Connectedness and local connectedness of topological groups and extensions

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Abstract. It is shown that both the free topological group $F(X)$ and the free Abelian topological group $A(X)$ on a connected locally connected space X are locally connected. For the Graev's modification of the groups $F(X)$ and $A(X)$, the corresponding result is more symmetric: the groups $F\Gamma(X)$ and $A\Gamma(X)$ are connected and locally connected if X is. However, the free (Abelian) totally bounded group $FTB(X)$ (resp., $ATB(X)$) is not locally connected no matter how "good" a space X is.

The above results imply that every non-trivial continuous homomorphism of $A(X)$ to the additive group of reals, with X connected and locally connected, is open.

We also prove that any dense in itself subspace of the Sorgenfrey line has a Urysohn connectification. If D is a dense subset of $\{0,1\}^{\mathfrak{c}}$ of power less than $\mathfrak{c}$, then D has a Urysohn connectification of the same cardinality as D.

We also strengthen a result of [1] for second countable Tychonoff spaces without open compact subspaces proving that it is possible to find a compact metrizable connectification of such a space preserving its dimension if it is positive.

Keywords: connected, locally connected, free topological group, Pontryagin's duality, pseudo-open mapping, open mapping, Urysohn space, connectification

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0. Introduction

A number of articles (for example, [1], [21] and [17]) have appeared recently on the subject of dense embeddings in connected spaces. Since the cases of dense embeddings in T_0- and in T_1-spaces are trivial, attention has been concentrated on dense embeddings in connected Hausdorff spaces, while in [1], some partial results were given concerning compact connectifications. Unfortunately, the methods used in the above-mentioned articles to construct Hausdorff connectifications are in general, not suitable for constructing spaces satisfying higher axioms of separation except for semiregularity; for example, the construction given in [21] of a connectification of a countable T_3-space with countable π-base and without isolated points, will never produce a Urysohn space.

In this paper we concentrate on embedding spaces into connected or locally connected ones. This embedding can be dense and in this case we give methods of constructing Urysohn connectifications and in particular, we will show that the Sorgenfrey line can be embedded densely in a connected Urysohn space with countable remainder; this connectification can also be made semiregular. Since it has been shown (see [5]) in response to a question of van Douwen, that the Sorgenfrey line has no T_3-connectification, this is in some sense, the best possible result.

We also consider (not necessarily dense) embeddings of Tychonoff spaces in connected and locally connected topological groups.

It is easy to verify (see Section 6 of [6]) that both $F\Gamma(X)$ and $A\Gamma(X)$, respectively the free topological and the free Abelian topological groups on a space X in the sense of Graev are connected iff X is connected. The situation in the case of the free (Abelian) topological group $F(X)$ (resp., $A(X)$) is different: neither $F(X)$ nor $A(X)$ is connected for a non-empty space X. However, the groups $F(X)$ and $A(X)$ contain open normal subgroups $F^*(X)$ and $A^*(X)$ resp., such that the quotient groups $F(X)/F^*(X)$ and $A(X)/A^*(X)$ are isomorphic to the group of integers, and both groups $F^*(X)$ and $A^*(X)$ are connected iff X is connected. The construction of these subgroups is very simple. Let f be a mapping of X to the discrete group of integers, $f(x) = 1$ for each $x \in X$. Then f extends to a continuous homomorphism $\hat{f}$ of $A(X)$ to the integers and we put $A^*(X) = \hat{f}^{-1}(0)$; the subgroup $F^*(X)$ of $F(X)$ is defined analogously.

Local connectedness is a more subtle property. We show that if a space X is connected and locally connected then each of the groups $F\Gamma(X)$, $A\Gamma(X)$, $F^*(X)$ and $A^*(X)$ has the same properties. The converse is not true: it is known ([15]) that there exist two connected metrizable compact spaces K_1 and K_2 such that the groups $F\Gamma(K_1)$ and $F\Gamma(K_2)$ are topologically isomorphic, but only one of the compact spaces K_1 and K_2 is locally connected. In particular, local connectedness is not an invariant of the M-equivalence relation.

The above results are applied to deduce the following: every non-trivial continuous homomorphism of $F\Gamma(X)$ as well as of $A\Gamma(X)$, $F^*(X)$, $A^*(X)$ to the additive group of the reals is open whenever X is connected and locally connected.

We also show that the free (Abelian) totally bounded group $FTB(X)$ (resp., $ATB(X)$) is never locally connected.

Finally, we study the possibility of embedding densely a second countable regular space X without open compact subsets into a metrizable connected Y with $\dim(Y) = \dim(X)$.

1. Notation and terminology

All spaces under consideration are assumed to be Hausdorff if no separation axioms are mentioned. If X is a space, then $\mathcal{T}(X)$ is its topology and $\mathcal{T}(x, X) = \{U \in \mathcal{T}(X) : x \in U\}$. The cardinals are identified with the relevant ordinals and are therefore, the sets of all preceding ordinals. In particular, $2 = \{0, 1\}$. The symbol $\mathfrak{c}$ serves to denote the cardinality of the continuum, ω is the set of natural numbers and $N^+ = \omega\backslash\{0\}$.

The end of a proof is denoted by $\square$. If a substatement of a non-proved statement is proved, then we use the symbol $\triangle$.

If X and Y are spaces and $f : X \to Y$ is a map, then $f^\#(A) = Y\backslash f(X\backslash A)$ for any $A \subset X$. For a subspace A of a space X we denote by $cl_X(A)$ and $Bd_X(A)$ the closure and the boundary of A in X respectively. If it is clear from the context where the closure of A is taken we denote it by $\overline{A}$. All other notions are standard.

2. Local connectedness of free groups

For our purpose we need a description of a base at the identity of $F\Gamma(X)$. Let X be a completely regular space with a fixed point $e \in X$ and suppose that h is a homeomorphism of X onto its copy X^{-1}. Denote by $\tilde{X}$ the quotient space of the free sum $X \oplus X^{-1}$ by identifying of two points e and $h(e)$. By Graev's construction ([6]), e is assigned to be the identity of $F\Gamma(X)$. If X is connected and/or locally connected then $\tilde{X}$ has the same property. For each integer n let $\triangle_n$ be the diagonal $\{(x,x) : x \in \tilde{X}^n\}$ of the space $\tilde{X}^{2n}$. Denote by i the natural embedding of $\tilde{X}$ to $F\Gamma(X)$ such that the image $i(X \setminus e)$ is an algebraic basis for $F\Gamma(X)$, $i(\{e, h(e)\}) = e$ and $i(h(x)) = i(x)^{-1}$ for every $x \in X \setminus \{e\}$. For every $n \in N^+$ and $z = (x,y) \in \tilde{X}^{2n}$, where $x = (x_1, \dots, x_n) \in \tilde{X}^n$, $y = (y_1, \dots y_n) \in \tilde{X}^n$, put $j_n(z) = i(x_1) \cdot \dots \cdot i(x_n) \cdot i(y_n)^{-1} \cdot \dots \cdot i(y_1)^{-1}$. The symbol $^{-1}$ stands here for the inverse operation in $F\Gamma(X)$.

Thus for every $n \in N^+$, we have defined the mapping $j_n : \tilde{X}^{2n} \to F\Gamma(X)$. The continuity of the group operation of $F\Gamma(X)$ implies that j_n is also continuous for each $n \in N^+$ (for example, see Section 5 of [3]). Let E be a sequence $\{U_n : n \in N^+\}$, where U_n is an open neighborhood of the diagonal $\triangle_n$ in $\tilde{X}^{2n}$ for every $n \in N^+$. We put

$$V(E) = \bigcup_{n \in N^+} \bigcup_{\pi \in S_n} j_{\pi(1)}(U_{\pi(1)}) \cdot \dots \cdot j_{\pi(n)}(U_{\pi(n)}),$$

where S_n is the permutation group of $\{1, \dots, n\}$. Denote by $\mathcal{E}$ the family of all sequences E of the above form.

We omit a proof of the following theorem which is similar to that of Theorem 1.1 of [20].

2.1 Theorem. *The family $\{V(E) : E \in \mathcal{E}\}$ is a base at the identity of $F\Gamma(X)$.*

We need one more auxiliary result.

2.2 Lemma. *Suppose that X is a connected locally connected space and $n \in N^+$. Then for every neighborhood U of the diagonal $\triangle_n$ in $\tilde{X}^{2n}$ there exists an open connected neighborhood U' of $\triangle_n$ lying in U.*

Proof: Let U be a neighborhood of $\triangle_n$ in $\tilde{X}^{2n}$. Making use of the local connectedness of $\tilde{X}$, for each point $x \in \tilde{X}$ fix an open connected set $U(x) \ni x$ with $U(x)^{2n} \subset U$. Put $U' = \cup\{U(x)^{2n} : x \in \tilde{X}\}$. It is clear that $\triangle_n \subset U' \subset U$. Furthermore, the sets $U(x)^{2n}$ are connected and all intersect the connected set $\triangle_n$ (homeomorphic to $\tilde{X}^n$), and hence U' is connected. □

2.3 Theorem. *Let X be a connected locally connected space. Then all the groups $F\Gamma(X)$, $A\Gamma(X)$, $F^*(X)$ and $A^*(X)$ are connected and locally connected, and the groups $F(X)$, $A(X)$ are locally connected.*

Proof: Connectedness of the groups $F\Gamma(X)$, $A\Gamma(X)$, $F^*(X)$ and $A^*(X)$ follows from the fact that a union of an increasing sequence of connected sets is connected (see Section 6 of [6]).

The identity mapping id_X extends to the open epimorphism $\varphi : F\Gamma(X) \to A\Gamma(X)$ and $\psi : F(X) \to A(X)$ (see [6] and [12]). It is easy to verify the equality $\psi^{-1}(A^*(X)) = F^*(X)$, therefore the restriction of ψ to $F^*(X)$ is an open homomorphism onto $A^*(X)$. Since local connectedness is preserved by open mappings and $F^*(X)$ is open in $F(X)$, $A^*(X)$ is open in $A(X)$, it suffices to prove that the groups $F\Gamma(X)$ and $F^*(X)$ are locally connected for a given connected locally connected space X. Let us show this for the group $F\Gamma(X)$.

Consider an arbitrary neighborhood O of the identity e in $F\Gamma(X)$. By Theorem 2.1, there exists a sequence $E = \{U_n : n \in N^+\} \in \mathcal{E}$ such that $V(E) \subset O$. Making use of Lemma 2.2, for each $n \in N^+$ find an open connected neighborhood U'_n of the diagonal $\triangle_n$ lying in U_n. Put $E' = \{U'_n : n \in N^+\}$. Then $E' \in \mathcal{E}$ and $V(E') \subset V(E) \subset O$. The continuity of the mappings j_n implies that all the sets $j_n(U'_n)$ are connected. For each permutation $\pi \in S_n$, therefore, the set

$$V'_{n,\pi} = j_{\pi(1)}(U'_{\pi(1)}) \cdot \dots \cdot j_{\pi(n)}(U'_{\pi(n)})$$

is also connected. Since $e \in V'_{n,\pi}$ for all $n \in N^+$ and $\pi \in S_n$, we conclude that the set

$$V(E') = \bigcup_{n \in N^+} \bigcup_{\pi \in S_n} V'_{n,\pi}$$

is connected. Thus, the group $F\Gamma(X)$ is locally connected.

As for the group $F^*(X)$, apply the above reasoning with the use of Theorem 1.1 of [20] instead of Theorem 2.1. The group $A^*(X)$ is a quotient space of $F^*(X)$ so it is connected and locally connected too. □

Denote by $FTB(X)$ (resp., $ATB(X)$) the free (Abelian) totally bounded topological group on a space X. Let also $FC(X)$ (resp., $AC(X)$) be the free (Abelian) compact topological group on X. It is known ([14]) that $ATB(X)$ embeds as a dense subgroup to $AC(X)$, so the completion of $ATB(X)$ is topologically isomorphic the group $AC(X)$; the same is true for the groups $FTB(X)$ and $FC(X)$. Let us show that these four groups are never locally connected.

2.4 Theorem. *The groups $FC(X)$, $AC(X)$, $FTB(X)$ and $ATB(X)$ are not locally connected for any non-empty space X.*

Proof: Note that $AC(X)$ is an open homomorphic image of $FC(X)$ and $ATB(X)$ is an open homomorphic image of $FTB(X)$. Furthermore, $AC(X)$ is the completion of $ATB(X)$ and $FC(X)$ is the completion of $FTB(X)$, so the local connectedness of the group $FTB(X)$ would imply the local connectedness of all others. Clearly, it suffices to show that the group $AC(X)$ is not locally connected, since completions preserve local connectedness.

We recall the following characterization of local connectedness in the class of compact Abelian groups (see Chapter 38, Theorem 48 of [16]):

Fact. *A compact Abelian group G is locally connected iff every finite subset of the dual group $G^\wedge$ is contained in a finitely generated pure subgroup of $G^\wedge$.*

One can easily verify that the dual group for $AC(X)$ is $C(X,T)$, the discrete group of continuous mappings from X to the circle group T (see [8]). Denote by f the constant function on X, $f(x) = e^{i\pi}$ for all $x \in X$. Assume that f belongs to some finitely generated pure subgroup H of $C(X,T)$. Being pure, H must contain a sequence $\{f_n : n \in N^+\}$ such that $2 \cdot f_{n+1} = f_n$ for each $n \in N^+$, where $f_1 = f$. [We cannot assert that $f_n \equiv e^{i\pi/2^n}$ because the space X may be disconnected.] Let K be the subgroup of H generated by the set $\{f_n : n \in N^+\}$. It is clear that every finitely generated subgroup of K is finite, and hence K is not finitely generated. This contradicts, however, the well-known result that every subgroup of an Abelian finitely generated group is also finitely generated ([18]). It remains to apply the above Fact to complete the proof. □

Following [2], we say that a continuous mapping $f : X \to Y$ is inductively perfect if there exists a closed subset $F \subset X$ such that $f(F) = f(X)$ and the restriction of f to F is perfect. The following result is a generalization of Theorem 2 of [10].

2.5 Proposition. *Let X be a space which is the union of an increasing sequence $\{B_n : n \in N^+\}$ of compact connected subsets of X. Then every continuous real-valued function on X is inductively perfect.*

Proof: The conditions of our proposition imply that X is connected, therefore $f(X)$ is an interval of the reals. If $f(X)$ contains its infimum and its supremum — let us denote them by a and b, then there is $n \in N^+$ such that $f(B_n)$ contains both points a and b. Therefore $f(B_n) = f(X)$ and we are done. Otherwise $f(X)$ can be represented as a union $f(X) = \cup\{I_k : k \in N^+\}$ of a locally finite family (in $f(X)$) of closed intervals I_k. For every $k \in N^+$ there exists $n(k) \in N^+$ such that $f(B_{n(k)}) \supseteq I_k$. Put $F_k = B_{n(k)} \cap f^{-1}(I_k)$ and $F = \cup\{F_k : k \in N^+\}$. It is clear that the family $\{F_k : k \in N^+\}$ of closed sets is locally finite in X, and hence F is closed in X. It remains to note that $f(F) = f(X)$ and that the restriction $f|_F$ is perfect. □

2.6 Corollary. *Let X be a compact connected space. Then every continuous real-valued function on each of the groups $F\Gamma(X)$, $A\Gamma(X)$, $F^*(X)$ and $A^*(X)$ is inductively perfect.*

Proof: The proof will be identical for all mentioned groups, so let us prove our corollary, say for $F\Gamma(X)$. For every $n \in N^+$ denote by $F\Gamma_n(X)$ the set of all words in $F\Gamma(X)$ having the reduced length at most n. One readily verifies that $F\Gamma_n(X)$ is an image of the connected compact space $\tilde{X}^n$ under the continuous mapping j_n (see Section 1). Since $F\Gamma(X)$ is the union of the increasing sequence $\{F\Gamma_n(X) : n \in N^+\}$ of compact connected subsets, the application of Proposition 2.5 completes the proof. □

Being inductively perfect is a strong property of a mapping. In some cases a somewhat weaker property can be proved. The following result seems to be part of the folklore.

2.7 Proposition. *Let X be a connected locally connected space. Then every continuous real-valued function on X is pseudo-open (= hereditarily quotient) if it is considered as a mapping onto its image.*

Proof: Suppose that f is a continuous real-valued function on X. It suffices to show that for every point $r \in f(X)$ and any given neighborhood O of $f^{-1}(r)$ in X the set $f(O)$ is a neighborhood of r in $f(X)$. Since X is connected, $f(X)$ is an interval in the reals. Assume that r is an interior point of $f(X)$. Put $U = (-\infty, r)$ and $V = (r, \infty)$. Since X is connected, the sets $A = f^{-1}(r) \cap cl_X(f^{-1}(U))$ and $B = f^{-1}(r) \cap cl_X(f^{-1}(V))$ are not empty. Consequently one can pick points $a \in A$ and $b \in B$. By the local connectedness of X there exist open connected neighborhoods $O_a \ni a$ and $O_b \ni b$ in X such that $O_a \cup O_b \subset O$. Then $O_a \cap f^{-1}(U) \neq \emptyset$ and $O_b \cap f^{-1}(V) \neq \emptyset$. Therefore the set $f(O_a \cup O_b)$ is a neighborhood of the point r. The case of an end point r of the interval $f(X)$ is treated analogously. □

Applying Theorem 2.3 and Proposition 2.7, we obtain the following.

2.8 Corollary. *Let X be a connected locally connected space. Then every continuous real-valued function on each of the groups $F\Gamma(X)$, $A\Gamma(X)$, $F^*(X)$ and $A^*(X)$ is a pseudo-open mapping (onto its image).*

2.9 Proposition. *Every continuous non-trival homomorphism of a connected locally connected topological group G to the additive group of reals is open.*

Proof: Let $\varphi : G \to R$ be a non-trivial continuous homomorphism to the group of reals R and U be a neighborhood of the identity in G. We show that $\varphi(U)$ is a neighborhood of zero in R. Choose a symmetric connected neighborhood V of the identity in G lying in U. The homomorphism φ is not constant on V, otherwise φ would be constant on every set $g \cdot V$, $g \in G$. This, however, together with connectedness of G would imply that the homomorphism φ be constant on G, i.e., trivial. Therefore, $\varphi(g) = a \neq 0$ for some $g \in V$. Since V is symmetric and connected, the image $\varphi(V)$ contains the open interval $(-|a|, |a|)$ and is a neighborhood of zero in R. □

2.10 Corollary. *If X is connected and locally connected then every non-trivial continuous homomorphism of $F\Gamma(X)$ or $A\Gamma(X)$ (respectively, $F^*(X)$ or $A^*(X)$) to the additive group of reals is open.*

Proof: Apply Proposition 2.9 and Theorem 2.3. □

It is obvious that the condition of connectedness in Proposition 2.9 is essential: the identity mapping of the reals R_d with the discrete topology onto R is a simple example of a continuous homomorphism of a locally compact, locally connected group onto the reals which is not open. Let us show that the local connectedness of the group G in Proposition 2.9 is also essential.

2.11 Example. *The additive group R of reals admits a connected topological group topology t^* strictly finer than the usual one.*

PROOF: The idea of our example is to construct a discontinuous homomorphism h of R to the circle group T such that the subgroup $G = \{(x, h(x)) : x \in R\}$ of $R \times T$ with the topology t^* induced from $R \times T$ be connected. It is clear that the restriction of the projection $p : R \times T \to R$ to G is a continuous one-to-one homomorphism of G onto R and that $p|_G$ is not a homeomorphism — otherwise h would be a continuous mapping. So, identifying G and R by means of $p|_G$, we get a connected topological group topology t^* on R strictly finer than the usual topology t. In fact, we will construct the group G to be dense in $R \times T$. To start with, we need the following fact about dense disconnected subsets of $R \times T$.

2.12 Claim. *Let G be a dense disconnected subset of $R \times T$ such that $p(G) = R$. Then there exists a non-empty closed subset F of $R \times T$ disjoint from G such that $p(F)$ has no isolated points.*

PROOF OF CLAIM 2.12: Since G is disconnected, there exist two non-empty disjoint open sets U and V in $R \times T$ such that $G \subset U \cup V$. We will show that the set $F = \overline{U} \cap \overline{V}$ is as required. Indeed, F is disjoint from U and V and, a fortiori, from G. The fact that $F \neq \emptyset$ follows from connectedness of the product $R \times T$. It remains to verify that $p(F)$ has no isolated points. Assume the contrary, and let x be an isolated point of $p(F)$; let W be an open interval containing x and disjoint from $p(F) \setminus \{x\}$. Then, on one hand, the open set $O = p^{-1}(W) \setminus F$ is disconnected: the closed sets $\overline{U}$ and $\overline{V}$ disconnect it (note that $O \cap U \neq \emptyset \neq O \cap V$ because F intersects $p^{-1}(x)$). On the other hand, O is arcwise connected, because $p^{-1}(W) \cap F$ is a proper subset of $p^{-1}(x)$ — the latter follows from the choice of the point x and the fact that $x \in R = p(G)$. In particular, O is connected, which is a contradiction. ⊿

Let us start the construction of a homomorphism $h : R \to T$. Denote by $\mathcal{K}$ the family of all non-empty closed subsets K of R without isolated points. Then, in particular, $|K| = c$ for each $K \in \mathcal{K}$. Clearly, $|\mathcal{K}| \leq c$ because R has countable base.

Since the additive group of reals R is torsion-free and divisible, it is algebraically isomorphic to the direct sum $\oplus_{\nu<\mathfrak{c}} Q_\nu$, where Q_ν is a subgroup of R isomorphic to the group Q of rationals for every $\nu < \mathfrak{c}$; (see Theorem 4.1.5 of [18]). It is possible to find the above representation of R and choose points $a_\nu \in Q_\nu \setminus \{0\}$, $\nu < \mathfrak{c}$, so that $|K \cap \{a_\nu : \nu < \mathfrak{c}\}| = \mathfrak{c}$ for all $K \in \mathcal{K}$. Indeed, let $\{K_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the family $\mathcal{K}$ such that every element $K \in \mathcal{K}$ occurs in this sequence $\mathfrak{c}$ times. Let $R = \{x_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the reals.

Suppose we have defined elements $a_\mu \in R$ for all $\mu < \nu$, where $\nu < \mathfrak{c}$, so that the set $\{a_\mu : \mu < \nu\}$ is independent over Q, i.e., a linear combination of arbitrary elements of this set with rational coefficients is equal to zero iff all the coefficients are equal to zero. Denote by H_ν the subgroup of R consisting of all rational linear combinations of elements a_μ, $\mu < \nu$. Then $|H_\nu| < \mathfrak{c}$, and hence we define a_ν to be the element x_α of the set $K_\nu \setminus H_\nu$ with the minimal index $\alpha < \mathfrak{c}$. Obviously, this construction gives us the equality $R = \oplus_{\mu<\mathfrak{c}} Q_\mu$, where $Q_\mu = a_\mu \cdot Q$, and the set $\{a_\mu : \mu < \mathfrak{c}\}$ intersects each element of $\mathcal{K}$ in $\mathfrak{c}$ many points.

Denote by $\mathcal{F}$ the family of all closed subsets F of $R \times T$ such that $p(F) \in \mathcal{K}$. We have $|\mathcal{F}| \leq \mathfrak{c}$ and $|p(F)| = \mathfrak{c}$ for each $F \in \mathcal{F}$. Since $|p(F) \cap \{a_\mu : \mu < \mathfrak{c}\}| = \mathfrak{c}$ for each $F \in \mathcal{F}$, one can easily define a function f from $A = \{a_\mu : \mu < \mathfrak{c}\}$ to T such that for each $F \in \mathcal{F}$ there exists $a \in A$ with $(a, f(a)) \in F$. Note that the set $\{(a, f(a)) : a \in A\}$ is dense in $R \times T$, because every non-empty open set in $R \times T$ contains elements of $\mathcal{F}$. Making use of divisibility of the group T, extend f to a homomorphism $h : R \to T$ and then define the subgroup $G = \{(x, h(x)) : x \in R\}$ of $R \times T$ mentioned above. Since G is dense in $R \times T$ and intersects all elements of the family $\mathcal{F}$, Claim 2.12 implies that G is a connected subset of $R \times T$. □

3. Urysohn and Tychonoff connectifications of dense subspaces of $2^{\mathfrak{c}}$

The methods used in this section to construct Urysohn connectifications, are modifications of a technique used by Roy to construct a countable connected Urysohn space with a dispersion point (see [19]).

Let D denote a dense subspace of $X = \{0,1\}^\omega$ (that is, the Cantor set). If D is countable, then it is well known that D is homeomorphic to the rationals and hence is homeomorphic to the topological union of ω copies of itself.

3.1 Lemma. *For every proper dense subset D of the Cantor set X, there exists a set S of the same cardinality as D, such that $D \subset S \subset X$ and such that S is homeomorphic to a topological union $\bigoplus\{S_n : n \in \omega\}$, where all the S_n are homeomorphic.*

Proof: Suppose that D is a proper dense subset of X; clearly D is Lindelöf but not compact. Furthermore, there is a sequence of $\{V_n : n \in \omega\}$ of pairwise disjoint basic clopen subsets of X such that $D \subset \cup\{V_n : n \in \omega\}$ and the V_n are all homeomorphic. For each $n \in \omega$, let $D_n = D \cap V_n$ and fix a homeomorphism $f_n : V_n \to V_{n+1}$. Now define $H = D_0 \cup {f_0}^{-1}[D_1] \cup {f_0}^{-1}[{f_1}^{-1}[D_2]] \cup \dots$.

Then,

$V_0 \supset H \supset D_0$,
$V_1 \supset f_0(H) \supset D_1$ and
$V_2 \supset f_1[f_0(H)] \supset D_2$ etc.

Finally, let S be the union of the subspaces H, $f_0[H]$, $f_1[f_0(H)]$, etc. Since each V_n is clopen, it follows that S is the topological union of pairwise homeomorphic subspaces of X and has the same cardinality as D. □

3.2 Remark. *Suppose that $|D| < \mathfrak{c}$. Clearly, H is dense in V_0 (which is homeomorphic to the Cantor set). Hence, there is some dense subset J of X which is homeomorphic to H. However, J has cardinality less than the continuum, and so there is a sequence of points $\{x_n : n \in \omega\}$ in X such that $(J + x_n) \cap (J + x_m) = \emptyset$ if $m \neq n$ (where $+$ denotes the usual group operation on $\{0,1\}^\omega$). Furthermore, $J + x_n$ is dense in X for each n and has the same cardinality as H and D.*

3.3 Theorem. *If D is a dense subset of $X = \{0,1\}^\omega$ of cardinality less than the continuum, then D has a Urysohn connectification of the same cardinality.*

Proof: If D is countable, then D is homeomorphic to the rationals, and the example of Roy gives a countable Urysohn connectification of D. Now suppose that $|D| = \kappa > \omega$. With the same notation as in Remark 3.2, we have $J + x_n$ is homeomorphic to $J + x_m$ for all $m, n \in \omega$. Now we use the construction of Roy, defining $C_n = J + x_n$ for all $n \in \omega$ and $Z = \cup\{C_n \times \{n\} : n \in \omega\} \cup \{p\}$ where $p \notin C_n$ for all $n \in \omega$ and a topology τ on Z as follows.

(a) A basic neighborhood of p is of the form $\{p\} \cup \bigcup\{C_n : n \geq m\}$ for $m \in \omega$.

(b) V is a neighborhood of $(x, 2n) \in C_{2n} \times \{2n\}$ if it contains a set of the form $(W \cap C_{2n}) \times \{2n\}$, where W is a neighborhood of x in X.

(c) V is a neighborhood of $(x, 2n+1) \in C_{2n+1} \times \{2n+1\}$ if it contains a set of the form $[(W \cap C_{2n+1}) \times \{2n+1\}] \cup [(W \cap C_{2n}) \times \{2n\}] \cup [(W \cap C_{2n+2}) \times \{2n+2\}]$, where W is a neighborhood of x in X.

The proof that (Z, τ) is a connected Urysohn space follows that of [19] and we omit it. Note that $\cup\{C_{2n} : n \in \omega\}$ is a dense open subset of Z which is homeomorphic to S (see Lemma 3.1). Hence Z is the required connectification of D. $\square$

As shown in [21, Lemma 3.10], the semiregularization of Z is a semiregular connectification of D and it is trivial that the semiregularization preserves the Urysohn property.

3.4 Corollary. *If D is a dense subspace of $X = \{0,1\}^{\mathfrak{c}}$ of cardinality less than the continuum, then D has a Urysohn connectification of the same cardinality as D.*

Proof: For $I \subset \mathfrak{c}$ denote the projection of D to the subproduct determined by I by $\Pi_I(D)$. Then D is dense in $\Pi_\omega(D) \times \Pi_{\mathfrak{c}\setminus\omega}(D)$ which, in turn, is dense in $\{0,1\}^\omega \times \{0,1\}^{\mathfrak{c}\setminus\omega}$. Now repeat the construction of the sets $\{C_n : n \in \omega\}$ for $\prod_I(D) \subset \{0,1\}^\omega$ and define

$$Z = \{p\} \cup \bigcup\{C_n \times \Pi_{\mathfrak{c}\setminus\omega}(D) \times \{n\} : n \in \omega\}$$

with a topology analogous to that defined in Theorem 3.3. We omit the straightforward details. $\square$

It turns out, that if we do not require the connectification in 3.4 to be of the same cardinality as D, then it can be made Tychonoff and locally connected.

3.5 Theorem. *Let κ be a cardinal number such that $\omega \le \kappa \le \mathfrak{c}$. If D is a dense subspace of $\{0,1\}^\kappa$ with $|D| < 2^\kappa$ then D can be densely embedded into $[0,1]^\kappa$.*

Proof: It suffices to prove that there is a continuous surjective map

$$f : \{0,1\}^\kappa \to [0,1]^\kappa,$$

such that $f^{-1}f(d) = \{d\}$ for every $d \in D$.

To find such an f let us first observe that $K_\kappa = \{0,1\}^\kappa = \prod\{C_\alpha : \alpha < \kappa\}$, where C_α is a Cantor set for all α. There exists a continuous onto map $g_\alpha : C_\alpha \to [0,1]$ such that there is only a countable number of points x with $|g_\alpha^{-1} g_\alpha(x)| > 1$. Bearing in mind that C_α is also a topological group denote by G_α some countable subgroup which contains all such x.

Now let $g : K_\kappa \to [0,1]^\kappa$ be the product of the maps g_α. It is clear that

$$I_g = \{x \in K_\kappa : g^{-1}g(x) \ne \{x\}\} \subset G = \prod\{G_\alpha : \alpha < \kappa\}.$$

But G is a subgroup of K_κ such that $|K_\kappa/G| = 2^\kappa$.

The cardinality of D is less than 2^κ, so there is an $x \in K_\kappa$ with $(x+G)\cap D = \emptyset$, or equivalently, $(x + D) \cap G = \emptyset$. Now we can put f(y)=g(x+y) for all $y \in K_\kappa$. $\square$

3.6 Corollary. *If D is a dense subspace of $\{0,1\}^\kappa$ with $|D| < 2^\kappa$, where $\omega \le \kappa \le \mathfrak{c}$, then D has a locally connected connectification.*

The connectification of D in 3.6 is homeomorphic to a Tychonoff cube and is therefore a homogeneous space. But the space $\{0,1\}^\kappa$ which is the original extension of D, is a topological group. Can we construct a topological group X, which would connectify D? The answer to this question is trivially yes, if we only require a dense embedding of D. Indeed, proving 3.5, we could substitute the segment $[0,1]$ by the circle everywhere in the proof and this would change nothing.

But it is also natural to ask what happens if a subgroup of $\{0,1\}^\kappa$ is topologically isomorphic to a subgroup of X. It turns out that X cannot be connected.

3.7 Proposition. *Let κ be an infinite cardinal number. If some subgroup D of $\{0,1\}^\kappa$ is topologically isomorphic to a dense subgroup of a topological group X, then X is not connected.*

Proof: It is well known that X can be assumed to be a subgroup of the completion of D. But D is a Boolean totally bounded group, so its completion is a compact Boolean group and hence is $\{0,1\}^\lambda$ for some λ [7, Theorem 25.9]. Therefore X is zero-dimensional. $\square$

In 1977, van Douwen raised the question as to whether or not the Sorgenfrey line has a compact connectification. Answering this in the negative, Emeryk and Kulpa [5] showed that, in fact, it has no T_3-connectification. At the same time, they constructed a Hausdorff connectification, the existence of which can be deduced from more general results in [21] and [17], but none of these constructions lead to a Urysohn space. Here we show that the Sorgenfrey line, and indeed, any of its dense-in-itself subspace, has a Urysohn (and semiregular) ω-connectification. The method used is also based on the construction of Roy [19], but deeper modifications are required in this case.

3.8 Theorem. *There exists a Urysohn connectification of the Sorgenfrey line.*

PROOF: Denote the set of real numbers by R and let $\{Q_n : n \in \omega\}$ be a family of disjoint dense (in the usual topology of R) subsets of the rationals. In what follows, the sets Q_n and all intervals are considered to be subsets of R. Our space is defined as follows.

Let $Z = \{p\} \cup \bigcup\{X_n \times \{n\} : n \in \omega\}$ where

(i) $X_n = Q_n$ if $n \not\equiv 0(mod\ 4)$,
(ii) $X_n = R$ if $n \equiv 0(mod\ 4)$ and
(iii) $p \notin X_n \times \{n\}$ for all $n \in \omega$,

with a topology τ defined as follows:

Let $\{a_k : k \in \omega\}$ be an enumeration of the countable set

$$A = \cup\{X_{4n+2} \times \{4n+2\} : n \in \omega\}.$$

Now for each $\xi \equiv 2(mod\ 4)$, each $x \in X_\xi$ and each $\varepsilon > 0$, let

$$T(x, \varepsilon) = \{m \in \omega : a_m \in ((x - \varepsilon, x] \cap X_\xi) \times \{\xi\}\}.$$

(a) For $\xi \equiv 2(mod\ 4)$, a basic open neighborhood of $(x, \xi) \in Z$ is of the form:

$$(((x - \varepsilon, x] \cap X_\xi) \times \{\xi\}) \cup \bigcup\{(X_{4m} \setminus [-j, j)) \times \{4m\} : m \in T(x, \varepsilon)\}$$

for some $j \in \omega$ and $\varepsilon > 0$.

(b) For $m \in \omega$, a basic open neighborhood of $(x, 4m) \in X_{4m} \times \{4m\}$ is of the form:

$$[x, x + \varepsilon) \times \{4m\} \text{ for some } \varepsilon > 0.$$

(c) An open neighborhood of (x, j) when $j \equiv 3(mod\ 4)$, is any open set containing a set of the form:

$$(((x - \varepsilon, x] \cap X_j) \times \{j\}) \cup ((x - \varepsilon, x) \times \{j + 1\}) \cup (((x - \varepsilon, x) \cap X_{j-1}) \times \{j - 1\}).$$

(Note that this set is not open, since it is not a neighborhood of any of its points which lie in $X_{j-1} \times \{j - 1\}$.)

(d) An open neighborhood of (x, j) when $j \equiv 1 (mod\ 4)$, is any open set containing a set of the form:

$$(((x-\varepsilon, x] \cap X_j) \times \{j\}) \cup ((x-\varepsilon, x) \times \{j-1\}) \cup (((x-\varepsilon, x) \cap X_{j+1}) \times \{j+1\}).$$

(e) A sub-basic open neighborhood of p is of the form:

$$\{p\} \cup \bigcup \{X_{4m} \times \{4m\} : m \notin T(x,1)\} \text{ for some } x \in X_\xi \text{ and } \xi \equiv 2 (mod\ 4).$$

Note that for each $m \in \omega$, the subspace $X_{4m} \times \{4m\}$ of (Z, τ), with the relative topology is homeomorphic to the Sorgenfrey line. Furthermore, since the topological union of a countable number of copies of the Sorgenfrey line is again homeomorphic to the Sorgenfrey line, it follows that the open subspace $\cup\{X_{4m} \times \{4m\} : m \in \omega\}$ is homeomorphic to the Sorgenfrey line. Furthermore, it is easy to see that this open set is dense in (Z, τ) and its complement is countable. Thus our construction will be complete if we show that (Z, τ) is connected and Urysohn.

To show that Z is connected, we note that

- (i) for each $m \in \omega$, $cl_Z((X_{4m} \times \{4m\}) \supset \cup\{X_\xi : |\xi - 4m| \leq 1\}$,
- (ii) for each $m \in \omega$, $cl_Z(X_{4m+2} \times \{4m+2\}) \supset \cup\{X_\xi : |\xi - 4m + 2| \leq 1\}$,
- (iii) an open neighborhood of X_{4m+1} must contain open dense subsets of X_{4m+2} and X_{4m},
- (iv) an open neighborhood of X_{4m+3} must contain open dense subsets of X_{4m+2} and $X_{4(m+1)}$.

It follows from the above observations, that if U is a clopen neighborhood of X_{4m}, then $U \supset \cup\{X_\xi : |\xi - 4m| \leq 4\}$.

Now to show that (Z, τ) is connected, suppose that U is a clopen subset of Z and $p \in U$. Then for some $m \in \omega$, $X_{4m} \times \{4m\} \subset U$. A repetition of the above argument shows that $U = Z$.

To show that (Z, τ) is Urysohn, suppose that s and t are distinct points of Z; there are a number of cases to consider:

(1) $s = (x, \xi)$ and $t = (y, \rho)$, where $x \neq y$,
(2) $s = (x, \xi)$ and $t = (x, \rho)$, where $\xi \neq \rho$,
(3) $s = p$ and $t = (x, \xi)$.

The cases are straightforward and we leave the proof to the reader. □

We have shown that the Sorgenfrey line S has a Urysohn ω-connectification. However, even more is true: Every dense in-itself subspace of the Sorgenfrey line has a Urysohn connectification. In order to prove this, it clearly suffices to prove the following:

3.9 Theorem. *The closure in S of every dense-in-itself subspace of S is homeomorphic to S.*

Proof: Let A be a closed, dense-in-itself subspace of S. In the order inherited from S, A can have no jumps (that is, a pair of consecutive elements), since if a

has an immediate successor in A, it would be isolated in A, contradicting the fact that A is dense-in-itself. Since S is a Lindelöf GO-space, $S \setminus A$ is the countable union of (topologically) open subintervals of S and since A has no jumps, each of these subintervals must be of the form $[a,b)$ (rather than (a,b)), for some $a, b \in S$. That is to say, $S \setminus A = \cup\{[a_n, b_n) : n \in \omega\}$. We claim that A (with the order inherited from S) is order complete. To show this, suppose that $B \subset A$ is bounded above in A. Clearly B has a supremum s in S, if $s \in A$, then we are done. If $s \notin A$, then $s = a_k$ for some $k \in \omega$. But then in the order on A, $sup(B) = b_k$ and again we are done. Thus A is an order complete subset of the real line with no jumps and no maximum element (which would be isolated) and hence is order isomorphic to $[0,1)$ or $(0,1)$. Furthermore, since $A \subset S$, A (with the relative Sorgenfrey topology) is a GO-space in which $[a, \rightarrow)$ is open for each $a \in A$. Since no element of A has a successor and for all $a \in A$, $(\leftarrow, a]$ is not open in S, it follows that $(\leftarrow, a]$ is not open in A for any $a \in A$ and so A is homeomorphic (and order isomorphic) to one of the two sets $[0,1)$ or $(0,1)$ with the Sorgenfrey topology. Since these two spaces are both homeomorphic to S we are done. □

4. Embedding second countable spaces into connected and locally connected compact spaces

Bowers proved in [4], that a second countable space X with $\dim(X) = n$ is nowhere locally compact iff it embeds densely into the n-dimensional Menger-Nöbeling space. The fact that this space is connected and locally connected for all $n > 0$ implies the existence of an n-dimensional connected (and locally connected) extension for any n-dimensional nowhere locally compact second countable space if $n > 0$.

We are going to prove the last statement (without local connectedness of the extension) for n-dimensional second countable spaces with no open compact subsets.

4.1 Theorem. *Let X be a second countable space without non-trivial open compact subspaces. Then*
(1) *if* $\dim(X) = 0$, *then there exists a one-dimensional connected metrizable compact extension of X;*
(2) *if* $\dim(X) = n > 0$, *then X can be densely embedded into a compact metrizable connected Y with* $\dim(Y) = n$.

Proof: Let us prove first the trivial parts of the theorem. In [1] it was proved that for any second countable compactification Z of the space X with $Z \backslash X$ dense in itself there is a continuous map $q : Z \to Y$ such that $q \restriction X$ is an embedding, Y is connected and $1 \leq |q^{-1}(y)| \leq 2$ for all $y \in Y$. This implies $\dim(Y) \leq \dim(Z) + 1$ ([13]). We will prove below (see Lemma 4.2) that for any second countable X without open compact subsets there is a metrizable compactification Z of the space X such that $\dim(Z) = n$ and $Z \backslash X$ is dense in itself.

Now it is clear that (1) is true as well as (2) in case $n = \infty$. Let us take up the most difficult case of $0 < \dim(X) = n < \infty$.

4.2 Lemma. *Let* $\dim(X) = n$ *(where* $n \in \omega$ *can be equal to zero). Then there is a compact metrizable extension* Z *of the space* X *such that* $\dim(Z) = n$ *and* $Z \setminus X$ *is dense in itself.*

Proof of Lemma 4.2: It was proved in [1] that for any metrizable compact extension T of X there is a metrizable compact extension $\hat{T}$ of X and a continuous map $e_T : \hat{T} \to T$ such that $\hat{T} \setminus X$ is dense in itself and $e_T \restriction X = id_X$.

Now take any metrizable compact extension Z_0 of X with $\dim(Z_0) = n$. Suppose that we have constructed second countable compact extensions $Z_0, Z_1, \ldots, Z_{2l}$ of the space X such that

(i) there are continuous maps $p_k^{k+1} : Z_{k+1} \to Z_k$ such that $p_k^{k+1} \restriction X = id_X$ for all $k < 2l$;

(ii) $\dim(Z_{2k}) = n$ for all $k \leq l$;

(iii) $Z_{2k+1} \setminus X$ is dense in itself for all $k \leq l - 1$.

Let $Z_{2l+1} = \hat{Z}_{2l}$ and $p_{2l}^{2l+1} = e_{Z_{2l}}$. Let $f : \beta X \to Z_{2l+1}$ be the natural map. It is well known that $\dim(\beta X) = n$, so by Mardešić factorization theorem ([11]) there is a compact metrizable Z_{2l+2} with $\dim(Z_{2l+2}) = n$ and maps $h : \beta X \to Z_{2l+2}$, $p_{2l+1}^{2l+2} : Z_{2l+2} \to Z_{2l+1}$ such that $p_{2l+1}^{2l+2} \circ h = f$.

The inductive construction being carried out we have the inverse system

$$\mathcal{L} : \quad Z_0 \xleftarrow{p_0^1} Z_1 \xleftarrow{p_1^2} Z_2 \xleftarrow{p_2^3} \ldots \xleftarrow{p_{2l-1}^{2l}} Z_{2l} \xleftarrow{p_{2l}^{2l+1}} Z_{2l+1} \xleftarrow{p_{2l+1}^{2l+2}} \ldots$$

of compact metrizable extensions of X with the following properties:

(iv) $\dim(Z_{2l}) = n$ for all $l \in \omega$;

(v) $Z_{2l+1} \setminus X$ is dense in itself for all $l \in \omega$;

(vi) $p_l^{l+1} \restriction X = id_X$ for all $l \in \omega$.

Let $Z = \varprojlim \mathcal{L}$. It is evident that Z is a compact metrizable extension of X.

The space Z is also a limit of a cofinal subsystem of $\mathcal{L}$ consisting of Z_{2l}, $l \in \omega$. Since $\dim(Z_{2l}) = n$ for all l, we have also $\dim(Z) = n$.

It is straightforward that $Z \setminus X = \varprojlim (Z_l \setminus X)$, so if there is an isolated point $z \in Z \setminus X$, then $p_{2l+1}(z)$ would be isolated in $Z_{2l+1} \setminus X$ for some $l \in \omega$ (where $p_{2l+1} : Z \to Z_{2l+1}$ is the relevant limit map) which gives a contradiction. ⊿

4.3 Lemma. *Let* Z *be a second countable space with* $\dim(Z) = n$ *and* $A \subset Z$*. Then there is a countable base* $\mathcal{B}$ *in* Z *such that for any* $V \in \mathcal{B}$ *we have* $\dim(Bd_Z(V)) \leq (n-1)$ *and* $Bd_Z(V) \cap A$ *is nowhere dense in* A*.*

Proof of Lemma 4.3: Let $x \in Z$ and $x \in U \in \mathcal{T}(x, Z)$. Denote by ρ some metric compatible with $\mathcal{T}(Z)$. There is an $\varepsilon > 0$ such that the closure of the x-ball $B(x, \varepsilon)$ lies in U. Let a regular open $V \in \mathcal{T}(x, X)$ be such that $\dim(Bd(V)) \leq (n-1)$ and $V \subset B(x, \frac{\varepsilon}{4})$. Denote by W the set $Z \setminus cl_Z(V)$. Then $F = Z \setminus (V \cup W)$ is nowhere dense in Z and $\dim(F) \leq (n-1)$. The set $F \cap A$ can have nonempty interior in A. Let $V_A = V \cap A$ and $W_A = W \cap A$. Put $H = W_A \cup Int_A(F \cap A)$. Now the set $F_A = A \backslash (V_A \cup H)$ is nowhere dense in A. Observe also that $G = cl_Z(V_A) \cap cl_Z(H) \subset cl_Z(V) \cap cl_Z(W) = F$. Hence $\dim(G) \leq (n-1)$. The space $Z_1 = Z \backslash G$ has the dimension $\leq n$ and $cl_Z(\{x\} \cup V_A) \cap Z_1$ and $cl_Z(H) \cap Z_1$ are closed and disjoint in Z_1. The diameter of $cl_Z(\{x\} \cup V_A) \cap Z_1$ is less than or equal to $\frac{\varepsilon}{2}$ so that there is a regular open set V_1 in Z_1 such that $V_1 \supset (\{x\} \cup V_A)$, $\operatorname{diam}(V_1) < \varepsilon$, $cl_{Z_1}(V_1) \cap H = \emptyset$ and $\dim(Bd_{Z_1}(V_1)) \leq (n-1)$.

Then $Bd_Z(V_1) \subset Bd_{Z_1}(V_1) \cup G$. Since G is a G_δ-set in Z it is easy to represent $Bd_{Z_1}(V_1) \cup G$ as a countable union of its closed subsets of dimension $\leq n-1$, so that $\dim(Bd_Z(V_1)) \leq (n-1)$. Evidently, $Bd_Z(V_1) \cap A \subset G \cap A \subset F_A$ and thus is nowhere dense in A. We have diam $V_1 < \varepsilon$ and $x \in V_1$ so that $V_1 \subset B(x, \varepsilon) \subset U$. The open set $U \ni x$ was chosen arbitrarily, so the sets V_1 constructed as above make a base in Z at x. Constructing this base for all $x \in Z$ we obtain a base $\mathcal{B}_1$ in Z such that $\dim(Bd_Z(U)) \leq (n-1)$ and $Bd_Z(U) \cap A$ is nowhere dense in A for any $U \in \mathcal{B}_1$. Now to finish the proof, choose a countable base $\mathcal{B} \subset \mathcal{B}_1$. ⊿

Returning to the proof of our theorem, use Lemma 4.2 to find a metrizable compact extension Z of X with $\dim(Z) = n$ and $Z \backslash X$ dense in itself.

Apply Lemma 4.3 to find a countable base $\mathcal{B} = \{U_l : l \in \omega\}$ in Z such that $\mathcal{B}$ is closed with respect to finite unions, $Bd(U_l) \cap (Z \backslash X)$ is nowhere dense in $Z \backslash X$ and $\dim(Bd(U_l)) \leq (n-1)$ for all $l \in \omega$. Denote by $\mathcal{C} = \{W_l : l \in \omega\}$ the family of all proper clopen subsets of Z. We will consider only the case when $\mathcal{C}$ is infinite because otherwise the proof is a simple exercise involving no considerations given below.

It was proved in [1] that

$$\lim_{l \to \infty} \rho(W_l \cap (Z \backslash X), (Z \backslash X) \backslash W_l) = 0,$$

so that for every $l \in \omega$ we can pick a $p_l \in W_l \cap (Z \backslash X)$ and $q_l \in (Z \backslash X) \backslash W_l$ with the following properties:

(vii) $p_l \notin \cup\{Bd(U_k) : k \leq l\} \cup \{p_k : k < l\} \cup \{q_k : k < l\}$;

(viii) $q_l \notin \cup\{Bd(U_k) : k \leq l\} \cup \{p_k : k \leq l\} \cup \{q_k : k < l\}$;

(ix) $\rho(p_l, q_l) \leq 2\rho(W_l \cap (Z \backslash X), (Z \backslash X) \backslash W_l)$.

Then $\rho(p_l, q_l) \to 0$ if $l \to \infty$ and identifying p_l with q_l for all $l \in \omega$ we obtain a quotient map $q : Z \to Y$, where the quotient space Y is a metrizable compactification of X. It was proved in [1] that Y has to be connected. Let us check that $\dim(Y) = n$.

It is clear that $\mathcal{D} = \{q^{\#}(U) : U \in \mathcal{B}\}$ is a base in Y. We are going to prove that $\dim(Bd_Y(V)) \leq (n-1)$ for all $V \in \mathcal{D}$.

Suppose that $V = q^{\#}(U_k)$ for some $k \in \omega$. Denote by P_l the set $\{p_l, q_l\}$. For every $l \in \omega$ let $z_l \in Y$ be such that $\{z_l\} = q(P_l)$.

1) *Claim.* $\dim(q(Bd_Z(U_k))) \leq (n-1)$. Indeed, only finitely many sets P_l intersect the set $F = Bd_Z(U_k)$ so that $q(F)$ is a union of a finite set G and a subset H which is homeomorphic to a subset of F. Since $\dim(H) \leq (n-1)$ we obtain the inequality $\dim(q(F)) \leq (n-1)$ using the countable sum theorem for the dimension dim. ⊿

2) *Claim.* $Bd_Y(V) \subset q(F) \cup \{z_l : l \in \omega\}$. To prove this, take any $z \in Bd_Y(V)$. If $z \notin \{z_l : l \in \omega\}$ then $z = q(t)$ for some $t \in Z$ and $\{t\} = q^{-1}q(t)$. If $t \in U_k$, then $z \in q^{\#}(U_k) = V$. If $t \in Z \backslash \overline{U}_k$, then $z \notin \overline{q^{\#}(U_k)} = \overline{V}$. Therefore $t \notin U_k$ and $t \notin Z \backslash \overline{U}_k$, so that $t \in Bd_Z(U_k) = F$ and $z \in q(F)$. ⊿

To finish the proof of our theorem, observe that $\dim(Bd_Y(V)) \leq (n-1)$ because $Bd_Y(V)$ lies in $q(F) \cup \{z_l : l \in \omega\}$ and the last set has the dimension $\leq (n-1)$ by Claim 1 and the countable sum theorem. □

4.4 Remark. *If X is a space like in* 4.1, *it is not true in general, that X has a locally connected extension. Indeed, if X is a locally compact, non-compact, connected but not locally connected space, then X has no non-trivial open compact subspaces and X will be open in any extension Y, so Y will not be locally connected at the points at which X is not locally connected. The subset*

$$P = (\{0\} \times [-1,1]) \cup \{(x, \sin(\tfrac{1}{x})) : x > 0\}$$

of the plane $\mathbf{R}^2$ serves as an example of such a space X.

Neither is it possible to connectify all second countable locally connected spaces preserving local connectedness, as the following example shows.

4.5 Example. *There exists a locally connected second countable Tychonoff space X without open compact subsets such that no connected extension of X is locally connected.*

Proof: Let Y be a countable metrizable hedgehog, that is $Y = \{p\} \cup \{I_n : n \in \omega\}$, where $I_n = (0,1]$ for all n and if $x, y \in Y$ then the distance $d(x,y)$ between x and y is defined as follows:

1) $d(p,y) = y$ if $y \in \cup\{I_n : n \in \omega\}$;
2) $d(x,y) = |x-y|$, if there exists an $n \in \omega$ such that $\{x,y\} \subset I_n$;
3) if x and y are in different spines I_k and I_l, then $d(x,y) = x + y$.

Let $X = Y \oplus Y$. Assume that Z is a Hausdorff extension of X with $Z \neq X$. Pick a point $z \in Z \backslash X$. Suppose that there is a connected neighborhood U of z in Z whose closure does not contain p. Being locally compact, any I_n is open in Z. There must be an $n \in \omega$ such that $H = U \cap I_n \neq \emptyset$. Now H is open in Z and hence in U. Being a proper subset of U the set H cannot be closed in U because of connectedness of U. Therefore $cl_U(H) \backslash H \neq \emptyset$ which is a contradiction with $cl_Z(H) \subset I_n$.

To finish the proof it suffices to observe that since the space X is disconnected there have to be points in $Z\setminus X$ for any connectification Z of the space X. $\square$

4.6 Remark. *On the other hand, if X is a second countable Tychonoff locally compact locally connected space, then its one-point compactification is locally connected by a theorem of R.L. Moore* (*see* [9, Chapter VI, §49, Theorem 3]). *This gives a reason to hope that metric spaces without points of local compactness have a locally connected connectification* (*see Problem* 5.10).

5. Unsolved problems

It usually happens that the more questions one solves the more new ones arise; this paper is by no means an exception. The following are only a sample of the questions we did not succeed in answering!

5.1 Problem. *Characterize those varieties $\mathcal{V}$ of topological groups for which the $\mathcal{V}$-free topological group $F\mathcal{V}(X)$ is locally connected whenever a space X is connected and locally connected.*

5.2 Problem. *Is there a maximal element in the family of all connected topological group topologies on the reals* (*ordered by inclusion*)*?*

5.3 Problem. *Let G be a connected totally bounded Abelian topological group. Can one find a strictly finer connected group topology for G? What if G is Lindelöf* (*or locally compact*)*?*

5.4 Problem. *Let X be a countable Hausdorff space without isolated points. Does X have a locally connected connectification?* (*The answer is "yes" if X is locally connected.*)

5.5 Problem. *Let X be a countable Tychonoff space without isolated points. Does X have a locally connected Tychonoff* (*Urysohn or Hausdorff*) *connectification?*

5.6 Problem. *Assume $MA+\neg CH$. Let X be a countable Hausdorff space without isolated points and such that $\pi w(X) < \mathfrak{c}$. Does X have a locally connected countable connectification?*

5.7 Problem. *Assume $MA + \neg CH$. Let X be a countable Tychonoff space without isolated points and such that $w(X) < \mathfrak{c}$. Does X have a Tychonoff connectification?*

5.8 Problem. *Let X be a countable Tychonoff non-discrete topological group. Does X have a locally connected Tychonoff* (*Urysohn or Hausdorff*) *connectification?*

5.9 Problem. *Let X be a countable Tychonoff non-discrete topological group. Does X have a locally connected Tychonoff (Urysohn or Hausdorff) countable connectification?*

5.10 Problem. *Let X be a metric space with no points of local compactness. Does X have a Tychonoff (or metrizable) locally connected connectification?*

5.11 Problem. *Does the Sorgenfrey line have a Hausdorff locally connected connectification?*

5.12 Problem. *Does the Sorgenfrey line have a Hausdorff locally connected connectification with countable remainder?*

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