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# Characteristic zero loop space homology for certain two-cones

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*Abstract.* Given a principal ideal domain $R$ of characteristic zero, containing $1/2$, and a two-cone $X$ of appropriate connectedness and dimension, we present a sufficient algebraic condition, in terms of Adams-Hilton models, for the Hopf algebra $FH(\Omega X; R)$ to be isomorphic with the universal enveloping algebra of some $R$-free graded Lie algebra; as usual, $F$ stands for free part, $H$ for homology, and $\Omega$ for the Moore loop space functor.

*Keywords:* two-cone, Moore loop space, differential graded Lie algebra, free Lie algebra on a graded module, universal enveloping algebra, Hopf algebra

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Throughout the paper modules and tensor products are taken over a principal ideal domain $R$ of characteristic zero, containing $1/2$; as usual, $\varrho(R)$ denotes the least prime (or $\infty$) not invertible in $R$. Following [2], [4], given integer $k \geq 1$ and real $r > 0$, a CW-complex will be called $(k, r)$-*mild* if it is finite, $k$-connected, and its dimension does not exceed $kr$. Halperin [4] shows that, if $R \subseteq \mathbb{Q}$, and $X$ is a $(k, \varrho(R))$-mild CW-complex (integer $k \geq 1$) whose Moore loop space $\Omega X$ has an $R$-torsion free homology $H(\Omega X; R)$ with primitives $P$, then the natural arrow $U(P) \to H(\Omega X; R)$ is an isomorphism of graded Hopf algebras; here and for the remainder of the paper, $U$ denotes, of course, the universal enveloping algebra functor over $R$. In particular, for $R = \mathbb{Q}$, Halperin recovers a well-known result established by Milnor and Moore [5] for simply connected pointed topological spaces $X$. On the other hand, if $R$ is a subring of $\mathbb{Q}$, containing $1/6$, and $X$ is a two-cone, Anick [1] gives reasonable algebraic conditions for $H(\Omega X; R)$ to be $R$-free and thereby to understand its algebra structure fairly well. His requirements are that all of the so-called implicit primes for $X$ be invertible in $R$, and a certain graded Lie algebra $L_X$, associated with $X$, be $R$-free. Our purpose here is to exhibit the Hopf algebra $FH(\Omega X; R)$ as the universal enveloping algebra of some $R$-free graded Lie algebra, whatever the implicit primes of $X$ and however $L_X$ might be, at the cost of restricting the class of two-cones $X$. Here and for the remainder of the paper, $F$ stands for free part, and the Hopf algebra structure on $FH(\Omega X; R)$ is acquired via the comultiplication

$$FH(\Omega X; R) \xrightarrow{FH(\Delta_X)} FH(\Omega X \times \Omega X; R) \xleftarrow{\cong} FH(\Omega X; R) \otimes FH(\Omega X; R),$$

where $\Delta_X : \Omega X \to \Omega X \times \Omega X$ is the diagonal, and the isomorphism $FH(\Omega X; R) \otimes FH(\Omega X; R) \xrightarrow{\cong} FH(\Omega X \times \Omega X; R)$ is given by the Künneth and Eilenberg-Zilber theorems.

In order to state our result, recall that a *two-cone* $X$ is the cofiber of a map between two finite type wedges of spheres,

$$V \xrightarrow{\phi} W \to X, \quad V = \vee_{i \in I} S^{m_i+1}, \quad W = \vee_{j \in J} S^{n_j+1}, \quad \phi = \vee_{i \in I} \phi_i,$$

subject to well-known reasonable standard constraints on the index sets $I$ and $J$, and the dimensions $m_i$ and $n_j$ ([1]); since we will only be concerned with simply connected spaces, we assume positive integers $m_i$ and $n_j$ throughout. Recall also that the tensor algebra $T(x_i)_{i \in I}$, with generators $x_i$ of dimension $m_i$ and trivial differential, is an Adams-Hilton model for $V$ over $R$, so the Pontrjagin algebra $H(\Omega V; R)$ is simply $T(x_i)_{i \in I}$ ([1]); likewise, $H(\Omega W; R) = T(y_j)_{j \in J}$, with generators $y_j$ of dimension $n_j$. Finally, for each $i \in I$, let $\xi_i$ denote the element $H(\Omega\phi; R)(x_i)$ of $H(\Omega W; R)_{m_i}$. The notation in this paragraph will be kept throughout the remainder of the paper.

We are now in a position to state our main result.

**Theorem 1.** *If $(T(x_i)_{i \in I}, 0)$ and $(T(y_j)_{j \in J}, 0)$ are Adams-Hilton models for $V$ and $W$, respectively, such that, for some partition $J = J' \cup J''$, with non-empty $J''$, each $\xi_i$ belongs to the $R$-span of the commutators $[y_{j_1}, [\dots, [y_{j_t}, y_j] \dots]]$, with (not necessarily distinct) $j_1, \dots, j_t \in J'$ and $j \in J''$, then the Hopf algebra $FH(\Omega X; R)$ is isomorphic with the universal enveloping algebra of some $R$-free graded Lie algebra, provided that the dimension of $X$ does not exceed $n\varrho(R)$, where $n = \min\{n_j : j \in J\}$.*

Recalling further that the differential tensor algebra $(T(x'_i, y_j)_{i \in I, j \in J}, d)$, with generators $x'_i$ of dimension $m_i + 1$, $dx'_i = \xi_i$ and $dy_j = 0$, may be taken to be an Adams-Hilton model for $X$ over $R$ ([1]), in the light of Anick's work [2], Theorem 1 is actually the topological counterpart of the following purely algebraic fact.

**Theorem 2.** *With reference to the previous notation, if for some partition $J = J' \cup J''$, with non-empty $J''$, each $\xi_i$ belongs to the $R$-span of the commutators $[y_{j_1}, [\dots, [y_{j_t}, y_j] \dots]]$, with (not necessarily distinct) $j_1, \dots, j_t \in J'$ and $j \in J''$, then the natural arrow $UFH(\mathbb{L}(x'_i, y_j)_{i \in I, j \in J}, d) \to FH(T(x'_i, y_j)_{i \in I, j \in J}, d)$ is an isomorphism of graded Hopf algebras.*

*In particular, the graded Lie algebra $FH(\mathbb{L}(x'_i, y_j)_{i \in I, j \in J}, d)$ maps isomorphically onto the sub Lie algebra of primitive elements of $FH(T(x'_i, y_j)_{i \in I, j \in J}, d)$.*

Of course, $\mathbb{L}$ denotes the free Lie algebra functor over $R$, and, writing for short $(A, d)$ instead of $(T(x'_i, y_j)_{i \in I, j \in J}, d)$, the Hopf algebra structure on $FH(A, d)$ is acquired by means of the comultiplication

$$FH(A, d) \xrightarrow{FH(\Delta)} FH((A, d) \otimes (A, d)) \xleftarrow{\cong} FH(A, d) \otimes FH(A, d),$$

where $\Delta : (A, d) \to (A, d) \otimes (A, d)$ is the comultiplication on $(A, d)$, and the isomorphism $FH(A, d) \otimes FH(A, d) \xrightarrow{\cong} FH((A, d) \otimes (A, d))$ is given by the Künneth theorem. It might also be worth remarking here that no mildness condition is required in the statement of Theorem 2.

We begin by proving Theorem 2.

Proof of Theorem 2: Since for an empty $J'$ the result is a special case of Theorem 1 in [6], we may (and will) henceforth assume $J'$ non-empty, as well. For the sake of brevity, write $(L, d)$ for $(\mathbb{L}(x_i', y_j)_{i\in I, j\in J}, d)$ and note that the inclusion $\iota : (\mathbb{L}(y_j)_{j\in J'}, 0) \hookrightarrow (L, d)$ is a right inverse for the surjection $\pi : (L, d) \to (\mathbb{L}(y_j)_{j\in J'}, 0)$, sending the generators $y_j$, $j \in J'$, identically onto themselves, and the remaining ones to zero. Then $H(\iota)$ is a right inverse for $H(\pi)$ and there results a trivial connecting morphism in the long exact homology sequence associated with the (right split) short exact sequence of differential graded Lie algebras

$$0 \to (K, d) \xrightarrow{\varkappa} (L, d) \xrightarrow{\pi} (\mathbb{L}(y_j)_{j\in J'}, 0) \to 0, \tag{1}$$

in which $(K, d)$ is, of course, the kernel of $\pi$. Consequently,

$$0 \to H(K, d) \xrightarrow{H(\varkappa)} H(L, d) \xrightarrow{H(\pi)} \mathbb{L}(y_j)_{j\in J'} \to 0$$

is a short exact sequence of graded Lie algebras with right $R$-splitting $H(\iota)$, yielding another short exact sequence of graded Lie algebras

$$0 \to FH(K, d) \xrightarrow{FH(\varkappa)} FH(L, d) \xrightarrow{FH(\pi)} \mathbb{L}(y_j)_{j\in J'} \to 0, \tag{2}$$

with a right $R$-splitting induced by $H(\iota)$. Since both (1) and (2) involve $R$-free objects of finite type, the corresponding universal enveloping algebras form, respectively, short exact sequences of homology Hopf algebras ([3]). Thus, (1) yields

$$U(K, d) \otimes (T(y_j)_{j\in J'}, 0) \xrightarrow{\cong} U(L, d),$$

as left $U(K, d)$-modules and right $(T(y_j)_{j\in J'}, 0)$-comodules, under $U(\varkappa) \otimes U(\iota)$ followed by multiplication; the fact that $dy_j = 0$, for $j \in J'$, is essential in identifying $U(K, d) \otimes (T(y_j)_{j\in J'}, 0)$ and $U(L, d)$ as *differential* objects. Consequently,

$$FHU(K, d) \otimes T(y_j)_{j\in J'} \xrightarrow{\cong} FHU(L, d), \tag{3}$$

by the Künneth theorem. Similarly, (2) yields

$$UFH(K, d) \otimes T(y_j)_{j\in J'} \xrightarrow{\cong} UFH(L, d), \tag{4}$$

as left $UFH(K, d)$-modules and right $T(y_j)_{j\in J'}$-comodules. We now turn to the natural arrow $UFH(K, d) \to FHU(K, d)$ and prove it an isomorphism of graded

Hopf algebras; the first statement in Theorem 2 then follows at once by (3) and (4). To show $UFH(K,d)$ and $FHU(K,d)$ isomorphic under the natural arrow, note that $K$ is the free graded Lie algebra on all commutators $[y_{j_1},[\dots,[y_{j_t},x'_i]\dots]]$ and $[y_{j_1},[\dots,[y_{j_t},y_j]\dots]]$, with (not necessarily distinct) $j_1,\dots,j_t \in J'$, $i \in I$ and $j \in J''$. Recalling that all $dy_j = 0$ and each $dx'_i = \xi_i$ was assumed to lie in the $R$-span of the second kind of those commutators, it follows that the $R$-free module spanned by the generators of $K$ is stable under $d$, so the natural arrow $UFH(K,d) \to FHU(K,d)$ is indeed an isomorphism of graded Hopf algebras ([6]). This proves the first statement in the theorem. The second now follows at once by the Poincaré-Birkhoff-Witt theorem. □

We are now in a position to derive Theorem 1.

Proof of Theorem 1: Recall the notation $(L,d)$, made in the preceding proof. Since $X$ is $(n,\varrho(R))$-mild, it turns out that $(L,d)$ is one of the ingredients of Anick's model for $X$ over $R$ ([2]). The object $(L,d)$ actually comes with a quasi-isomorphism of differential graded algebras $\theta : U(L,d) \xrightarrow{\simeq} C_*(\Omega X;R)$, which preserves the diagonal up to differential graded algebra homotopy. Thus, $H(\theta) : HU(L,d) \to H(\Omega X;R)$ is an isomorphism of graded algebras, identifying $H(\Delta_L)$ with $H(\Delta_X)$, where $\Delta_L : U(L,d) \to U(L,d)\otimes U(L,d)$ and $\Delta_X : \Omega X \to \Omega X \times \Omega X$ are the respective diagonals. Consequently, $FH(\theta)$ identifies $FHU(L,d)$ and $FH(\Omega X;R)$ as Hopf algebras, and nothing remains but apply Theorem 2. □

We now consider some examples. In what follows, the condition $\dim X \le n_1\varrho(R)$ is, of course, to be implicitly assumed, whenever the Hopf algebra $FH(\Omega X;R)$ is under consideration. As one might expect, a first application is related to the Whitehead product.

**Example 3.** Let $I$ be finite, let $J = \{1,2\}$, and set all $m_i = n_1 + n_2$, with $n_2 > n_1$. As an attaching map $\phi_i$ we might, for instance, take the Brouwer degree $r_i$ self-map on $S^{n_1+n_2+1}$, followed by the Whitehead bracket $\omega_{n_1,n_2} : S^{n_1+n_2+1} \to S^{n_1+1} \vee S^{n_2+1}$. Thus, $X$ is the cofiber of a map $\vee_{i\in I} S^{n_1+n_2+1} \to S^{n_1+1}\vee S^{n_2+1}$. By a degree argument, $\xi_i = r_i[y_1,y_2]$, for some $r_i \in R$, whatever $i \in I$, so the theorems apply with $J' = \{1\}$ and $J'' = \{2\}$. For $I = \{1\}$ or $I = \{1,2\}$, the computations are quite feasible. Since the contribution of the generators with indices $i$ for which $r_i = 0$ is obvious, assume all $r_i \ne 0$.

Thus, for $I = \{1\}$, $FH(\mathbb{L}(x'_1,y_1,y_2),d) = \mathbb{L}(y_2) \oplus \mathbb{L}(y_1)$, as Lie algebras, and $UFH(\mathbb{L}(x'_1,y_1,y_2),d)$, $FH(T(x'_1,y_1,y_2),d)$ and $FH(\Omega X;R)$ may all appropriately be identified to $T(y_2)\otimes T(y_1)$.

For $I = \{1,2\}$, let $x = r'_2 x'_1 - r'_1 x'_2$, where $r'_i = r_i/r$, $r$ being the greatest common divisor of $r_1$ and $r_2$. Then $FH(\mathbb{L}(x'_1,x'_2,y_1,y_2),d) = \mathbb{L}(y_2, \mathrm{ad}^k(y_1)(x))_{k=0,1,2,\dots} \oplus \mathbb{L}(y_1)$, as $R$-modules, but *not* as Lie algebras, for $[y_1,\mathrm{ad}^k(y_1)(x)] = \mathrm{ad}^{k+1}(y_1)(x)$; as for $UFH(\mathbb{L}(x'_1,x'_2,y_1,y_2),d)$, $FH(T(x'_1,x'_2,y_1,y_2),d)$ and $FH(\Omega X;R)$, they may now all appropriately be identified to $T(y_2,\mathrm{ad}^k(y_1)(x))_{k=0,1,2,\dots}\otimes T(y_1)$. □

The next example is a generalization of the previous one.

**Example 4.** Let again $I$ be finite and $J = \{1,2\}$, and now set all $m_i = n_1p + n_2$, with $n_2 > n_1p$ and integer $p \geq 1$. Thus, $X$ is the cofiber of a map $\vee_{i\in I} S^{n_1p+n_2+1} \to S^{n_1+1} \vee S^{n_2+1}$, $\xi_i = r_i \operatorname{ad}^p(y_1)(y_2)$, for some $r_i \in R$, whatever $i \in I$, so the theorems apply again with $J' = \{1\}$ and $J'' = \{2\}$. Once more, the cases $I = \{1\}$ and $I = \{1,2\}$ are fairly tractable. As before, the indices $i$ for which $r_i = 0$ may be left aside, and all $r_i$ assumed non-zero.

Thus, for $I = \{1\}$, $UFH(\mathbb{L}(x'_1, y_1, y_2), d)$, $FH(T(x'_1, y_1, y_2), d)$ and $FH(\Omega X; R)$ may all appropriately be identified to

$$T(\operatorname{ad}^k(y_1)(y_2))_{k=0,\dots,p-1} \otimes T(y_1);$$

and for $I = \{1,2\}$, $UFH(\mathbb{L}(x'_1, x'_2, y_1, y_2), d)$, $FH(T(x'_1, x'_2, y_1, y_2), d)$ and $FH(\Omega X; R)$ may appropriately be identified to

$$T((\operatorname{ad}^k(y_1)(y_2))_{k=0,\dots,p-1}, (\operatorname{ad}^k(y_1)(x))_{k=0,1,2,\dots}) \otimes T(y_1),$$

where, as before, $x = r'_2 x'_1 - r'_1 x'_2$, $r'_i = r_i/r$, $r$ being the greatest common divisor of $r_1$ and $r_2$. $\square$

Our last example deals with somewhat richer wedges of spheres.

**Example 5.** Given integers $k, q \geq 2$, let $I$ be finite, with $m_i \in \{4k, 6k\}$, let $J = \{1, \dots, q\}$, with $n_1 = 2k$ and $n_j = 2k+1$, $j = 2, \dots, q$, and note, by a simple degree argument, that the $\xi_i$ must lie in the $R$-span of the commutators $[y_j, y_1]$ and $[y_{j_1}, [y_{j_2}, y_1]]$, with $j$, $j_1$ and $j_2$ in $\{2, \dots, q\}$. Our theorems now clearly apply with $J' = \{2, \dots, q\}$ and $J'' = \{1\}$. $\square$

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