The Tamano Theorem in $\mathcal{MAP}$

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Abstract. In this paper we continue with the study of paracompact maps introduced in [1]. We give two external characterizations for paracompact maps including a characterization analogous to The Tamano Theorem in the category $\mathcal{TOP}$ (of topological spaces and continuous maps as morphisms). A necessary and sufficient condition for the Tychonoff product of a closed map and a compact map to be closed is also given.

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Classification: Primary 54C05, 54C10; Secondary 54B30, 54C99

1. Introduction

The study of General Topology is usually concerned with the category $\mathcal{TOP}$ of topological spaces as objects, and continuous maps as morphisms. One of the most important classes of topological spaces is the class of *paracompact spaces*. Paracompact spaces simultaneously generalize both compact spaces and metrizable spaces and were introduced by J. Dieudonné in 1944 [5].

The concepts of *space* and *map* are equally important and one can even look at a space as a map from this space onto a singleton space and in this manner identify these two concepts. With this in mind, a branch of General Topology which has become known as General Topology of Continuous Maps, or Fibrewise General Topology, was initiated. This field of research is concerned most of all in extending the main notions and results concerning topological spaces to those of continuous maps. In this way one can see some well-known results in a new and clearer light and one can also be led to further developments which otherwise would not have suggested themselves. The fibrewise viewpoint is standard in the theory of fibre bundles, however, it has been recognized relatively recently that the same viewpoint is also as important in other areas such as General Topology.

For an arbitrary topological space Y one considers the category $\mathcal{TOP}_Y$, the objects of which are continuous maps into the space Y, and for the objects $f : X \to Y$ and $g : Z \to Y$, a morphism from f into g is a continuous map $\lambda : X \to Z$ with the property $f = g \circ \lambda$. This situation is a generalization of the category $\mathcal{TOP}$, since the category $\mathcal{TOP}$ is isomorphic to the particular case of $\mathcal{TOP}_Y$ in which the space Y is a singleton space.

The research carried out showed a strong analogy in the behaviour of spaces and maps and it was possible to extend the main notions and results of spaces to that of maps. Since the considered case is of a wider generality (compared to that of spaces), the results obtained for maps are technically more complicated. Moreover there are moments which are specific for maps. For example, there is no analogue to Urysohn's Lemma for maps and so normality and functional normality do not coincide and, as a consequence, there exist two theories of compactifications, one for Hausdorff compactifications and one for Tychonoff compactifications.

Some results in the General Topology of Continuous Maps were obtained quite some time ago. For example, in 1947, I.A. Vainstein [21] proposed the name of compact maps to perfect maps, G.T. Whyburn in 1953 [22], [23], as did G.L. Cain, N. Krolevets, V.M. Ulyanov [20] and others, considered compactifications of maps. In the meantime, until quite recently, there was no consistent unified theory for maps. One of the main reasons might have been the lack of separation axioms for maps, especially that of Tychonoffness (and complete regularity) and also that of (functional) normality and collectionwise normality.

Completely regular and Tychonoff maps, as well as (functionally) normal maps, were defined by B.A. Pasynkov in 1984 [15]. These definitions made it possible to generalize and obtain an analogue to the theorem on the embedding of Tychonoff spaces of weight τ into I^τ and to the existence of a compactification for a Tychonoff space having the same weight. It was also possible to construct a maximal Tychonoff compactification for a Tychonoff map (i.e. construct an analogue to the Stone-Čech compactification). Collectionwise normal maps were defined by the author [4] and enabled the definition of *metrizable type maps*, giving a satisfactory fibrewise version of the theory of metrizable spaces.

In most cases there is some choice in defining properties on maps and one usually prefers the simplest and the one that gives the most complete generalization of the corresponding results in the category $\mathcal{TOP}$. It would be beneficial to have a more systematic way of extending definitions and results from the category $\mathcal{TOP}$ to the category $\mathcal{TOP}_Y$ and some hope is provided by the link between Fibrewise Topology and Topos Theory [9], [10], [12], [13]. Unfortunately, as was noted in [8], this approach has several drawbacks. In defining *compact maps* [16, Proposition 2.2 (V.P. Norin)], *paracompact maps* [1], *metacompact maps*, *subparacompact maps*, *submetacompact maps* [3] and *metrizable type maps* [4], one can see a systematic method in defining notions in the category $\mathcal{TOP}_Y$ (or more general in the category $\mathcal{MAP}$) corresponding to definitions which involve coverings or bases of topological spaces. This construction gave satisfactory definitions which can be seen from the results obtained for such maps [1], [3], [4], [16]. One can also add that the definitions of paracompact maps, metacompact maps, subparacompact maps and submetacompact maps strengthened the result that paracompactness, metacompactness, subparacompactness and submetacompactness are all inverse invariant of perfect maps. Namely, it was proved that the inverse image of a paracompact T_2 (resp. subparacompact, metacompact, submetacompact) space by a paracompact T_2 (resp. subparacompact, metacompact, submetacompact) map is paracompact T_2 (resp. subparacompact, metacompact, submetacompact) [1], [3].

In [2], a category of maps $\mathcal{MAP}$ in which one does not restrain oneself with a fixed base space Y was introduced. The objects of $\mathcal{MAP}$ are continuous maps from any topological space into any topological space. For two objects $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, a morphism from f_1 into f_2 is a pair of continuous maps $\{\lambda_T, \lambda_B\}$, where $\lambda_T : X_1 \to X_2$ and $\lambda_B : Y_1 \to Y_2$, such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\lambda_T} & X_2 \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow[\lambda_B]{} & Y_2 \end{array}$$

is commutative. It is not difficult to see that this definition of a morphism in $\mathcal{MAP}$ satisfies the necessary axioms that morphisms should satisfy in any category (see, for example, [17]).

Let $\mathcal{P}_T$ and $\mathcal{P}_B$ be two topological/set theoretic properties of maps (for example: closed, open, 1-1, onto, etc.). If λ_T has property $\mathcal{P}_T$ and λ_B has property $\mathcal{P}_B$ then we say that $\{\lambda_T, \lambda_B\}$ is a $\{\mathcal{P}_T, \mathcal{P}_B\}$-*morphism*. If $\mathcal{P}_T$ is the *continuous* property, then we say that $\{\lambda_T, \lambda_B\}$ is a $\{*, \mathcal{P}_B\}$-*morphism*, similarly for $\mathcal{P}_B$. Therefore, a $\{*, *\}$-morphism is just a morphism. Also, if $\mathcal{P}_T = \mathcal{P}_B = \mathcal{P}$ then a $\{\mathcal{P}_T, \mathcal{P}_B\}$-morphism is called a $\mathcal{P}$-*morphism*.

For more details and undefined terms on the General Topology of Continuous Maps one can consult [1], [2], [3], [4], [7], [8], [11], [15], [16].

2. Preliminary definitions in the category $\mathcal{MAP}$

Separation axioms for maps had already been defined in the category $\mathcal{TOP}_Y$ and since these axioms involve only one map, they were also taken to be defined for the category $\mathcal{MAP}$. For completeness, we give the definitions of those separation axioms which appear in the following sections.

Definition 2.1. The subsets A and B of the space X are said to be *functionally separated in* $U \subset X$, if the sets $A \cap U$ and $B \cap U$ are functionally separated in U (that is, there exists a continuous function $\phi : U \to [0, 1]$ such that $A \cap U \subset \phi^{-1}(0)$ and $B \cap U \subset \phi^{-1}(1)$).

Definition 2.2. A continuous map $f : X \to Y$ is said to be *functionally Hausdorff or* $T_{2\frac{1}{2}}$, if for every two distinct points x and x' in X lying in the same fibre, there exists a neighborhood O of the point $f(x)$, such that the sets $\{x\}$ and $\{x'\}$ are functionally separated in $f^{-1}O$.

Definition 2.3. A continuous map $f : X \to Y$ is said to be *completely regular*, if for every point $x \in X$ and every closed set F in X, not containing the point x, there exists a neighborhood O of the point $f(x)$, such that the sets $\{x\}$ and F are functionally separated in $f^{-1}O$. A completely regular T_0-map is called a *Tychonoff* (or $T_{3\frac{1}{2}}$-) map, where a map $f : X \to Y$ is said to be a T_0-map if for every two distinct points $x, x' \in X$ lying in the same fibre, at least one of the points x, x' has a neighborhood in X which does not contain the other point.

It can be easily verified that every Tychonoff map is functionally Hausdorff.

Definition 2.4. For a continuous map $f : X \to Y$, the subsets A and B of the space X are said to be *f-functionally separated* (resp. *f-neighborhood separated*), if for every $y \in Y$ there exists a neighborhood O of y, such that the sets A and B are functionally separated (resp. neighborhood separated) in $f^{-1}O$.

Definition 2.5. A continuous map $f : X \to Y$ is said to be *functionally prenormal* (resp. *prenormal*) if every two disjoint closed sets in X are f-functionally separated (resp. f-neighborhood separated).

Therefore, a functionally prenormal map is prenormal.

Definition 2.6. A continuous map $f : X \to Y$ is said to be *functionally normal* (resp. *normal*) if for every open set O in Y the map $f : f^{-1}O \to O$ is functionally prenormal (resp. prenormal).

It is evident that every normal map is prenormal. Also, a functionally normal map is normal and functionally prenormal. A normal T_3-map is called a T_4-map, and a functionally normal $T_{3\frac{1}{2}}$-map is called a $T_{4\frac{1}{2}}$-map.

If the map f is closed we have the following two results.

Proposition 2.1. *For a closed map $f : X \to Y$ we have:*

1. *if for every $y \in Y$, every $x \in f^{-1}y$ and every closed set A in $f^{-1}y$ such that $x \notin A$, the sets $\{x\}$ and A are functionally separated in some neighborhood of $f^{-1}y$, then f is completely regular;*
2. *if for every $y \in Y$, every two disjoint closed sets in $f^{-1}y$ are functionally separated in some neighborhood of $f^{-1}y$ in X, then f is functionally normal.*

We now give the definition of a submap as an analogue of subspace. Since we do not restrict ourselves to a fixed base space Y our definition slightly differs from that given in the category $\mathcal{TOP}_Y$ ([15]). This definition was introduced in [2].

Definition 2.7. A map $g : A \to B$ is said to be a (closed, open, everywhere dense, etc.) submap of the map $f : X \to Y$, if g is the restriction of the map f on the (closed, open, everywhere dense, etc.) subset A of the space X and $g(A) = f(A) \subset B \subset Y$.

The following result is known ([2]).

Proposition 2.2. *Any submap of a T_i-map is a T_i-map for $i \leq 3\frac{1}{2}$. Prenormality, functional prenormality, normality and functional normality are hereditary with respect to closed submaps.*

The proof of the following proposition for the case $B = Y$ can be found in [16]. For the situation given below the proof is analogous and so is omitted. Remember that in $\mathcal{TOP}_Y$ (and also in $\mathcal{MAP}$), by a *compact map* we mean a *perfect map*, namely, a closed map with compact fibres. It is evident that a closed submap of a compact map is compact.

Proposition 2.3. *Let a compact map $g : A \to B$ be a submap of a T_2-map $f : X \to Y$ and let B be a closed subset of Y, then g is a closed submap of f.*

Finally, we give the definition of a compactification for a continuous map ([22], [23]).

Definition 2.8. A compact map $bf : b_f X \to Y$ is said to be a compactification of $f : X \to Y$ if there exists a {dense homeomorphic embedding}-morphism $\{\lambda, \mathrm{id}_Y\} : f \to bf$.

In the above situation we usually identify X with $\lambda(X)$ and so $b_f X = [X]_{b_f X}$ and $f = bf|_X$, where by $[X]_{b_f X}$ we mean the closure of X in $b_f X$. For details concerning compactifications of Tychonoff maps, in particular the construction of the maximal Tychonoff compactification $\beta f : \beta_f X \to Y$ of a Tychonoff map $f : X \to Y$, one can consult [15], [16], [11].

3. Paracompact maps

Paracompact maps were defined in [1].

Let $f : X \to Y$ be a continuous map of a topological space X into a topological space (Y, τ). For $y \in Y$, a collection of subsets of X is said to be y-locally finite if for every $x \in f^{-1}y$, there exists a neighborhood O_x of x in X, such that O_x meets finitely many elements of the collection. If the collection $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ is a y-locally finite open (in X) collection, then $\mathcal{U}$ is locally finite in $\bigcup_{x\in f^{-1}y} O_x$, i.e. for every $z \in \bigcup_{x\in f^{-1}y} O_x$, z has a neighborhood in X which meets finitely many elements of $\mathcal{U}$. In particular, if f is closed and $\mathcal{U}$ covers $f^{-1}y$, then there exists a neighborhood O_y of y such that $\mathcal{U}$ is a cover of $f^{-1}O_y$ and is locally finite in $f^{-1}O_y$, that is for every $z \in f^{-1}O_y$, z has a neighborhood in $f^{-1}O_y$ (and so in X) such that it intersects finitely many elements of $\mathcal{U}$.

Definition 3.1. We call a continuous map $f : X \to Y$ *paracompact* if for every point $y \in Y$ and every open (in X) cover $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ of the fibre $f^{-1}y$ (i.e. $f^{-1}y \subset \bigcup\{U_\alpha : \alpha \in \mathcal{A}\}$), there exists a neighborhood O_y of y such that $f^{-1}O_y$ is covered by $\mathcal{U}$ and $(f^{-1}O_y \wedge \mathcal{U})$ has an open (in X) y-locally finite refinement in $f^{-1}O_y$.

Note that if f is paracompact then it is a closed map and it is fibrewise paracompact, i.e. for every $y \in Y$, $f^{-1}y$ is paracompact. The converse is not true, that is there exists an example of a Tychonoff closed map with paracompact fibres which is not paracompact ([1]). Every compact map is paracompact, and every closed submap of a paracompact map is paracompact. In [1] it was proved that a paracompact T_2-map is regular and normal (and, hence, a T_4-map).

We now prove the following result.

Proposition 3.1. *A paracompact $T_{2\frac{1}{2}}$-map is completely regular (and so is Tychonoff) and functionally normal.*

Proof: We first show that if f is paracompact then it is a completely regular map. Since f is closed, by Proposition 2.1 it is enough to show that for every point $y \in Y$, every point $x \in f^{-1}y$ and every closed in $f^{-1}y$ set F, such that $x \notin F$, the sets $\{x\}$ and F are functionally separated in some neighborhood of the fibre $f^{-1}y$.

Thus, let y be an arbitrary point of Y and let $x \in f^{-1}y$. Consider a closed in $f^{-1}y$ set F such that $x \notin F$. For every $z \in F$ there exists a neighborhood $O_z(y)$ of y such that $\{x\}$ and $\{z\}$ are functionally separated in $f^{-1}O_z(y)$, that is there exists a function $f_z : f^{-1}O_z(y) \to [0,1]$ with $f_z x = 0$ and $f_z z = 1$.

Let $H \subset X$ be a closed set with $H \cap f^{-1}y = F$ and let

$$U_0(z) = f_z^{-1}([0,1[) \cap X \setminus H, \quad U_1(z) = f_z^{-1}(]\tfrac{1}{2},1]).$$

The collection $\mathcal{U} = \{U_0(z), U_1(z) : z \in F\}$ covers $f^{-1}y$ and so there exists a neighborhood O of y such that $f^{-1}O$ is covered by $\mathcal{U}$ and there exists a closed locally finite in $f^{-1}O$ refinement $\mathcal{A}$ of $\mathcal{U} \wedge f^{-1}O$. For every $A \in \mathcal{A}$ fix an element $U(A) \in \mathcal{U}$ with $A \subset U(A)$. We now construct a function $g : f^{-1}O \to \mathbb{R}$ in the following manner. Let p be an arbitrary element of $f^{-1}O$. Since $\mathcal{A}$ is locally finite and closed in $f^{-1}O$ there exists only a finite number of elements of $\mathcal{A}$ containing p, say $p \in A_1, \ldots, A_k$. Now let $g(p) = \sum_{i=1}^{k} f_{z(i)}(p)$, where $U(A_i) = U_s(z(i))$, $s = 0$ or $s = 1$ for every $i = 1, \ldots, k$. It is not difficult to see that the function g is continuous. Finally by letting $h = 2\min(\frac{1}{2}, g)$ we get the desired function.

We have thus proved that f is completely regular and so is a Tychonoff map. Analogously, using the above proof and Proposition 2.1, one can show that f is functionally normal. □

The following definition for the fibrewise analogue of star refinements was given in [1].

Definition 3.2. Let $f : X \to Y$ be a continuous map and $y \in Y$. Let $\mathcal{U}$ be an open (in X) cover of $f^{-1}y$. The collection $\mathcal{V}$ of subsets of X is said to be a y-star refinement of $\mathcal{U}$ if $V \cap f^{-1}y \neq \emptyset$ for every $V \in \mathcal{V}$, $f^{-1}y \subset \bigcup\{V : V \in \mathcal{V}\}$ and there exists a neighborhood $O(y)$ of y in Y such that $\mathcal{U}$ covers $f^{-1}O(y)$ and $\{St(V, \mathcal{V}) : V \in \mathcal{V}\} < \mathcal{U} \wedge f^{-1}O(y)$.

This definition made it possible to prove the following theorem.

Theorem 3.2. *For a T_1-map $f : X \to Y$ the following are equivalent:*

1. *the map f is paracompact T_2;*
2. *for every $y \in Y$ and every open (in X) cover $\mathcal{U}$ of $f^{-1}y$, there exists an open y-star refinement $\mathcal{V}$ of $\mathcal{U}$;*
3. *the map f is regular and for every $y \in Y$ and every open (in X) cover $\mathcal{U}$ of $f^{-1}y$, there exists a neighborhood $O(y)$ of y in Y such that $f^{-1}O(y)$ is covered by $\mathcal{U}$ and $(f^{-1}O(y) \wedge \mathcal{U})$ has a y-σ-locally finite open refinement;*
4. *the map f is regular and for every $y \in Y$ and every open (in X) cover $\mathcal{U}$ of $f^{-1}y$, there exists a neighborhood $O(y)$ of y in Y such that $f^{-1}O(y)$ is covered by $\mathcal{U}$ and $(f^{-1}O(y) \wedge \mathcal{U})$ has a y-σ-discrete open refinement.*

The following weaker definition of star refinements will be needed.

Definition 3.3. Let $f : X \to Y$ be a continuous map and $y \in Y$. Let $\mathcal{U}$ be an open (in X) cover of $f^{-1}y$. The collection $\mathcal{V}$ of subsets of X is said to be a weak y-star refinement of $\mathcal{U}$ if $V \cap f^{-1}y \neq \emptyset$ for every $V \in \mathcal{V}$, $f^{-1}y \subset \bigcup\{V : V \in \mathcal{V}\}$ and there exists a neighborhood $O(y)$ of y in Y such that $\mathcal{U}$ covers $f^{-1}O(y)$ and $\{St(V, \mathcal{V}) \cap f^{-1}O_V : V \in \mathcal{V}\} < U \wedge f^{-1}O(y)$, where for every $V \in \mathcal{V}$ we have that O_V is a neighborhood of y in Y, $O_V \subset O(y)$ and $V \subset f^{-1}O(y)$.

The aim of this section is to prove the following theorem which is used to prove The Tamano Theorem in $\mathcal{MAP}$ in Section 5. Remember that a finite collection $\mathcal{H}$ is said to be *centered* if $\bigcap \mathcal{H} \neq \emptyset$ and *centered at* x if $x \in \bigcap \mathcal{H}$.

Theorem 3.3. *For a T_1-map $f : X \to Y$ the following are equivalent:*

1. *the map f is paracompact T_2;*
2. *for every $y \in Y$ and every open (in X) cover $\mathcal{U}$ of $f^{-1}y$, there exists an open weak y-star refinement $\mathcal{V}$ of $\mathcal{U}$;*
3. *for every $y \in Y$ and every open (in X) cover $\mathcal{U}$ of $f^{-1}y$, there exists a neighborhood $O(y)$ of y in Y and an open cover $\mathcal{V}$ of $f^{-1}y$ such that $\mathcal{U}$ covers $f^{-1}O(y)$, $\mathcal{V} < \mathcal{U} \wedge f^{-1}O(y)$ and for every $V \in \mathcal{V}$, $V \cap f^{-1}y \neq \emptyset$ and there exists a finite collection $\mathcal{W} \subset \mathcal{U}$ and a neighborhood of $O_V \subset O(y)$ of y satisfying $St(V, \mathcal{V}) \cap f^{-1}O_V \subset \bigcup \mathcal{W}$ and $V \subset f^{-1}O_V \cap (\bigcap \mathcal{W})$;*
4. *for every $y \in Y$ and every open (in X) cover $\mathcal{U}$ of $f^{-1}y$, there exists a neighborhood $O(y)$ of y in Y and an open cover $\mathcal{V}$ of $f^{-1}y$ such that $V \cap f^{-1}y \neq \emptyset$, $\mathcal{U}$ covers $f^{-1}O(y)$, $\mathcal{V} < \mathcal{U} \wedge f^{-1}O(y)$ and for every $x \in f^{-1}O(y)$, there exists an open set $V \subset f^{-1}O(y)$ containing x, a neighborhood $O_V \subset O(y)$ of y and a finite collection $\mathcal{W} \subset \mathcal{U}$ centered at x satisfying $St(V, \mathcal{V}) \cap f^{-1}O_V \subset \bigcup \mathcal{W}$ and $V \subset f^{-1}O_V$.*

The proof of Theorem 3.3 is preceded by two lemmas.

Lemma 3.4. *A T_1-map $f : X \to Y$ satisfying* (3) *of Theorem* 3.3 *is regular* (*and, therefore, Tychonoff*).

Proof: We first note that under the above hypothesis, the map f is closed. Consider a point $x \in f^{-1}y$ and a closed set $F \subset f^{-1}y$ such that $x \notin F$. Consider $A = f^{-1}y \setminus \{x\}$ and $B = f^{-1}y \setminus F$ which are open in $f^{-1}y$. Take open sets U, V in X such that $U \cap f^{-1}y = A$ and $V \cap f^{-1}y = B$. Then $\mathcal{U} = \{U, V\}$ is an open (in X) cover of $f^{-1}y$. Let $O(y)$ be a neighborhood of y in Y and let $\mathcal{W}$ be an open in X cover of $f^{-1}y$ satisfying property (3). Let $W(x) \in \mathcal{W}$ contain x. There exists a finite collection $\mathcal{P} \subset \mathcal{U}$ and a neighborhood $O_{W(x)} \subset O(y)$ of y satisfying $St(W(x), \mathcal{W}) \cap f^{-1}O_{W(x)} \subset \bigcup \mathcal{P}$ and $W(x) \subset f^{-1}O(y) \cap (\bigcap \mathcal{P})$. Since $x \in W(x)$ and $W(x) \subset \bigcap \mathcal{P}$ it follows that $\mathcal{P} = \{V\}$, which shows that $St(W(x), \mathcal{W}) \cap f^{-1}O_{W(x)} \subset V$. Therefore, $[W(x)]_X \cap F = \emptyset$ since $F \subset f^{-1}O_{W(x)}$, which shows that x and F are neighborhood separated in X. Consequently f is regular (since f is closed). $\square$

Lemma 3.5. *Let $y \in Y$. If every open (in X) cover of $f^{-1}y$ has an open weak y-star refinement, then every open (in X) cover of $f^{-1}y$ has also an open y-σ-discrete refinement. Moreover, if f is T_1 then it is also regular.*

PROOF: Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ be an open (in X) cover of $f^{-1}y$. Let $\mathcal{U}_0 = \mathcal{U}$ and take a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open (in X) covers of $f^{-1}y$ such that $\mathcal{U}_{i+1}$ is a weak y-star refinement of $\mathcal{U}_i$ for $i < \omega$. Thus there exists a sequence of open neighborhoods $O_i(y)$ of y in Y such that $O_{i+1}(y) \subset O_i(y)$ for $i < \omega$, and

$$\{St(U,\mathcal{U}_{i+1}) \cap f^{-1}O_U : U \in \mathcal{U}_{i+1}\} < \mathcal{U}_i' \qquad \text{for } i < \omega, \tag{3.1}$$

where $\mathcal{U}_i' = \mathcal{U}_i \wedge f^{-1}O_i(y)$, $O_U \subset O_i(y)$ and $U \subset f^{-1}O_U$. Note that for $i < \omega$ we have that $\bigcup\{U : U \in \mathcal{U}_i'\} = f^{-1}O_i(y)$.

For every $\alpha \in \mathcal{A}$ and $0 < i < \omega$ take the collection of open sets

$$\mathcal{U}_{\alpha,i} = \{U \in \mathcal{U}_i : St(U,\mathcal{U}_i) \cap f^{-1}O(U) \subset U_\alpha \text{ for some } O(U) \subset O_{i-1}(y)\} \tag{3.2}$$

and let $U_{\alpha,i} = \bigcup \mathcal{U}_{\alpha,i}$, where $O(U)$ is a neighborhood of y in Y.

The collection $\{U_{\alpha,i} \cap U_\alpha : \alpha \in \mathcal{A}\}$ is an open refinement of $\mathcal{U} \wedge f^{-1}O_{i-1}(y)$ for $0 < i < \omega$ and it can be easily seen that $U_{\alpha,i} \cap f^{-1}y = (U_{\alpha,i} \cap U_\alpha) \cap f^{-1}y$.

We now show that

$$\begin{aligned} &\text{if } U \in \mathcal{U}_{i+1} \text{ and } U \cap U_{\alpha,i} \neq \emptyset, \\ &\text{then } U \cap (f^{-1}O_i(y) \setminus U_{\alpha,i+1}) = \emptyset. \end{aligned} \tag{3.3}$$

Indeed, it follows from (3.2) that for every $U \in \mathcal{U}_{i+1}$ there exists a $W \in \mathcal{U}_i$ and $O_U \subset O_i(y)$, $U \subset f^{-1}O_U$ such that $St(U,\mathcal{U}_{i+1}) \cap f^{-1}O_U \subset W$. Let $U \cap V \neq \emptyset$ for some $V \in \mathcal{U}_{\alpha,i}$. By definition, $V \in \mathcal{U}_i$ and $St(V,\mathcal{U}_i) \cap f^{-1}O(V) \subset U_\alpha$ for some $O(V) \subset O_{i-1}(y)$. Let $O(U) = O_U \cap O(V)$. Then $St(U,\mathcal{U}_{i+1}) \cap f^{-1}O(U) \subset U_\alpha$ and so $U \in \mathcal{U}_{\alpha,i+1}$. Therefore, $U \cap (f^{-1}O_i(y) \setminus U_{\alpha,i+1}) = \emptyset$.

We now take a well-ordering relation $<$ on the set $\mathcal{A}$ and let

$$G_{\alpha_0,i} = \left(U_{\alpha_0,i} \setminus \left[\bigcup_{\alpha<\alpha_0} U_{\alpha,i+1}\right]_X\right) \cap f^{-1}O_i(y). \tag{3.4}$$

For every pair α_1, α_2 of distinct elements of $\mathcal{A}$ we have either $\alpha_1 < \alpha_2$ or $\alpha_2 < \alpha_1$, and therefore either $G_{\alpha_2,i} \subset f^{-1}O_i(y) \setminus U_{\alpha_1,i+1}$ or $G_{\alpha_1,i} \subset f^{-1}O_i(y) \setminus U_{\alpha_2,i+1}$. Hence, from (3.3) it follows that each $U \in \mathcal{U}_{i+1}$ intersects at most one element of $\{G_{\alpha,i} : \alpha \in \mathcal{A}\}$. Therefore, the collection of open sets $\{G_{\alpha,i} : \alpha \in \mathcal{A}\}$ is discrete in $f^{-1}O_i(y)$ for $0 < i < \omega$.

We now show that the collection $\{G_{\alpha,i} : \alpha \in \mathcal{A}, 0 < i < \omega\}$ is a cover of $f^{-1}y$. Let $x \in f^{-1}y$ and denote by $\alpha(x)$ the smallest element in $\mathcal{A}$ such that $x \in U_{\alpha(x),i}$ for some positive integer i. Such an $\alpha(x)$ exists since for $0 < i < \omega$ the collection

$\{U_{\alpha,i} : \alpha \in \mathcal{A}\}$ is a cover of $f^{-1}y$. Since $x \notin U_{\alpha,i+2}$ for $\alpha < \alpha(x)$, it follows from (3.3) that

$$St(x, \mathcal{U}_{i+2}) \cap \left(\bigcup_{\alpha<\alpha(x)} U_{\alpha,i+1} \right) = \emptyset \tag{3.5}$$

and this shows that $x \in G_{\alpha(x),i}$. Since $(G_{\alpha,i} \cap U_\alpha) \cap f^{-1}y = G_{\alpha,i} \cap f^{-1}y$, the collection $\{G_{\alpha,i} \cap U_\alpha : \alpha \in \mathcal{A}, 0 < i < \omega\}$ is a y-σ-discrete refinement of $\mathcal{U}$.

Finally, from Lemma 3.4 it follows that if f is a T_1-map then it is regular. $\square$

Proof of Theorem 3.3: Since any y-star refinement is a weak y-star refinement, the implication (1) $\Rightarrow$ (2) follows from Theorem 3.2.

To see that (2) $\Rightarrow$ (4) one only has to take an open weak y-star refinement $\mathcal{V}$ of $\mathcal{U}$.

We now show that (4) $\Rightarrow$ (3). Let $\mathcal{U}$ be an open cover of $f^{-1}y$. There exists a neighborhood $O(y)$ of y in Y and an open cover $\mathcal{V}$ of $f^{-1}y$ satisfying (4) of the theorem. Then $\mathcal{U}$ covers $f^{-1}O(y)$ and $\mathcal{V} < \mathcal{U} \wedge f^{-1}O(y)$. For every $x \in f^{-1}O(y)$ pick an open set $V(x) \subset f^{-1}O(y)$ (one can assume that $V(x) \subset V$ for some $V \in \mathcal{V}$) containing x, a neighborhood $O_{V(x)} \subset O(y)$ of y and a finite collection $\mathcal{W}(x) \subset \mathcal{U}$ centered at x satisfying $St(V(x), \mathcal{V}) \cap f^{-1}O_{V(x)} \subset \bigcup \mathcal{W}(x)$ and $V(x) \subset f^{-1}O_{V(x)}$. Let $H(x) = V(x) \cap (\bigcap \mathcal{W}(x))$ and let $\mathcal{H} = \{H(x) : x \in f^{-1}O(y)\}$. Then $\mathcal{H}$ is an open cover of $f^{-1}O(y)$, and every $H(x) \in \mathcal{H}$ satisfies

$$St(H(x), \mathcal{H}) \cap f^{-1}O_{V(x)} \subset St(V(x), \mathcal{V}) \cap f^{-1}O_{V(x)} \subset \bigcup \mathcal{W}(x)$$
$$\text{and} \quad H(x) \subset f^{-1}O_{V(x)} \cap (\bigcap \mathcal{W}(x)).$$

Therefore, the cover $\mathcal{H}$ satisfies (3) of the theorem.

Finally we show that (3) $\Rightarrow$ (1). Consider the open in X covers $\mathcal{U}_3$, $\mathcal{U}_2$ and $\mathcal{U}_1$ of $f^{-1}y$ and the neighborhoods $O_1(y)$ and $O_2(y)$ of y in Y satisfying

$$\begin{gathered} \mathcal{U}_2 < \mathcal{U}_1 \wedge f^{-1}O_1(y) = \mathcal{U}_1', \\ \mathcal{U}_3 < \mathcal{U}_2 \wedge f^{-1}O_2(y) = \mathcal{U}_2', \\ O_2(y) \subset O_1(y), \end{gathered}$$

in the manner of (3) of the theorem.

Let $\mathcal{U}_3^{FC}$, $\mathcal{U}_2^{FC}$ and $\mathcal{U}_1^{FC}$ denote the covers of $f^{-1}y$ obtained by taking all unions of finite centered subcollections from $\mathcal{U}_3$, $\mathcal{U}_2$ and $\mathcal{U}_1$ respectively. For every $U \in \mathcal{U}_3^{FC}$ let $\mathcal{G}_U$ denote a finite centered subcollection from $\mathcal{U}_3^{FC}$ such that $U = \bigcup \mathcal{G}_U$. We want to show that $\mathcal{U}_3^{FC}$ is a weak y-star refinement of $\mathcal{U}_1^{FC}$. Let $V \in \mathcal{U}_3^{FC}$, for each $S \in \mathcal{G}_V$ pick a finite $\mathcal{W}(S) \subset \mathcal{U}_2$ and a neighborhood $O_S \subset O_2$ of y satisfying $St(S, \mathcal{U}_3) \cap f^{-1}O_S \subset \bigcup \mathcal{W}(S)$ and $S \subset f^{-1}O_S \cap (\bigcap \mathcal{W}(S))$. Let

$\mathcal{H} = \bigcup\{\mathcal{W}(S) : S \in \mathcal{G}_V\}$. The collection $\mathcal{H}$ is a finite subcollection of $\mathcal{U}_2$ satisfying $St(V,\mathcal{U}_3) \cap f^{-1}\left(\bigcap_{S\in\mathcal{G}_V} O_S\right) \subset \bigcup\mathcal{H}$ and $\bigcap\mathcal{G}_V \subset f^{-1}\left(\bigcap_{S\in\mathcal{G}_V} O_S\right) \cap (\bigcap\mathcal{H})$. Let $O_V = \bigcap_{S\in\mathcal{G}_V} O_S$ and $V' = V \cap f^{-1}O_V$. Then $St(V',\mathcal{U}_3) \cap f^{-1}O_V \subset \bigcup\mathcal{H}$.

For each $H \in \mathcal{H}$ there exists a finite $\mathcal{W}(H) \subset \mathcal{U}_1$ and a neighborhood $O_H \subset O_1(y)$ of y satisfying $St(H,\mathcal{U}_2) \cap f^{-1}O_H \subset \bigcup\mathcal{W}(H)$ and $H \subset f^{-1}O_H \cap (\bigcap\mathcal{W}(H))$. Let $\mathcal{K} = \bigcup\{\mathcal{W}(H) : H \in \mathcal{H}\}$. The collection $\mathcal{K}$ is a finite centered subcollection of $\mathcal{U}_1$ and $St(\bigcup\mathcal{H},\mathcal{U}_2) \cap f^{-1}\left(\bigcap_{H\in\mathcal{H}} O_H\right) \subset \bigcup\mathcal{K}$. Let $O_{\mathcal{H}} = \bigcap_{H\in\mathcal{H}} O_H$ and $O'_V = O_V \cap O_{\mathcal{H}}$. It follows that $St(V'',\mathcal{U}_2) \cap f^{-1}O'_V \subset \bigcup\mathcal{K} \in \mathcal{U}_1^{FC}$, where $V'' = V' \cap f^{-1}O'_V$.

We need to show that the open collection $\{V'' : V \in \mathcal{U}_3^{FC}\}$ is a cover of $f^{-1}O_2(y)$. But this can be readily checked by noting that each element $S \in \mathcal{U}_3$ is also an element of $\mathcal{U}_3^{FC}$. Therefore, the open collection $\mathcal{U}_3^{FC}$ is a weak y-star refinement of $\mathcal{U}_1^{FC}$. Thus, by repeated application and using the proof of Lemma 3.5, the collection $\mathcal{U}_1^{FC}$ has a y-σ-discrete refinement. Consequently, $\mathcal{U}_1$ has a y-σ-locally finite refinement. Also, by Lemma 3.4, the map f is regular and so paracompact by Theorem 3.2. $\square$

4. Tychonoff products

Tychonoff products of maps is taken to be the Tychonoff product of objects in the category $\mathcal{MAP}$ ([2]). We recall the definition.

Definition 4.1. Let $\{f_\alpha : \alpha \in \mathcal{A}\}$ be a collection of continuous maps, where $f_\alpha : X_\alpha \to Y_\alpha$. The Tychonoff product of the maps $\{f_\alpha : \alpha \in \mathcal{A}\}$, which is denoted by $\prod\{f_\alpha : \alpha \in \mathcal{A}\}$, is the continuous map which assigns to the point $x = \{x_\alpha\} \in \prod\{X_\alpha : \alpha \in \mathcal{A}\}$ the point $\{f_\alpha(x_\alpha)\} \in \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$.

If $pr_T^\alpha : \prod\{X_\alpha : \alpha \in \mathcal{A}\} \to X_\alpha$ and $pr_B^\alpha : \prod\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ are the projections, then the diagram

$$\begin{array}{ccc} \prod\{X_\alpha : \alpha \in \mathcal{A}\} & \xrightarrow{pr_T^\alpha} & X_\alpha \\ {\scriptstyle \prod\{f_\alpha:\alpha\in\mathcal{A}\}}\Big\downarrow & & \Big\downarrow{\scriptstyle f_\alpha} \\ \prod\{Y_\alpha : \alpha \in \mathcal{A}\} & \xrightarrow[pr_B^\alpha]{} & Y_\alpha \end{array}$$

is commutative. Therefore, the pair $\{pr_T^\alpha, pr_B^\alpha\}$ is a $\{\text{onto}, \text{onto}\}$-morphism of $\prod\{f_\alpha : \alpha \in \mathcal{A}\}$ into f_α.

The following result was proved in [2].

Proposition 4.1. *The Tychonoff product $\prod\{f_\alpha : \alpha \in \mathcal{A}\}$ of T_i-maps f_α is a T_i-map for $i \leq 3\frac{1}{2}$.*

Proposition 4.2. *Let $f_1 : X_1 \to Y_1$ be a paracompact map and $f_2 : X_2 \to Y_2$ a compact map. If $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ is an open in $X_1 \times X_2$ cover of $(f_1 \times f_2)^{-1}(y_1, y_2)$, where $y_1 \in Y_1$ and $y_2 \in Y_2$, then there exists a (y_1, y_2)-locally finite open in $X_1 \times X_2$ cover $\mathcal{V} = \{V_\beta : \beta \in \mathcal{B}\}$ of $(f_1 \times f_2)^{-1}(y_1, y_2)$ such that for every $\beta \in \mathcal{B}$ there exists an $\alpha \in \mathcal{A}$ satisfying $V_\beta \subset U_\alpha$.*

Proof: Denote by K the fibre $(f_1 \times f_2)^{-1}(y_1, y_2) = f_1^{-1}y_1 \times f_2^{-1}y_2$. For every $(x_1, x_2) \in K$ fix an elementary neighborhood $U_{x_2}(x_1) \times U_{x_1}(x_2) \subset U_\alpha$ for some $\alpha \in \mathcal{A}$. The open in $X_1 \times X_2$ collection $\mathcal{U}' = \{U_{x_2}(x_1) \times U_{x_1}(x_2) : (x_1, x_2) \in K\}$ covers K.

Fix $x_1 \in f_1^{-1}y_1$. There exists a finite number of points $x_2^k(x_1), k = 1, \dots, n(x_1)$, such that the collection $\{U_{x_2^k(x_1)}(x_1) \times U_{x_1}(x_2^k(x_1)) : k = 1, \dots, n(x_1)\}$ covers $x_1 \times f_2^{-1}y_2$. Let $U(x_1) = \bigcap_{k=1}^{n(x_1)} U_{x_2^k(x_1)}(x_1)$. Then it can be easily seen that $\{U(x_1) \times U_{x_1}(x_2^k(x_1)) : k = 1, \dots, n(x_1)\}$ also covers $x_1 \times f_2^{-1}y_2$. Consider the open in X_1 cover $\mathcal{U}'_{y_1} = \{U(x) : x \in f^{-1}y_1\}$ of $f^{-1}y_1$. There exists an open neighborhood O of y_1 in Y_1 and a y_1-locally finite open refinement $\mathcal{W} < f_1^{-1}O \wedge \mathcal{U}'_{y_1}$, say $\mathcal{W} = \{W_\beta : \beta \in \mathcal{B}\}$. Fix some $U(x_\beta) \in \mathcal{U}'_{y_1}$ such that $W_\beta \subset U(x_\beta)$ and let $\mathcal{V}_\beta = \{W_\beta \times U_{x_\beta}(x_2^k(x_\beta)) : k = 1, \dots, n(x_\beta)\}$ and $\mathcal{V} = \bigcup_{\beta \in \mathcal{B}} \mathcal{V}_\beta$. Evidently, the collection $\mathcal{V}$ covers $f_1^{-1}y_1 \times f_2^{-1}y_2$. For any point $(x_1, x_2) \in K$, the point x_1 has a neighborhood $G(x_1) \subset f_1^{-1}O$ which intersects finitely many elements of $\mathcal{W}$. Then it is not difficult to see that the neighborhood $G(x_1) \times X_2$ of the point (x_1, x_2) intersects finitely many elements of $\mathcal{V}$. $\square$

Corollary 4.3. *If the Tychonoff product of a paracompact map and a compact map is closed then it is paracompact.*

The following example shows that the Tychonoff product of a paracompact map and a compact map is not necessarily a closed map.

Example 4.2. Let ω be the ordinal number of the set of positive integers with their natural order. Let $\Omega_{+1} = [0, \omega + 1]$ and $\mathbb{R}$ have the usual order and topology (as LOTS). Consider the set $\alpha X = \mathbb{R} \times \Omega_{+1}$ with lexicographic order and topology (as LOTS), and let $\alpha f = \mathrm{pr}_{\mathbb{R}} : \alpha X \to \mathbb{R}$ be the projection. Denote by X the subspace $\alpha X \setminus \{(y, \omega) : y \in \mathbb{R}\}$ and let $f = \alpha f|_X = \mathrm{pr}_{\mathbb{R}}|_X : X \to \mathbb{R}$. As proved in [4] the space X is not an M-space but is Lindelöf, hereditary paracompact and MT-space. Both f and αf are closed maps, in fact f is a $T_{2\frac{1}{2}}$ MT-map and, hence, paracompact and Tychonoff, while αf is a $T_{2\frac{1}{2}}$ compact map (and so also Tychonoff). It is not difficult to see that αf is a compactification of f.

We next consider the product $f \times \alpha f : X \times \alpha X \to \mathbb{R} \times \mathbb{R}$. We show that this map is not closed. Let y_0 be any point in $\mathbb{R}$. The set $W_n = [(y_0 - \frac{1}{n}, \omega + 1), \to [\ \cap] \leftarrow, (y_0 + \frac{1}{n}, 0)]$ is open in αX and contains the fibre $(\alpha f)^{-1}y_0$ for every $n < \omega$. We next consider the following open in X sets: $U_0 = [(y_0 - \epsilon, \omega + 1), (y_0, 0)]$, $U_n = \{(y_0, n)\}$ and $U_{\omega+1} = [(y_0, \omega + 1), (y_0 + \epsilon, 0)]$, where $\epsilon > 0$. The set

$V = \{U_0 \times W_1\} \cup \{U_n \times W_n : 1 \leq n < \omega\} \cup \{U_{\omega+1} \times W_1\}$ is an open set in $X \times \alpha X$ which covers the fibre $(f \times \alpha f)^{-1}(y_0, y_0)$. It is not difficult to see that there does not exist an open neighborhood O of (y_0, y_0) in $\mathbb{R} \times \mathbb{R}$ such that $(f \times \alpha f)^{-1}O \subset V$ and therefore $f \times \alpha f$ is not closed.

On the other hand, as it is well known, the following result holds ([6]).

Proposition 4.4. *If the Tychonoff product $f = \prod\{f_\alpha : \alpha \in \mathcal{A}\}$, where $f_\alpha : X_\alpha \to Y_\alpha$ and $X_\alpha \neq \emptyset$ for every $\alpha \in \mathcal{A}$, is closed, then all the maps f_α are closed.*

We end this section by a necessary and sufficient condition for the Tychonoff product of a closed map and a compact map to be closed.

Proposition 4.5. *Let $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ be the product of a closed map f_1 and a compact map f_2. The Tychonoff product map $f_1 \times f_2$ is closed if and only if for every point $(y_1, y_2) \in Y_1 \times Y_2$, every open cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ of $f_1^{-1}y_1$ and every collection of open sets $\{W_\alpha : \alpha \in \mathcal{A}\}$ satisfying $f_2^{-1}y_2 \subset W_\alpha$ for every $\alpha \in \mathcal{A}$, there exists a subset $\mathcal{A}' \subset \mathcal{A}$ and a neighborhood $O(y_2)$ of y_2 such that $\{U_\alpha : \alpha \in \mathcal{A}'\}$ covers $f_1^{-1}y_1$ and $f_2^{-1}O(y_2) \subset W_\alpha$ for every $\alpha \in \mathcal{A}'$.*

Proof: Suppose that the map $f_1 \times f_2$ is closed. Take any point $(y_1, y_2) \in Y_1 \times Y_2$ and consider an open cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ of $f_1^{-1}y_1$ and every collection of open sets $\{W_\alpha : \alpha \in \mathcal{A}\}$ satisfying $f_2^{-1}y_2 \subset W_\alpha$ for every $\alpha \in \mathcal{A}$. The open set $G = \bigcup\{U_\alpha \times W_\alpha : \alpha \in \mathcal{A}\}$ contains the fibre $(f_1 \times f_2)^{-1}(y_1, y_2)$ and so there exists neighborhoods $O(y_1), O(y_2)$ of y_1 and y_2 respectively, such that $f_1^{-1}O(y_1) \times f_2^{-1}O(y_2) \subset G$. This shows that for every $x \in f_1^{-1}y_1$ one can choose $\alpha(x)$ such that $x \in U_{\alpha(x)}$ and $f_2^{-1}O(y_2) \subset W_{\alpha(x)}$. Consequently, the subset $\mathcal{A}' = \{\alpha(x) : x \in f_1^{-1}y_1\}$ has the desired properties.

Conversely, suppose that the above property holds and we want to show that $(f_1 \times f_2)$ is closed. Let U be any open set in $(X_1 \times X_2)$ containing the fibre $K = (f_1 \times f_2)^{-1}(y_1, y_2) = f_1^{-1}y_1 \times f_2^{-1}y_2$. For every $(x_1, x_2) \in K$ there exists an elementary neighborhood $V_{x_2}(x_1) \times V_{x_1}(x_2) \subset U$. The open in $X_1 \times X_2$ collection $\mathcal{V} = \{V_{x_2}(x_1) \times V_{x_1}(x_2) : (x_1, x_2) \in K\}$ covers K. Fix $x_1 \in f_1^{-1}y_1$. There exists a finite number of points $x_2^k(x_1), k = 1, \ldots, n(x_1)$, such that the collection $\{V_{x_2^k(x_1)}(x_1) \times V_{x_1}(x_2^k(x_1)) : k = 1, \ldots, n(x_1)\}$ covers $x_1 \times f_2^{-1}y_2$. Let $U(x_1) = \bigcap_{k=1}^{n(x_1)} V_{x_2^k(x_1)}(x_1)$ and $W(x_1) = \bigcup_{k=1}^{n(x_1)} V_{x_1}(x_2^k(x_1))$. Then by the hypothesis, there exists an indexing set $\mathcal{A}$ and a neighborhood $O(y_2)$ of y_2 such that $\{U(x_\alpha) : \alpha \in \mathcal{A}\}$ covers $f_1^{-1}y_1$ and $f_2^{-1}O(y_2) \subset W(x_\alpha)$ for every $\alpha \in \mathcal{A}$. Since f_1 is closed, there exists a neighborhood $O(y_1)$ of y_1 such that $f_1^{-1}O(y_1) \subset \bigcup\{U(x_\alpha) : \alpha \in \mathcal{A}\}$ and so

$$f_1^{-1}O(y_1) \times f_2^{-1}O(y_2) \subset \left(\bigcup\{U(x_\alpha) : \alpha \in \mathcal{A}\}\right) \cup \left(\bigcap\{W(x_\alpha) : \alpha \in \mathcal{A}\}\right) \subset U.$$

$\square$

Corollary 4.6. *The Tychonoff product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a compact map if and only if $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are compact maps.*

Corollary 4.7. *Let $f_1 : X_1 \to Y_1$ be a closed map and $f_2 : X_2 \to Y_2$ a compact map. Also, let $\chi(f_2^{-1}y_2, X_2) = \kappa$, where κ is some infinite cardinal. If the Tychonoff product map $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is closed then there are no minimal open covers of $f_1^{-1}y_1$ of cardinality $\geq \kappa$.*

[Note: Here by $\chi(f_2^{-1}y_2, X_2)$ we understand the *character of the set $f_2^{-1}y_2$ in X_2* and by a minimal cover of $f_1^{-1}y_1$ we understand a cover $\mathcal{U}$ no subcover of which covers $f_1^{-1}y_1$.]

Proof: Let $\{U_\alpha : \alpha < \kappa\}$ be a base for $f_2^{-1}y_2$ in X_2 of cardinality κ and suppose that there exists a minimal open cover $\mathcal{V}$ of $f_1^{-1}y_1$ of cardinality $\geq \kappa$. For every $V \in \mathcal{V}$ let V^* be an open set in X_1 such that $V = V^* \cap f^{-1}y_1$. Let $\{V_\alpha : \alpha < \tau\}$ be a well ordering of $\mathcal{V}$, where $\tau \geq \kappa$. Since $\{V_\alpha^* : \alpha < \tau\}$ is minimal on the fibre $f_1^{-1}y_1$, by Proposition 4.5, there exists a neighborhood $O(y_2)$ of y_2 such that $f_2^{-1}O(y_2) \subset U_\alpha$ for every $\alpha < \kappa$. This implies that there must exist some $\alpha_0 < \kappa$ satisfying $f_2^{-1}O(y_2) = U_{\alpha_0}$ from which follows that $\chi(f_2^{-1}y_2, X_2) = 1$ which contradicts the assumption on κ. $\square$

5. The Tamano Theorem

External characterizations of paracompact spaces in the category $\mathcal{TOP}$ provide contrast to the open cover characterizations of paracompactness. An interesting external characterization of paracompactness is the Tamano Theorem.

Theorem 5.1 (The Tamano Theorem)**.** *For a Tychonoff space X the following conditions are equivalent:*

1. *the space X is paracompact;*
2. *for every compactification bX of the space X the Tychonoff product $X \times bX$ is normal;*
3. *the Tychonoff product $X \times \beta X$ is normal;*
4. *there exists a compactification bX of the space X such that the Tychonoff product $X \times bX$ is normal.*

The equivalence of conditions (1) and (3) was established by H. Tamano [18] while the equivalence of conditions (1) and (4) was obtained independently by K. Morita [14] and by H. Tamano [19]. We now prove an analogue to this external characterization for paracompact maps.

Theorem 5.2. *For a Tychonoff map $f : X \to Y$, where Y is a T_2-space, and its compactification $bf : b_fX \to Y$ such that the product $f \times bf : X \times b_fX \to Y \times Y$ is closed, the following are equivalent:*

1. *f is paracompact;*
2. *$f \times bf$ is functionally normal;*
3. *$f \times bf$ is normal.*

Proof: (1) $\Rightarrow$ (2). Since the map f is paracompact, bf is compact and the product $f \times bf$ is closed, the product map $f \times bf$ is even paracompact by Lemma 4.3. Therefore, by Proposition 4.1 and Proposition 3.1 it follows that $f \times bf$ is functionally normal.

(2) $\Rightarrow$ (3) is evident.

(3) $\Rightarrow$ (1). Suppose that the product $f \times bf$ is closed and normal, and that Y is a T_2-space. In particular we have that $f : X \to Y$ is a closed map and X and $b_f X$ are T_2-spaces. We show that f satisfies condition (4) of Theorem 3.3. Let y be an arbitrary point in Y and let $\mathcal{U}_0$ be an open in X collection covering the fibre $f^{-1}y$. By the closedness of f, there exists an open neighborhood O of y in Y such that $\mathcal{U}_0$ covers $f^{-1}O$. Let $\mathcal{U} = \mathcal{U}_0 \wedge f^{-1}O$. For every open set U in X we assign an open set U^* in $b_f X$ satisfying $U^* \cap X = U$. Let $W = \bigcup\{U \times U^* : U \in \mathcal{U}\}$ which is open in $X \times b_f X$. Since both X and $b_f X$ are Hausdorff, the set $\Delta_O = \{(x,x) : x \in f^{-1}O\} \subset W$ is closed in $(f \times bf)^{-1}(O \times O)$ and therefore, by the normality of $f \times bf$ there exists a neighborhood $O' \subset O$ of y and an open set $V \subset (f \times bf)^{-1}(O' \times O')$ such that

$$\Delta_O \cap (f \times bf)^{-1}(O' \times O') \subset V \subset [V]_{(f\times bf)^{-1}(O'\times O')} \subset W \cap (f \times bf)^{-1}(O' \times O').$$

One may assume that $V = \bigcup\{V_\alpha \times V_\alpha^* : \alpha \in \mathcal{A}\}$, where V_α is open in X for every $\alpha \in \mathcal{A}$. Let $\mathcal{U}' = \mathcal{U} \wedge f^{-1}O'$ and $\mathcal{U}'^* = \{U^* : U \in \mathcal{U}'\}$. Consider an arbitrary $x \in f^{-1}O'$ and let $E = (bf)^{-1}O' \setminus St(x,\mathcal{U}'^*)$. It is not difficult to see that $(\{x\} \times E) \cap W = \emptyset$ and so $(\{x\} \times E) \cap [V]_{(f\times bf)^{-1}(O'\times O')} = \emptyset$. Since the fibre $(bf)^{-1}y$ is compact, by the normality of $f \times bf$ there exist a neighborhood $O'' \subset O'$ of y and sets $W_1 \subset f^{-1}O''$, $W_2 \subset (bf)^{-1}O''$ open in X and $b_f X$ respectively such that $(\{x\} \times E) \cap (f \times bf)^{-1}(O'' \times O'') \subset W_1 \times W_2$ and $W_1 \times W_2 \cap [V]_{(f\times bf)^{-1}(O''\times O'')} = \emptyset$. We thus have $W_1 \times W_2 \cap (V_\alpha \times V_\alpha^*) = \emptyset$ for every $\alpha \in \mathcal{A}$ and so $W_2 \cap V_\alpha^* = \emptyset$ whenever $W_1 \cap V_\alpha \neq \emptyset$. This implies that $W_2 \cap V_\alpha^* = \emptyset$ whenever $W_1^* \cap V_\alpha^* \neq \emptyset$ or

$$W_2 \cap St(W_1^*, \mathcal{V}^*) = \emptyset,$$

where $\mathcal{V}^* = \{V_\alpha^* : \alpha \in \mathcal{A}\}$.

Hence $\big(E \cap (bf)^{-1}O''\big) \bigcap \big[St(W_1^*, \mathcal{V}^*) \cap (bf)^{-1}O''\big]_{(bf)^{-1}O''} = \emptyset$, and so

$$\Big[St(W_1^*, \mathcal{V}^*) \cap (bf)^{-1}O''\Big]_{(bf)^{-1}O''} \subset St(x,\mathcal{U}^*).$$

In fact, one can find a finite subcollection $\mathcal{U}_F^* \subset \mathcal{U}^*$, centered at x, satisfying

$$St(W_1^*, \mathcal{V}^*) \cap (bf)^{-1}O'' \subset \bigcup \mathcal{U}_F^*.$$

If $\mathcal{W} = \{U \cap X : U \in \mathcal{U}_F^*\}$, then $St(W_1, \mathcal{V}) \cap f^{-1}O'' \subset \bigcup \mathcal{W}$, where $\mathcal{V} = \{V_\alpha : \alpha \in \mathcal{A}\}$, and the conditions of Theorem 3.3 (4) are satisfied. ☐

Theorem 5.3. *For a $T_{3\frac{1}{2}}$-map $f : X \to Y$ the following are equivalent:*

1. *f is paracompact;*
2. *f is closed and for every compactification $bf : b_f X \to Y$, $y \in Y$ and every closed (in $(bf)^{-1}y$) set $F \subset (bf)^{-1}y \setminus f^{-1}y$, there exists a neighborhood O_y of y and a locally finite in $f^{-1}O_y$ cover $\mathcal{V}$ of $f^{-1}O_y$ such that $[V]_{(bf)^{-1}y} \cap F = \emptyset$ for every $V \in \mathcal{V}$.*

Proof: Let $f : X \to Y$ be a paracompact Tychonoff map and y an arbitrary point of Y. Let $F \subset (bf)^{-1}y \setminus f^{-1}y$ be some closed set in $(bf)^{-1}y$, where $bf : b_f X \to Y$ is some compactification of the map f. For every $x \in f^{-1}y$ there exists a neighborhood $O_y(x)$ of y such that x and F are neighborhood separated in $(bf)^{-1}O_y(x)$ by U_x^* and V_x^* respectively. Consider the collection $\mathcal{U} = \bigcup_{x\in X} U_x$, where $U_x = U_x^* \cap X$ for every $x \in X$. This collection covers $f^{-1}y$ and so, since f is paracompact, there exists a neighborhood $\tilde{O}_y$ of y and a locally finite in $f^{-1}\tilde{O}_y$ cover $\mathcal{V}$ of $f^{-1}\tilde{O}_y$ which is a refinement of $f^{-1}\tilde{O}_y \wedge \mathcal{U}$. It is not difficult to see that $[V]_{(bf)^{-1}y} \cap F = \emptyset$ for every $V \in \mathcal{V}$.

Conversely, let $f : X \to Y$ be a Tychonoff map satisfying property (2) above. Let $y \in Y$, O_y a neighborhood of y and $\mathcal{U}$ an open in X cover of $f^{-1}O_y$. Consider an arbitrary compactification $bf : b_f X \to Y$ and a collection $\mathcal{U}^*$ with $\mathcal{U}^* \wedge X = \mathcal{U}$. The set $F = (bf)^{-1}y \setminus \bigcup \mathcal{U}^*$ is closed in $(bf)^{-1}y$ and so there exists a neighborhood $O'_y \subset O_y$ and a locally finite in $f^{-1}O'_y$ cover $\mathcal{V}$ of $f^{-1}O'_y$ such that $[V]_{(bf)^{-1}y} \cap F = \emptyset$ for every $V \in \mathcal{V}$. For every $V \in \mathcal{V}$, the set $[V]_{(bf)^{-1}y}$ is compact and is a subset of $\bigcup \mathcal{U}^*$. Thus, there exists a finite subcollection $U_1^*(V), \dots, U_k^*(V) \in \mathcal{U}^*$ such that $[V]_{(bf)^{-1}y} \subset \bigcup_{i=1}^k U_i^*(V)$. Consider the collection $\mathcal{W}(V) = \{V \cap U_i^*(V) : i = 1, \dots, k\}$ and let $\mathcal{W} = \bigcup \{\mathcal{W}(V) : V \in \mathcal{V}, V \cap f^{-1}y \neq \emptyset\}$. Using the closedness of f we get that f is paracompact. □

Note that if the space Y is a T_1-space, then property (2) is true for any closed in X set $F \subset (bf)^{-1}y \setminus f^{-1}y$. If Y is a T_3-space, then so is X and we get the following.

Theorem 5.4. *For a $T_{3\frac{1}{2}}$-map f from a space X into a regular space Y the following are equivalent:*

1. *f is paracompact;*
2. *f is closed and for every compactification $bf : b_f X \to Y$, $O \subset Y$, $y \in O$ and every closed (in $(bf)^{-1}O$) set $F \subset (bf)^{-1}O \setminus f^{-1}O$, there exists a neighborhood $O_y \subset O$ of y and an open locally finite in $f^{-1}O_y$ cover $\mathcal{V}$ of $f^{-1}O_y$ such that $[V]_{(bf)^{-1}O_y} \cap F = \emptyset$ for every $V \in \mathcal{V}$.*

Proof: The proof is analogous to the proof of Theorem 5.3. □

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