An example of strongly self-homeomorphic dendrite not pointwise self-homeomorphic

Pavel Pyrih

Abstract. Such spaces in which a homeomorphic image of the whole space can be found in every open set are called *self-homeomorphic*. W.J. Charatonik and A. Dilks asked if any strongly self-homeomorphic dendrite is pointwise self-homeomorphic. We give a negative answer in Example 2.1.

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1. Introduction

W.J. Charatonik and A. Dilks introduced four types of self-homeomorphic spaces (see [1, p. 217]).

Definition 1.1. A topological space X is called *self-homeomorphic* if for any open set $U \subseteq X$ there is a set $V \subseteq U$ such that V is homeomorphic to X.

Definition 1.2. A topological space X is called *strongly self-homeomorphic* if for any open set $U \subseteq X$ there is a set $V \subseteq U$ with nonempty interior such that V is homeomorphic to X.

Definition 1.3. A topological space X is called *pointwise self-homeomorphic at a point* $x \in X$ if for any neighborhood U of x there is a set V such that $x \in V \subseteq U$ and V is homeomorphic to X. The space X is called *pointwise self-homeomorphic* if it is pointwise self-homeomorphic at each of its points.

Definition 1.4. A topological space X is called *strongly pointwise self-homeomorphic at a point* $x \in X$ if for any neighborhood U of x there is a neighborhood V of x such that $x \in V \subseteq U$ and V is homeomorphic to X. The space X is called *strongly pointwise self-homeomorphic* if it is strongly pointwise self-homeomorphic at each of its points.

W.J. Charatonik and A. Dilks asked in [1, p. 237] in Problem 6.21 and Problem 6.23 the following questions

Question 1.5. *If X is a self-homeomorphic dendrite, is X pointwise self-homeomorphic?*

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Question 1.6. *If X is a strongly self-homeomorphic dendrite, is X pointwise self-homeomorphic?*

We give a negative answer to both questions in Example 2.1.

By a *continuum* we mean a compact, connected metric space. By a *dendrite* we mean a locally connected continuum containing no simple closed curves. For a dendrite X, the *order of a point* $x \in X$ is the number of components of $X \setminus \{x\}$. It is denoted by $\mathrm{ord}(x)$. If there are infinitely many components of $X \setminus \{x\}$ we say $\mathrm{ord}(x) = \omega$, where $\omega > n$ for every natural number n. Points of order one are called *endpoints*, and points of order three or more are called *ramification* points.

Recall that metric spaces X and Y are called *similar* if there is a surjection $f : X \to Y$ such that there is a constant c satisfying $d(f(x), f(y)) = cd(x, y)$.

2. Counterexample

Example 2.1. *There exists a strongly self-homeomorphic dendrite which is not pointwise self-homeomorphic.*

Proof: Let $a = (1, 0)$, b and c be three equidistant points of the unit circle. We denote by T the triod consisting of three segments va, vb and vc joining the origin $v = (0, 0)$ with the endpoints a,b and c, respectively. We say that the circle S with center $(-1, 0)$ and radius 2 is the *supporting circle* for the triod T and the point a is the *attaching point* for the triod T.

Recall that a *locally connected fan* is the union of countable many straight line segments in the plane, any two of which intersect at their common point v only and such that for each $\varepsilon > 0$ at most finitely many segments have lengths greater than ε. The common point v is called the *vertex* or the *top*.

We denote by F the *locally connected fan* $F = \bigcup\{ve_n : n \geq 0\}$, where ve_n is the segment joining $v = (0, 0)$ with $e_n = (1/3n, 1/3n + 1/9n^2)$, $n \in \mathbb{N}$, $e_0 = (0, 1)$. We denote by G the circle with center $e_0 = (0, 1)$ and radius 1 and we say that G is the *supporting circle* for the locally connected fan F and the origin is the *attaching point* for the locally connected fan F.

Construction of the "fanned-triod" T_{fans}.

We attach similar copies of the locally connected fan F with their attaching points perpendicularly to all maximal free arcs in T (we attach to the midpoints of these free arcs) in such a way that

- $(*)$ all these copies together with their supporting circles are contained in the convex hull $T_H = \mathrm{conv}\ [T]$;
- $(**)$ all these copies have mutually positive distance between their supporting circles;
- $(***)$ all these copies with their supporting circles meet T just in the copy of the attaching point v of the locally connected fan F.

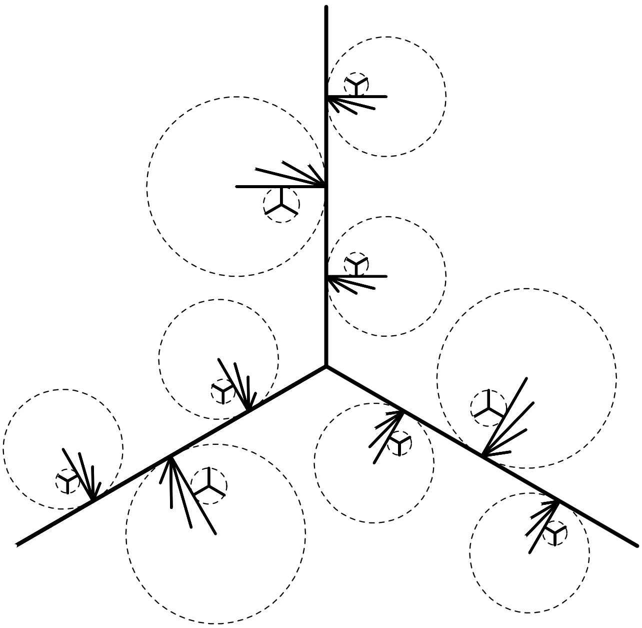

Figure 1 (Idea of Example 2.1: triod-fans-triods-fans- …)

We obtain the continuum T_1. Now we attach again a similar copy of the locally connected fan F perpendicularly to all midpoints of all maximal free arcs of T_1 contained in $T \cap T_1$. We attach sufficiently small copies such that the 'rules' $(*)$–$(***)$ are satisfied (we consider now also all supporting circles from all previous steps). We obtain a continuum T_2 and proceed with attaching to all midpoints of all maximal free arcs of T_2 in $T \cap T_2$. And so on. We use sufficiently small copies so that we finally obtain a locally connected continuum $T_{fans} = \bigcup_1^\infty T_n$. In fact T_{fans} is a dendrite due to the construction. T_{fans} is called the *fanned-triod.*

Construction of the "trioded-fan" F_{triods}.

We attach a similar copy of the triod T with their attaching point perpendicularly to all maximal free arcs in F (we attach to the midpoints of these free arcs) in such a way that

$(*)'$ all these copies with their supporting circles are contained in the convex hull $F_H = \operatorname{conv}[F]$;

$(**)'$ all these copies have mutually positive distance between their supporting circles;

$(***)'$ all these copies with their supporting circles meet F just in the copy of the attaching point a of the triod T.

We obtain the continuum F_1. Now we attach again a similar copy of the triod T perpendicularly to all midpoints of maximal free arcs of F_1 in $F \cap F_1$. We again apply the 'rules' $(*)'$–$(***)'$ (we consider now also all supporting circles from all previous steps). We obtain a continuum F_2 and proceed with attaching to all midpoints of all maximal free arcs of F_2 in $F \cap F_1$. And so on. We use sufficiently small copies so that we finally obtain a locally connected continuum $F_{triods} = \bigcup_1^\infty F_n$, in fact F_{triods} is a dendrite due to the construction. F_{triods} is called the *trioded-fan*.

Construction of the space X.

Let X_1 be the triod T. Given X_n, $n \geq 1$, we replace each maximal free triod in X_n which is similar to T with the fanned-triod T_{fans} and we also replace each maximal free fan in X_n which is similar to F with the trioded-fan F_{triods} and obtain the space X_{n+1}. We denote by X the closure $\operatorname{cl}(\bigcup_{n=1}^\infty X_n)$.

We claim:

(i) *Each X_n is a dendrite.*

PROOF: Clearly each X_n is an arcwise connected locally connected plane continuum containing no simple closed curve due to the construction.

(ii) *X is a dendrite.*

PROOF: We denote by Y the *inverse limit* of the sequence $\{X_n, f_n\}_{n=1}^\infty$ where the *bonding maps* $f_n : X_{n+1} \to X_n$ are the mappings replacing all similar copies of T_{fans} and F_{triods} in $X_{n+1} \setminus X_n$ with their attaching points reversing the process of constructing X_{n+1}. We write (see [2, 2.2]).

$$Y = \varprojlim \{X_n, f_n\}_{n=1}^\infty .$$

Then Y is a dendrite due to the fact that all the bonding maps are monotone (see [2, 10.36]). Moreover due to the construction and the 'rules' $(*)$–$(***)$, $(*)'$–$(***)'$ the conditions of (1) and (2) of [2, Theorem 2.2] are satisfied, hence Y is homeomorphic to the closure $X = \operatorname{cl}(\bigcup_{n=1}^\infty X_n)$. Hence X is a dendrite.

(iii) *X is strongly self-homeomorphic.*

PROOF: Each maximal similar copy $\hat{T}$ of the fanned-triod T_{fans} in X_n is located inside of its supporting circle $\hat{S}$ in the same way as X_1 in S. Hence there is a homeomorphic copy of X in the set $\mathrm{conv}[\hat{S}]$ with nonempty interior. Such similar copies of the fanned-triod T_{fans} are dense in X. Hence X is strongly self-homeomorphic.

(iv) X *is not pointwise self-homeomorphic.*

Proof: We focus now to the structure of X. Clearly X is arcwise connected, see [2, 10.1, 8.23] and contains no simple closed curve. We conclude that any point $x \in X \setminus \bigcup_{n=1}^{\infty} X_n$ is an endpoint of some arc which is contained, except of the end point x, in $\bigcup_{n=1}^{\infty} X_n$.

Let X be pointwise self-homeomorphic at $a = (1, 0) \in X$. Let for any neighborhood U of $a \in X$ there is a set V such that $a \in V \subseteq U$ and V is homeomorphic to X. We conclude a contradiction.

Notice that all similar to T triods in X have densely the points of infinite order in X, whereas all maximal similar copies of F in X have the points of order at most 3 in X, the exceptions are the attaching points of these copies.

Let $f : X \to V$ is a homeomorphic mapping between topological spaces X and V with the induced topology. All tree arcs va, vb, and vc have densely points of infinite order. The same must be true for their images $f(va)$, $f(vb)$ and $f(vc)$. These arcs cannot meet any maximal similar copy of F in more than one point, because any subarc of this copy have at most one point of infinite order. We conclude that the arcs $f(va)$, $f(vb)$ and $f(vc)$ must be contained in one maximal similar copy of T. Denote this copy by $\hat{T}$.

The points a, b and c are not interior points for any arc in X, hence the same is true for the points $f(a)$, $f(b)$ and $f(c)$ in V. These points are joined with the point $f(v)$ by the arcs $f(va)$, $f(vb)$ and $f(vc)$ in V. Hence any point outside the convex hull $\mathrm{conv}[\hat{T}]$ cannot be joined with $f(v)$ by an arc in V. This is true because starting outside the convex hull $\mathrm{conv}[\hat{T}]$ we can reach the 'midpoint' $f(v)$ in V only using the arcs contained in the triod $\hat{T}$. The points $f(a)$, $f(b)$ and $f(c)$ serves as the 'closed gates' to the 'city' $f(v)$, because they are not interior points for any arc in V.

Clearly the origin v is not in V for sufficiently small U. Hence $\hat{T} \neq T$. But then the point $a \in X \cap V$ is not in the convex hull $\mathrm{conv}[\hat{T}]$ and cannot be joined with $f(c)$ by an arc in V. Hence V is not arcwise connected — a contradiction.

The proof is complete. □

Remark 2.2. Figure 1 shows just the idea of Example 2.1. Notice that in fact the locally connected fan F is in fact 'sharper' at its attaching point and that the copies of F and T in Figure 1 should be smaller. The small copies of T in Figure 1 are on the wrong side of the arc in the copies of F. The construction must give a dendrite, so there must be uniqueness between 'paths' and their future 'endpoints' in X (see the conditions of (1) and (2) of [2, Theorem 2.2]).

References

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Department of Mathematical Analysis, Charles University, Sokolovská 83, CZ-186 75 Prague, Czech Republic

E-mail: pyrih@karlin.mff.cuni.cz

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