omega)\rangle\, d\mu$ for all $h \in E^\varphi(X)$. Thus $F(h) = F_{g_0}(h)$ for all $h \in E^\varphi(X)$, where $F_{g_0} \in L^\varphi(X)^\sim_n$ (see Theorem 2.1). Let $F_s(f) = F(f) - F_{g_0}(f)$ for all $f \in L^\varphi(X)$. Thus $F_s(h) = 0$ for all $h \in E^\varphi(X)$, so $F_s \in E^\varphi(X)^\perp = L^\varphi(X)^\sim_s$ (see Corollary 3.3). Since $L^\varphi(X)^\sim_n \cap L^\varphi(X)^\sim_s = \{0\}$ (see [27, Theorem 2.9]), the identity $L^\varphi(X)^* = L^\varphi(X)^\sim_n \oplus L^\varphi(X)^\sim_s$ holds, as desired.

(ii) In view of Theorem 4.1(ii) the Mackey topology $\tau_{L^\varphi(X)}$ is generated by the seminorm K_φ, so by Corollary 3.3, $L^\varphi(X)^* = (L^\varphi(X), K_\varphi)^* = L^\varphi(X)^\sim_s$. □

Corollary 4.4. *The following statements are equivalent:*

(i) $\liminf_{t\to\infty} \frac{\varphi(t)}{t} = 0$;

(ii) $L^\varphi(X)^* = L^\varphi(X)^\sim_s$;

(iii) $L^\varphi(X)^\sim_n = \{0\}$.

PROOF: (i) ⇒ (ii) See Theorem 4.3.

(ii) ⇒ (iii) We have $L^\varphi(X)^\sim_n = L^\varphi(X)^* \cap L^\varphi(X)^\sim_n = \{0\}$.

(iii) ⇒ (i) Let $\liminf_{t\to\infty} \frac{\varphi(t)}{t} > 0$. Then by Theorem 2.1. $L^\varphi(X)^\sim_n \neq \{0\}$. □

Corollary 4.5. *The following statements are equivalent:*

(i) $\varphi \in \Delta_2$;

(ii) $L^\varphi(X)^\sim_s = \{0\}$;

(iii) $L^\varphi(X)^* = L^\varphi(X)^\sim_n$.

Proof: (i) $\Rightarrow$ (ii) We know that $E^\varphi(X) = L^\varphi(X)$, so $L^\varphi(X)^\sim_s = E^\varphi(X)^\perp = \{0\}$ (see Corollary 3.3).

(ii) $\Rightarrow$ (iii) It follows from Theorem 4.3.

(iii) $\Rightarrow$ (ii) We have $L^\varphi(X)^\sim_s = L^\varphi(X)^* \cap L^\varphi(X)^\sim_s = \{0\}$.

(ii) $\Rightarrow$ (i) Assume that $\varphi \notin \Delta_2$. Then $E^\varphi(X) \subsetneqq L^\varphi(X)$. Since $E^\varphi(X) = \ker K_\varphi$, $E^\varphi(X)$ is K_φ-closed subspace of $L^\varphi(X)$. Hence for each $f \in L^\varphi(X) \setminus E^\varphi(X)$ there exists $F \in (L^\varphi(X), K_\varphi)^* = L^\varphi(X)^\sim_s$ such that $F(f) = 1$ and $F(h) = 0$ for all $h \in E^\varphi(X)$ (see [36, 2.3.9]). Thus $L^\varphi(X)^\sim_s \neq \{0\}$. □

Combining Theorem 4.3, Corollary 4.4 and Corollary 4.5 we get:

Corollary 4.6 (cf. [10], [11], [34]). *The following statements are equivalent:*

(i) $L^\varphi(X)^* = \{0\}$;

(ii) $\liminf\limits_{t\to\infty} \frac{\varphi(t)}{t} = 0$ *and* $\varphi \in \Delta_2$.

The basic properties of the conjugate modular $\overline{M}_\varphi$ and the norms $\|\cdot\|_{\overline{M}_\varphi}$, $|||\cdot|||_{\overline{M}_\varphi}$ and P_{M_φ} on $L^\varphi(X)^*$ are described by the following theorem.

Theorem 4.7. *Assume that* $\liminf\limits_{t\to\infty} \frac{\varphi(t)}{t} > 0$ *and* $\varphi \notin \Delta_2$. *Let* $F = F_g + F_s$, *where* $g \in L^{\varphi^*}(X^*, X)$ *and* $F_s \in L^\varphi(X)^\sim_s$. *Then*

(i) $\overline{M}_\varphi(F) = \overline{M}_\varphi(F_g) + \overline{M}_\varphi(F_s)$;

(ii) $|||F|||_{\overline{M}_\varphi} = \inf\{\lambda > 0\colon m_{\varphi^*}(\vartheta(g)/\lambda) + \lambda^{-1}\overline{M}_\varphi(F_s) \le 1\}$;

(iii) $\|F\|_{\overline{M}_\varphi} = \|F_g\|_{\overline{M}_\varphi} + \|F\|_{\overline{M}_\varphi}$;

(iv) $P_{\overline{M}_\varphi}(F) = P_{M_\varphi}(F_g) + P_{M_\varphi}(F_s)$.

Proof: (i) To prove that $\overline{M}_\varphi(F) \ge \overline{M}_\varphi(F_g) + \overline{M}_\varphi(F_s)$ let $\varepsilon > 0$ be given. Then there exists $h \in L^\varphi(X)$ with $M_\varphi(h) < \infty$ and such that $\overline{M}_\varphi(F_g) - \varepsilon \le F_g(h) - M_\varphi(h)$. Since $\operatorname{supp} E^\varphi = \Omega$, there exists a sequence (Ω_n) in Σ such that $\Omega_n \uparrow \Omega$, $\chi_{\Omega_n} \in E^\varphi$ for $n \in \mathbb{N}$. Let

$$h^{(n)}(\omega) = \begin{cases} h(\omega) & \text{if } \widetilde{h}(\omega) \le n \text{ and } \omega \in \Omega_n, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $h^{(n)} \in E^\varphi(X)$ for $n \in \mathbb{N}$ and $\widetilde{h^{(n)}} \uparrow \widetilde{h}$. Hence $F_g(h^{(n)}) \to F_g(h)$ and $M_\varphi(h^{(n)}) \uparrow M_\varphi(h)$, so there exists $n_0 \in \mathbb{N}$ such that for $h_1 = h^{(n_0)} \in E^\varphi(X)$ we have

$$\overline{M}_\varphi(F_g) - \frac{\varepsilon}{4} \le F_g(h_1) - M_\varphi(h_1). \tag{1}$$

Moreover, there exists $h_2 \in L^\varphi(X)$ with $M_\varphi(h_2) < \infty$ and such that

$$\overline{M}_\varphi(F_s) - \frac{\varepsilon}{4} \le F_s(h_2) - M_\varphi(h_2). \tag{2}$$

Since $\operatorname{supp} E^\varphi = \Omega$ there exists a sequence (u_n) in E^φ such that $0 \le u_n \uparrow \widetilde{h}_2$ (see [17, Lemma 4.3.1]). Let $B_n = \{\omega \in \Omega\colon 2u_n(\omega) \ge \widetilde{h}_2(\omega)\}$. Then $B_n \uparrow_\mu \Omega$, $\chi_{B_n}\widetilde{h}_2 \uparrow \widetilde{h}_2$ and $\chi_{B_n}\widetilde{h}_2 \le 2u_n \in E^\varphi$, so $\chi_{B_n} h_2 \in E^\varphi(X)$. Let $A_n = \Omega \setminus B_n$ for $n \in \mathbb{N}$. Then $A_n \searrow_\mu \emptyset$, so $F_g(\chi_{A_n} h_1) \to 0$ and $F_g(\chi_{A_n} h_2) \to 0$, because $\chi_{A_n}\widetilde{h}_1 \xrightarrow{(o)} 0$, $\chi_{A_n}\widetilde{h}_2 \xrightarrow{(o)} 0$. Choose $n_0 \in \mathbb{N}$ such that

$$|F_g(\chi_{A_{n_0}} h_1)| \le \frac{\varepsilon}{4} \quad \text{and} \quad |F_g(\chi_{A_{n_0}} h_2)| \le \frac{\varepsilon}{4}\,. \tag{3}$$

Let us put

$$h_0(\omega) = \begin{cases} h_1(\omega) & \text{if } \omega \in \Omega \setminus A_{n_0} = B_{n_0}, \\ h_2(\omega) & \text{if } \omega \in A_{n_0}. \end{cases}$$

Then $M_\varphi(h_0) \le M_\varphi(\chi_{\Omega\setminus A_{n_0}} h_1) + M_\varphi(\chi_{A_{n_0}} h_2)$ and since $h_1 \in E^\varphi(X)$ and $\chi_{\Omega\setminus A_{n_0}} h_2 \in E^\varphi(X)$, by (1), (2) and (3) and Corollary 3.3 we get

$$\begin{aligned}
\overline{M}_\varphi(F) &\ge F(h_0) - M_\varphi(h_0) \\
&\ge F_g(\chi_{\Omega\setminus A_{n_0}} h_1) + F_g(\chi_{A_{n_0}} h_2) + F_s(\chi_{\Omega\setminus A_{n_0}} h_1) + F_s(\chi_{A_{n_0}} h_2) \\
&\qquad - M_\varphi(\chi_{\Omega\setminus A_{n_0}} h_1) - M_\varphi(\chi_{A_{n_0}} h_2) \\
&\ge (F_g(h_1) - M_\varphi(h_1)) + F_g(\chi_{A_{n_0}} h_1) + F_g(\chi_{A_{n_0}} h_2) + (F_s(\chi_{A_{n_0}} h_2) \\
&\qquad + F_s(\chi_{\Omega\setminus A_{n_0}} h_2) - M_\varphi(h_0)) \\
&\ge \overline{M}_\varphi(F_g) - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} + \overline{M}_\varphi(F_s) - \frac{\varepsilon}{4} = \overline{M}_\varphi(F_g) + \overline{M}_\varphi(F_s) - \varepsilon.
\end{aligned}$$

On the other hand, since $\overline{M}_\varphi(F_s) = \sup\{|F_s(f)| : M_\varphi(f) < \infty\}$ (see Theorems 3.2 and 3.4) we get

$$\begin{aligned}
\overline{M}_\varphi(F) &\le \sup\{(|F_g(f)| - M_\varphi(f)) + |F_s(f)| : f \in L^\varphi(X), M_\varphi(f) < \infty\} \\
&\le \sup\{|F_g(f)| - M_\varphi(f) : M_\varphi(f) < \infty\} + \sup\{|F_s(f)| : M_\varphi(f) < \infty\} \\
&= \overline{M}_\varphi(F_g) + \overline{M}_\varphi(F_s).
\end{aligned}$$

Thus the identity (i) is proved.

By making use of (i) and Theorem 2.4, and Theorem 3.4 we have

$$\begin{aligned}
|||F|||_{\overline{M}_\varphi} &= \inf\{\lambda > 0 : \overline{M}_\varphi(\lambda^{-1} F_g) + \overline{M}(\lambda^{-1} F_s) \le 1\} \\
&= \inf\{\lambda > 0 : m_{\varphi^*}(\lambda^{-1}\vartheta(g)) + \lambda^{-1}\overline{M}_\varphi(F_s) \le 1\}
\end{aligned}$$

(iii) Similarly, in view of (i) and Theorem 2.4 and Theorem 3.4 we have:

$$\begin{aligned}\|F\|_{\overline{M}_\varphi} &= \inf_{\lambda>0}\{\lambda^{-1}(\overline{M}_\varphi(\lambda F_g)+\lambda\overline{M}_\varphi(F_s)+1)\}\\ &= \inf_{\lambda>0}\{\lambda^{-1}(\overline{M}_\varphi(\lambda F_g)+1)+\overline{M}_\varphi(F_s)\}\\ &= \|F_g\|_{\overline{M}_\varphi}+\overline{M}_\varphi(F_s)=\|F_g\|_{\overline{M}_\varphi}+\|F_s\|_{\overline{M}_\varphi}.\end{aligned}$$

(iv) To prove that $P_{M_\varphi}(F)\le P_{M_\varphi}(F_g)+P_{M_\varphi}(F_s)$ let $\varepsilon>0$ be given. Then in view of Theorem 2.1 there exists $f_1\in E^\varphi(X)$ with $M_\varphi(f_1)\le 1$ and such that

$$P_{M_\varphi}(F_g)-\frac{\varepsilon}{8}\le F_g(f_1). \tag{4}$$

In view of Theorem 2.2, F_g is modular continuous, so there exists $0<\delta<M_\varphi(h_1)$ such that $|F_g(f)|\le\frac{\varepsilon}{8}$ whenever $M_\varphi(f)\le\delta$. Moreover, there exists a subset $A\in\Sigma$ such that $M_\varphi(\chi_A f_1)=\delta$, because μ is atomless. Hence $M_\varphi(f_1)=M_\varphi(\chi_A f_1)+M_\varphi(\chi_{\Omega\setminus A}f_1)$, so $\eta=M_\varphi(\chi_{\Omega\setminus A}f_1)<1$. Let us put $f_1'=\chi_{\Omega\setminus A}f_1$. Then $F_g(f_1')=F_g(f_1)-F(\chi_A f_1)\ge F_g(f_1)-\frac{\varepsilon}{8}$. Thus by (4)

$$P_{M_\varphi}(F_g)-\frac{\varepsilon}{4}\le F_g(f_1'). \tag{5}$$

Moreover, there exists $f_2\in L^\varphi(X)$ with $M_\varphi(f_2)\le 1$ and such that

$$P_{M_\varphi}(F_s)-\frac{\varepsilon}{4}\le F_s(f_2). \tag{6}$$

Then there exists a sequence (v_n) in E^φ such that $0\le v_n\uparrow\widetilde{f}_2$ (see [17, Lemma 11.3.1]). Let $C_n=\{\omega\in\Omega\colon 2v_n(\omega)\ge\widetilde{f}_2(\omega)\}$. Then $C_n\uparrow\Omega$, $\chi_{C_n}\widetilde{f}_2\uparrow\widetilde{f}_2$ and $\chi_{C_n}\widetilde{f}_2\le 2v_n\in E^\varphi$, so $\chi_{C_n}f_2\in E^\varphi(X)$. Let $D_n=\Omega\setminus C_n$. Since $D_n\searrow_\mu\emptyset$, there exists $n_0\in\mathbb{N}$ such that

$$|F_g(\chi_{D_{n_0}}f_1')|\le\frac{\varepsilon}{4}\ \text{ and }\ |F_g(\chi_{D_{n_0}}f_2)|\le\frac{\varepsilon}{4} \tag{7}$$

and

$$M_\varphi(\chi_{D_{n_0}}f_2)\le 1-\eta. \tag{8}$$

Let us put

$$f_0(\omega)=\begin{cases} f_1'(\omega) & \text{if } \omega\in\Omega\setminus D_{n_0}=C_{n_0}\\ f_2(\omega) & \text{if } \omega\in D_{n_0}.\end{cases}$$

Then $M_\varphi(f_0)\le M_\varphi(\chi_{\Omega\setminus D_{n_0}}f_1')+M_\varphi(\chi_{D_{n_0}}f_2)\le M_\varphi(f_1')+M_\varphi(\chi_{D_{n_0}}f_2)\le\eta+(1-\eta)=1$. Since $f_1'\in E^\varphi(X)$ and $\chi_{\Omega\setminus D_{n_0}}f_2=\chi_{C_{n_0}}f_2\in E^\varphi(X)$ by (5), (6),

(7) and (8) we get

$$\begin{aligned} P_{M_\varphi}(F) &\geq F(h_0) = F_g(h_0) + F_s(h_0) \\ &= F_g(\chi_{\Omega\setminus D_{n_0}} f_1') + F_g(\chi_{D_{n_0}} f_2) + F_s(\chi_{\Omega\setminus D_{n_0}} f_1') + F_s(\chi_{D_{n_0}} f_2) \\ &= F_g(f_1') - F_g(\chi_{D_{n_0}} f_1') + F_g(\chi_{D_{n_0}} f_2) + F_s(\chi_{D_{n_0}} f_2) \\ &\quad + F_s(\chi_{\Omega\setminus D_{n_0}} f_2) \\ &\geq P_{M_\varphi}(F_s) - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} + P_{M_\varphi}(F_s) - \frac{\varepsilon}{4} \\ &= P_{M_\varphi}(F_g) + P_{M_\varphi}(F_s) - \varepsilon. \end{aligned}$$

Since $P_{M_\varphi}(F) \leq P_{M_\varphi}(F_g) + P_{M_\varphi}(F_s)$, the proof is complete. □

We know that $(L^\varphi(X), P_{M_\varphi})$ is a Banach space (see Theorem 1.3). Assume now that $\liminf_{t\to\infty} \frac{\varphi(t)}{t} > 0$. In view of Theorem 4.3 one can define a projection $P_n\colon L^\varphi(X)^* \to L^\varphi(X)^*$ by $P_n(F) = F_g$, whenever $F = F_g + F_s \in L^\varphi(X)^* = L^\varphi(X)^\sim_n \oplus L^\varphi(X)^\sim_s$, and for each $F \in L^\varphi(X)^*$ by Theorem 4.7 we have $P_{M_\varphi}(F) = P_{M_\varphi}(P_n(F)) + P_{M_\varphi}(F - P_n(F))$ (resp. $\|F\|_{\overline{M}_\varphi} = \|P_n(F)\|_{\overline{M}_\varphi} + \|F - P_n(F)\|_{\overline{M}_\varphi}$). It means that P_n is a continuous *L-projection* in $(L^\varphi(X)^*, P_{M_\varphi})$ (resp. $(L^\varphi(X)^*, \|\cdot\|_{\overline{M}_\varphi})$) (see [3, Definition 1.3]).

Moreover, $L^\varphi(X)^\sim_n$ and $L^\varphi(X)^\sim_s$ are topologically complementary in $(L^\varphi(X)^*, P_{M_\varphi})$ (resp. $(L^\varphi(X)^*, \|\cdot\|_{\overline{M}_\varphi})$); see [36, 5.3]. It follows that both $L^\varphi(X)^\sim_n$ and $L^\varphi(X)^\sim_s$ are closed in $(L^\varphi(X)^*, P_{M_\varphi})$ (resp. $(L^\varphi(X)^*, \|\cdot\|_{\overline{M}_\varphi})$) (see [36, Remark 5.3.9]).

5. Applications

In this section we present some consequences of Theorems 4.3 and 4.7. First we shall show that continuous linear functionals on $E^\varphi(X)$ have the unique P_{M_φ}-norm preserving extension to $L^\varphi(X)$ (cf. [32, Theorem 5.3])

Theorem 5.1. *Assume that* $\liminf_{t\to\infty} \frac{\varphi(t)}{t} > 0$ *and* $\varphi \notin \Delta_2$. *Let* G *be a* $|\cdot|_\varphi$-*continuous linear functional on* $E^\varphi(X)$. *Then there exists a unique* $|\cdot|_\varphi$-*continuous linear functional* F *on* $L^\varphi(X)$ *such that* $F(h) = G(h)$ *for all* $h \in E^\varphi(X)$ *and*

$$P_{M_\varphi}(F) = P_{M_\varphi}(G) = \sup\{|G(h)| : h \in E^\varphi(X), M_\varphi(h) \leq 1\}.$$

Proof: In view of Theorem 4.1(i) $(E^\varphi(X), \mathcal{T}_\varphi(X)|_{E^\varphi(X)})^* = (E^\varphi(X), \mathcal{T}_{\overline{\varphi}}(X)|_{E^\varphi(X)})^*$. It is known that $(E^\varphi)' = L^{\varphi^*}$ (see [25, Theorem 3.1]), so by [6, Corollary 4.1] there exists $g \in L^{\varphi^*}(X^*, X)$ such that $G(h) = \int_\Omega \langle h(\omega), g(\omega)\rangle\, d\mu$ for $h \in E^\varphi(X)$. Let us put

$$F(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle\, d\mu \quad \text{for all} \quad f \in L^\varphi(X).$$

Then $F(h) = G(h)$ for all $h \in E^{\varphi}(X)$ and $F \in L^{\varphi}(X)^{\sim}_n$ (see Theorem 2.2). Moreover, by Theorem 2.4(iv) $P_{M_\varphi}(F) = P_{M_\varphi}(G)$.

Now assume that $\overline{F}$ is another such extension of G, and let $F_1 = \overline{F} - F$. Then $F_1 \in L^{\varphi}(X)^{\sim}_s$ (see Corollary 3.3) and by Theorem 4.7 $P_{M_\varphi}(F) = P_{M_\varphi}(\overline{F}) = P_{M_\varphi}(F) + P_{M_\varphi}(F_1)$, so $P_{M_\varphi}(F_1) = 0$. Hence $F_1 = 0$, so $F = \overline{F}$, as desired. □

The next theorem gives an inner characterization of singular functionals on $L^{\varphi}(X)$ in terms of their norms $\|\cdot\|_{\overline{M}_\varphi}$ and $|||\cdot|||_{\overline{M}_\varphi}$ (cf. [32, Theorem 3.5]).

Theorem 5.2. *Assume that* $\lim\limits_{t\to\infty} \frac{\varphi(t)}{t} = \infty$. *Then for* $F \in L^{\varphi}(X)^*$ *the following statements are equivalent:*

(i) $F \in L^{\varphi}(X)^{\sim}_s$;
(ii) $\|F\|_{\overline{M}_\varphi} = |||F|||_{\overline{M}_\varphi}$.

Proof: (i) $\Rightarrow$ (ii) See Theorem 3.4.

(ii) $\Rightarrow$ (i) Let $F = F_g + F_s$, where $g \in L^{\varphi}(X)^{\sim}_n$ and $F_s \in L^{\varphi}(X)^{\sim}_s$. Then by Theorem 4.3

$$|||F|||_{\overline{M}_\varphi} \le |||F_g|||_{\overline{M}_\varphi} + |||F_s|||_{\overline{M}_\varphi} \le \|F_g\|_{\overline{M}_\varphi} + \|F_s\|_{\overline{M}_\varphi} = \|F\|_{\overline{M}_\varphi}.$$

Since $|||F_s|||_{\overline{M}_\varphi} = \|F_s\|_{\overline{M}_\varphi}$ (see Theorem 3.4) we conclude that $|||F_g|||_{\overline{M}_\varphi} = \|F_g\|_{\overline{M}_\varphi}$, so in view of Theorem 2.4 the identity $|||\vartheta(g)|||_{\varphi^*} = \|\vartheta(g)\|_{\varphi^*}$ holds. It follows that $\vartheta(g) = 0$ (see [32, Lemma 1]). Hence $g = 0$, i.e., $F = F_s \in L^{\varphi}(X)^{\sim}_s$, as desired. □

Theorem 5.3. *Assume that* $\liminf\limits_{t\to\infty} \frac{\varphi(t)}{t} > 0$ *and* $\varphi \notin \Delta_2$. *Let* $F = F_{g_0} + F^0_s$, *where* $g_0 \in L^{\varphi^*}(X^*, X)$ *and* $F^0_s \in L^{\varphi}(X)^{\sim}_s$. *Then* F_{g_0} *(resp.* F^0_s*) is the unique best approximant of* F *with respect to* $L^{\varphi}(X)^{\sim}_n$ *(resp.* $L^{\varphi}(X)^{\sim}_s$*), whenever* $L^{\varphi}(X)^*$ *is provided with the norms* P_{M_φ} *and* $\|\cdot\|_{\overline{M}_\varphi}$.

Proof: In view of Theorem 4.7, for any $g \in L^{\varphi^*}(X^*, X)$ we have $P_{M_\varphi}(F - F_g) = P_{M_\varphi}(F_{g_0} + F^0_s - F_g) = P_{M_\varphi}(F_{g_0} - F_g) + P_{M_\varphi}(F^0_s)$. Hence $\mathrm{dist}_{P_{M_\varphi}}(F, L^{\varphi}(X)^{\sim}_n) = P_{M_\varphi}(F^0_s) = P_{M_\varphi}(F - F_{g_0})$.

On the other hand, assume that $\mathrm{dist}_{P_{M_\varphi}}(F, L^{\varphi}(X)^{\sim}_n) = P_{M_\varphi}(F - F_g)$ for some $g \in L^{\varphi^*}(X^*, X)$. Hence $P_{M_\varphi}(F^0_s) = P_{M_\varphi}(F - F_g) = P_{M_\varphi}(F_{g_0} + F^0_s - F_g) = P_{M_\varphi}(F_{g_0} - F_g) + P_{M_\varphi}(F^0_s)$. It follows that $P_{M_\varphi}(F_{g_0} - F_g) = 0$, so $F_{g_0} = F_g$, as desired.

Similarly in the other cases. □

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