Infinitesimal characterization of almost Hermitian homogeneous spaces

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Abstract. In this note it is shown that almost Hermitian locally homogeneous manifolds are determined, up to local isometries, by an integer k_H, the covariant derivatives of the curvature tensor up to order $k_H + 2$ and the covariant derivatives of the complex structure up to the second order calculated at some point. An example of a Hermitian locally homogeneous manifold which is not locally isometric to any Hermitian globally homogeneous manifold is given.

Keywords: almost Hermitian homogeneous spaces, Singer invariant

Classification: 53C30, 53C55

Introduction

The curvature tensor and its covariant derivatives are a complete set of local invariants for analytic Riemannian metrics. Indeed, if (M, g) is an analytic manifold and $(U, x_1, \dots, x_n)$ a normal coordinate system centered at $p \in M$, the coefficients of the Taylor series expansion around p of $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are polynomials in the components of the curvature tensor R_p and its covariant derivatives $D^s R_p$, $s = 0, 1, \dots$, with respect to the Levi Civita connection. As a consequence, if (M, g) and (M', g') are (analytic) Riemannian manifolds and if for $p \in M$ and $q \in M'$ there is a linear isometry $F : T_pM \to T_qM'$ so that

$$F^*{}_p D'^s R'_q = D^s R_p \tag{0.1}$$

for any $s = 0, 1 \dots$, then $f := \exp_q \circ F \circ \exp_p^{-1}$ is a local isometry such that $f(p) = q$ and $f_{*|p} = F$.

For instance, if (M, g) is locally homogeneous, (0.1) holds for any $p, q \in M$ $(= M')$. Hence one can read local homogeneity by means of the curvature tensor together with all its covariant derivatives.

A stronger result holds. In fact, I.M. Singer [Si] (see also [NT]) proved that there is an integer k_M (the *Singer invariant*), with $0 \le k_M \le \frac{n(n-1)}{2} - 1$ ($n = \dim M$) such that, if for any $p, q \in M$ there is a linear isometry $F : T_pM \to T_qM$ for which (0.1) holds for $0 \le s \le k_M + 1$, then M is locally homogeneous. From

Work partially supported by the GNSAGA of CNR, Italy.

this result one can prove that a locally homogeneous space is completely determined by the curvature and its covariant derivatives up to some order at a fixed point, and that one can algebraically recapture the Lie algebra of infinitesimal isometries from the infinitesimal data $\{D^s R_p\}_{s \leq k_M+2}$ (see [NT] for a proof).

This note deals with the almost Hermitian case. In Section 3 we prove the following.

Main Theorem. *Let $(V, \langle\ ,\ \rangle, J)$ be a Hermitian vector space and k_H a positive integer. Let $J^0 = J, J^1, J^2, R^0, R^1, \ldots, R^{k_H+2}$ be tensors of types* (1,1), (1,2), (1,3), (0,4), ... ,(0, $k_H + 6$) *for which the identities* (3.1), ... (3.7), (3.9), (3.10) (*see Section* 3 *below*) *hold.*
Then there exists an almost Hermitian locally homogeneous space (M, g, J) with (*Hermitian*) *Singer invariant k_H such that $\langle\ ,\ \rangle$, $J^0, J^1, J^2, R^0, R^1, \ldots, R^{k_H+2}$ coincide, respectively with g_p, J_p, DJ_p, $D^2 J_p$, R_p, ... , $D^{k_H+2} R_p$ at a point p, where V is identified with T_pM.*

We now briefly describe the methods we use. In analogy with the Riemannian case (cf. [Tr] for a review on the subject), one can associate an algebraic object to any almost Hermitian locally homogeneous manifold, the *almost Hermitian infinitesimal model* (Definition 1.1). This object brings together the algebraic identities (see (1.1) ... (1.8)) satisfied by a curvature tensor and an almost complex structure at one point. Conversely, to any almost Hermitian infinitesimal model corresponds a uniquely defined (up to local isometries) almost Hermitian locally homogeneous manifold. The proof of the main Theorem is obtained by constructing an almost Hermitian infinitesimal model from the data $(\langle\ ,\ \rangle, J^0, J^1, J^2, R^0, R^1, \ldots, R^{k_H+2})$.

Almost Hermitian globally homogeneous manifolds are obtained from the *regular* almost Hermitian infinitesimal models (see Definition 1.2). In Section 4, following a similar construction as in [Tr] and [K2] we give an example of a Hermitian manifold of (complex) dimension 7 which is not locally isometric to any Hermitian *globally* homogeneous manifold.

We thank O. Kowalski and L. Vanhecke for useful discussions.

1. Preliminaries

A. Almost Hermitian homogeneous manifolds. An almost Hermitian manifold (M, g, J) is *Hermitian globally homogeneous* if for any $p, q \in M$ there is a Hermitian isometry f (i.e. a diffeomorphism which is compatible both with the almost complex structure and the Hermitian metric) such that $f(p) = q$. M is almost Hermitian *locally* homogeneous if f is a local isometry. In the global case, there is a Lie group G which acts transitively and effectively on M. Let K be the isotropy subgroup at $p \in M$. Then $M = G/K$. Moreover (if $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{k}$ the Lie algebra of K) there is a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$

with

$$\begin{aligned}
&[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}, \\
&\mathfrak{m} \simeq T_pM \qquad (J\mathfrak{m} = \mathfrak{m}), \\
&ad_{\mathfrak{k}}J = Jad_{\mathfrak{k}}.
\end{aligned}$$

Theorem 1 [Se]**.** *(M, g, J) is almost Hermitian locally homogeneous if and only if there is a connection ∇ on M such that*

(1) $\nabla g = 0$ *(i.e., ∇ is metric),*
(2) $\nabla T = 0$ *(where T is the torsion of ∇),*
(3) $\nabla K = 0$ *(where K is the curvature of ∇),*
(4) $\nabla J = 0$.

∇ is called a canonical connection. *The difference tensor $S := \nabla - D$ is called an* almost Hermitian homogeneous structure.

Moreover, if M is complete, simply connected and admits an almost Hermitian homogeneous structure, it is almost Hermitian globally homogeneous.

Note that a homogeneous structure depends on the reductive decomposition of the homogeneous manifold (M, g). Theorem 1 was proved by Sekigawa [Se] in the simply connected case, while the local version follows by a more general result of Kiričenko [Ki]. We will call (1), ... , (4) the Hermitian Ambrose-Singer equations. (1), (2) and (3) are the Ambrose-Singer equations, which hold for real locally homogeneous spaces. For the real Riemannian version of the theorem we refer to [TV].

Remark. The torsion T of ∇ and the homogeneous structure S are related by

$$T_XY = S_YX - S_XY.$$

The curvature tensor K of ∇ and the Riemannian curvature tensor R are related by

$$K_{XY} = R_{XY} + [S_X, S_Y] + S_{T_XY}.$$

For $(T, K, J) := (T_p, K_p, J_p)$, $p \in M$ the following identities hold:

(1.1) $T_XY = -T_YX$,

(1.2) $K_{XY}Z = -K_{YX}Z$,

(1.3) $g(K_{XY}Z, W) + g(Z, K_{XY}W) = 0$,

(1.4) $K_{XY} \cdot T = 0$,

(1.5) $K_{XY} \cdot K = 0$,

(1.6) $K_{XY} \cdot J = 0$,

(1.7) $\mathfrak{S}_{XYZ}(K_{XY}Z + T_{T_XY}Z) = 0$ (first Bianchi identity),

(1.8) $\mathfrak{S}_{X,Y,Z}K_{T_XYZ} = 0$ (second Bianchi identity).

(1.1), (1.2), (1.3) are the usual algebraic identities satisfied by the curvature and torsion tensors of a metric connection. (1.7) and (1.8) follow from the Bianchi identities taking into account the Sekigawa-Ambrose-Singer equations. Finally, as already indicated above, (1.4), (1.5), (1.6) are integrability conditions of the Sekigawa-Ambrose-Singer equations and follow from (2), (3), (4) in Theorem 1.

B. Almost Hermitian infinitesimal models. Let $(V, \langle\ ,\ \rangle, J)$ be a Hermitian vector space and consider tensors

$$\begin{aligned} T &: V \to \mathrm{End}V, \\ K &: V \times V \to \mathrm{End}V. \end{aligned}$$

Definition 1.1. *A triple (T, K, J) is an almost Hermitian infinitesimal model on V if* (1.1), ... , (1.8) *hold.*

If (T, K, J) and (T', K', J') are almost Hermitian infinitesimal models on vector spaces V and V', an isomorphism between them is a linear Hermitian isometry preserving the tensors $T, .., J'$.

It follows from the above discussion that one can associate with any almost Hermitian locally homogeneous space an almost Hermitian infinitesimal model. Conversely, Y. Watanabe and F. Tricerri [TW] proved the following

Theorem 2. *Let (T, K, J) be an almost Hermitian infinitesimal model on V. Then there exists an almost Hermitian locally homogeneous manifold (M, g, J) and a canonical connection ∇ on M, so that $(T_p, K_p, J_p) \cong (T, , K, J)$, for any $p \in M$. In particular, (M, g, J) is determined uniquely up to local Hermitian isometries.*

So far our considerations have been local. On the other hand, it is natural to look for conditions on an almost Hermitian infinitesimal model (T, K, J) in order that the associated almost Hermitian locally homogeneous manifold (M, g, J) is locally isometric to an almost Hermitian globally homogeneous manifold. To this aim we adapt a construction due to Nomizu [No].

C. The Nomizu construction. Let (T, K, J) be an almost Hermitian infinitesimal model on V. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{so}(V)$ given by

$$\mathfrak{h} := \{A \in \mathfrak{so}(V) \mid A \cdot T = A \cdot K = A \cdot J = 0\},$$

(note that, since $A \cdot J = 0$, $\mathfrak{h}$ is a subalgebra of the Lie algebra $\mathfrak{u}(V)$ of the unitary group). Let $\mathfrak{g} := \mathfrak{h} \oplus V$ with brackets defined by

$$[X, Y] = -T_X Y + K_{XY}, \tag{1.9}$$

$$[A, X] = A(X), \tag{1.10}$$

$$[A, B] = AB - BA, \tag{1.11}$$

where $X, Y \in V$ and $A, B \in \mathfrak{h}$. Remark that $ad_{\mathfrak{h}} J = J ad_{\mathfrak{h}}$ as a consequence of the definition of $\mathfrak{h}$.

Observe that, by (1.4), (1.5) and (1.6), $K_{XY} \in \mathfrak{h}$ for any $X, Y \in V$. In particular, (1.5) implies that the operators K_{XY} span a subalgebra $\mathfrak{h}'$ of $\mathfrak{h}$. One can repeat the construction above with $\mathfrak{h}'$ in the place of $\mathfrak{h}$. The Lie algebra $\mathfrak{g}' = \mathfrak{h}' \oplus V$ so determined is called the *transvection algebra* ([K1]).

Let G (G') be the (connected) simply connected Lie group whose Lie algebra is $\mathfrak{g}$ ($\mathfrak{g}'$) and H (H') its connected Lie subgroup with Lie algebra $\mathfrak{h}$ ($\mathfrak{h}'$).

Definition 1.2. *An almost Hermitian infinitesimal model is regular if H is closed in G.*

If (T, K, J) is regular, then $\widetilde{M} := G/H$ is an almost Hermitian (globally) homogeneous manifold. If M is the almost Hermitian locally homogeneous manifold constructed starting from (T, K, J) according to Theorem 2, one can show that M and $\widetilde{M}$ are locally isometric (the argument is similar to the real case, cf. [Tr]). Note that, if we replace the Lie algebra $\mathfrak{g}$ with the transvection algebra $\mathfrak{g}'$, the construction above leads to the same almost Hermitian homogeneous manifold $\widetilde{M}$ represented as coset space by G'/H' (cf. [K1]).

Conversely, if (N, g, J) is locally isometric to an almost Hermitian globally homogeneous manifold, then, with the same arguments as in [Tr], one can show that *all* infinitesimal models associated with it are regular. Moreover, one can show that the transvection algebra of each infinitesimal model is regular, i.e. H' is closed in G'.

Thus, in order to produce an almost Hermitian locally homogeneous manifold which cannot be locally isometric to any almost Hermitian globally homogeneous manifold it is sufficient to construct an infinitesimal model whose transvection algebra is not regular. This will be done in Section 4.

2. The Hermitian Singer invariant

Let (M, g, J) be an almost Hermitian manifold, and $\dim_{\mathbb{R}} M = 2n$. For any $p \in M$, $s \geq 0$, let

$$\mathfrak{g}_H(p, s) := \{A \in \mathfrak{u}(T_pM) \mid A \cdot R_p = A \cdot DR_p = \cdots = A \cdot D^s R_p = 0\}, \tag{2.1}$$

where R is the Riemannian curvature tensor and D the Levi Civita connection. $\mathfrak{g}_H(p, s)$ is a subalgebra of $\mathfrak{u}(T_pM)$. Note that $\mathfrak{g}_H(p, s) \supseteq \mathfrak{g}_H(p, s+1)$. Clearly, there exists a smallest integer s, denoted by $k_H(p)$, such that

$$\mathfrak{g}_H(p, k_H(p)) = \mathfrak{g}_H(p, k_H(p) + 1). \tag{2.2}$$

Definition 2.1. *(M, g, J) is said to be* almost Hermitian infinitesimally homogeneous *if for any $p, q \in M$ there is a linear Hermitian isometry $F : T_pM \to T_qM$ so that $F_p^* D^i R_q = D^i R_p$ for $i = 0, \ldots, k_H(p) + 1$.*

If M is almost Hermitian infinitesimally homogeneous, then the $\mathfrak{g}_H(p, i)$ and $\mathfrak{g}_H(q, i)$ are conjugate for $i = 0, \ldots, k_H(p) + 1$. This implies that $k_H(p)$ does not depend on p. Hence, we set $k_H := k_H(p)$ and, in analogy with [NT], we call k_H the *Hermitian Singer invariant.*

If (M, g, J) is almost Hermitian infinitesimally homogeneous one can show that M is almost Hermitian locally homogeneous. The idea is to construct a canonical connection ∇ (or equivalently a homogeneous structure S, cf. [NT] and [KT]) and then use Theorem 1. As we will see in the next section, this construction is purely algebraic. As a matter of fact, S_p ($p \in M$) can be obtained from R_p, a finite number of its covariant derivatives and finitely many covariant derivatives of J_p.

3. Infinitesimal data

Let (M, g, J) be an almost Hermitian locally homogeneous space and $p \in M$. In this section we use the Riemannian curvature tensor $R_{XYZW} = g(R_{XY}Z, W)$. Set

$$V := T_pM, \quad J^0 := J_p, J^1 := DJ_p, \dots \quad R^0 := R_p, R^1 := DR_p, \dots, R^s := D^sR_p.$$

The following identities hold for any almost Hermitian manifold

$$R^0_{XYZW} = -R^0_{YXZW} = R^0_{ZWXY}, \tag{3.1}$$

$$\mathfrak{S}_{XYZ} R^0_{XYZW} = 0 \quad \text{(first Bianchi identity)}, \tag{3.2}$$

$$R^1_{XYZVW} = -R^1_{XZYVW} = R^1_{XVWYZ}, \tag{3.3}$$

$$\mathfrak{S}_{YZV} R^1_{XYZVW} = 0, \tag{3.4}$$

$$\mathfrak{S}_{XYZ} R^1_{XYZVW} = 0 \quad \text{(second Bianchi identity)}, \tag{3.5}$$

$$R^0_{XY} \cdot R^s = R^{s+2}_{YX..} - R^{s+2}_{XY..} \quad \text{(Ricci identities for } R), \tag{3.6}$$

$$R^0_{XY} \cdot J^0 = J^2(Y, X, \dots) - J^2(X, Y, \dots) \quad \text{(Ricci identities for } J). \tag{3.7}$$

If, in addition, M is almost Hermitian infinitesimally homogeneous and ∇ is a canonical connection, then

$$\begin{aligned}
&\nabla_X D^s R = 0, \quad 0 \le s \le k_H + 1 && \text{or equivalently } i_X R^{s+1} = S_X \cdot R^s, \\
&\nabla_X J = 0, && \text{or equivalently } i_X J^1 = S_X \cdot J^0, \\
&\nabla_X DJ = 0, && \text{or equivalently } i_X J^2 = S_X \cdot J^1,
\end{aligned} \tag{3.8}$$

where i_X denotes the contraction with X. Note that the formulas on the right hand side of (3.8) have the advantage of being tensorial. Hence they depend only on the values of D^tR and D^iJ at p. If we define the following maps

$$\mu_s : \mathfrak{so}(V) \to (V^* \otimes V) \oplus (\overset{2}{\otimes} V^* \otimes V) \oplus \left(\sum_{\alpha=0}^{s} \overset{s+4}{\otimes} V^* \right)$$
$$A \mapsto (A \cdot J^0, A \cdot J^1, A \cdot R^0, \dots, A \cdot R^s),$$

$$\begin{aligned}\nu_t : V &\to (\overset{2}{\otimes} V^* \otimes V) \oplus (\overset{3}{\otimes} V^* \otimes V) \oplus \left(\sum_{\alpha=0}^{t} \overset{t+4}{\otimes} V^* \right)\\ A &\mapsto (i_X J^1, i_X J^2, i_X R^1, \ldots, i_X R^{t+1}),\end{aligned}$$

then (3.8) is equivalent to

$$\nu_{k_H+1}(V) \subseteq \mu_{k_H+1}(\mathfrak{so}(V)). \tag{3.9}$$

Note that $\ker \mu_s = \mathfrak{g}_H(p,s) \cap \ker\{A \mapsto A \cdot J^1\}$. Then $\mathfrak{g}_H(p,k_H) = \mathfrak{g}_H(p,k_H+1)$ implies

$$\ker \mu_{k_H+1} = \ker \mu_{k_H}. \tag{3.10}$$

Proof of the main Theorem: By (3.9), for any X, there exists an $A(X) \in \mathfrak{so}(V)$ so that $i_X J^{h+1} = A(X) \cdot J^h$ $(h = 1,2)$, $i_X R^{s+1} = A(X) \cdot R^s$ $(s = 0, \ldots, k_H + 2)$. Set $\mathfrak{h} := \ker \mu_{k_H+1}$. Then one can split $\mathfrak{so}(V)$ as $\mathfrak{h} \oplus \mathfrak{h}^\perp$ (with respect to the inner product $\langle\ ,\ \rangle$). Thus

$$A(X) = A_1(X) + A_2(X), \qquad A_1(X) \in \mathfrak{h}, A_2(X) \in \mathfrak{h}^\perp.$$

We can now set $S_X := A_2(X)$ and define

$$\begin{aligned}T_X Y &:= S_Y X - S_X Y,\\ K_{XY} &:= R^0_{XY} + [S_X, S_Y] + S_{T_X Y}.\end{aligned}$$

Then one can show that (T, K, J) is an almost Hermitian infinitesimal model. This can be done exactly as in [NT], except for the proof of (1.6). For the latter, using (3.7) and (3.8), we have

$$\begin{aligned}R^0_{XY} J^0(Z) &= J^2(Y,X,Z) - J^2(X,Y,Z)\\ &= i_Y J^2(X,Z) - i_X J^2(Y,Z) = S_Y J^1(X,Z) - S_X J^1(Y,Z)\\ &= S_Y i_X J^1(Z) - J^1(S_Y X, Z) - S_X i_Y J^1(Z) + J^1(S_X Y, Z)\\ &= S_Y S_X J^0(Z) - S_{S_Y X} J^0(Z) - S_X S_Y J^0(Z) + S_{S_X Y} J^0(Z)\\ &= -([S_X, S_Y] + S_{T_X Y}) J^0(Z).\end{aligned}$$

It suffices to use Theorem 2 to finish the proof. □

4. An example of non-regular Hermitian infinitesimal model

A. General construction ([Tr]). Let G be a simply connected Lie group and H a connected Lie subgroup of G for which one has a reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus V, \qquad [\mathfrak{h}, V] \subseteq V, \tag{4.1}$$

where $\mathfrak{g}$, $\mathfrak{h}$ is the Lie algebra of G, H, respectively. Suppose V is endowed with a complex structure J and a $Ad(H)$-invariant Hermitian inner product $< \quad , \quad >$. Let $ad : \mathfrak{h} \to ad_{\mathfrak{h}}, X \mapsto ad_X$ be an isomorphism and

$$ad_{\mathfrak{h}} J = J ad_{\mathfrak{h}}. \tag{4.2}$$

Define

$$T_X Y = [X, Y]_V, \tag{4.3}$$

$$K_{XY} = ad_{[X,Y]_{\mathfrak{h}}}. \tag{4.4}$$

By (4.2), (T, K, J) is an almost Hermitian infinitesimal model. Moreover its transvection algebra is isomorphic to $\mathfrak{g}$ and therefore *(T, K, J) is non-regular if H is not closed in G.*

B. An example. Let G be the simply connected Lie group $SU(4)$ and H its one parameter subgroup generated by

$$\begin{pmatrix} e^{it} & & & \\ & e^{-it} & & \\ & & e^{iat} & \\ & & & e^{-iat} \end{pmatrix},$$

with $a \in \mathbb{R} - \mathbb{Q}$. The Lie algebra $\mathfrak{h}$ of H is generated by

$$v = \begin{pmatrix} i & & & \\ & -i & & \\ & & ai & \\ & & & -ai \end{pmatrix}.$$

Let $\mathfrak{t}$ be the Lie subalgebra of the Lie algebra of G, $\mathfrak{g}$, given by the diagonal matrices with zero trace. One can identify $\mathfrak{t}$ with a hyperplane of $\mathbb{R}^4$ in a natural way. Let $v^\perp$ be the orthogonal complement of v in $\mathfrak{t}$ with respect to the euclidean metric. Since $\dim_{\mathbb{R}} v^\perp = 2$, given an orthonormal basis (e_1, e_2) of $v^\perp$, $v^\perp$ is endowed with the complex structure $J_1 : \begin{cases} e_1 \mapsto \ \ e_2 \\ e_2 \mapsto -e_1 \end{cases}$. Note that the euclidean inner product on $v^\perp$ (inherited by $\mathbb{R}^4$) is Hermitian and $Ad(H)$-invariant (the latter is trivial, since $Ad(H)$ is the identity on $v^\perp$). Let W be the subspace of $\mathfrak{g}$ of the matrices of $\mathfrak{g}$ having all entries on the diagonal equal to zero. Note that W is the tangent space (at I) of the flag manifold $SU(4)/T$, where T is the maximal torus. Since $SU(4)/T$ is a Hermitian homogeneous manifold, there is a natural complex structure J_2 on W given by multiplication by i for the elements above the diagonal and by $-i$ for the ones below the diagonal. Moreover any negative multiple of the Killing form of $\mathfrak{su}(4)$ is a Hermitian (with respect to J_2) $Ad(H)$-invariant inner product on W.

Let $V := v^{\perp} \oplus W$ be the Hermitian space, endowed with the complex structure $J := J_1 + J_2$. Then

$$\mathfrak{g} = \mathfrak{h} \oplus V$$

is a reductive decomposition, like in the general construction above, since $ad_{\mathfrak{h}} J = J ad_{\mathfrak{h}}$ (on the $v^{\perp}$-part this is trivial, while on W it follows from the fact that $SU(4)/T$ is a Hermitian homogeneous manifold).

If one defines T and K by means of (4.3) and (4.4) one gets an almost Hermitian infinitesimal model (T, K, J). Since H is not closed in G, such a model is not regular. Moreover, one can verify by direct computation that the Nijenhuis tensor N vanishes, hence the locally homogeneous almost Hermitian manifold with infinitesimal model (T, K, J) has integrable complex structure (i.e. it is Hermitian).

References

[Ki] Kiričenko V., *On homogeneous Riemannian spaces with invariant tensor structure*, Soviet Math. Dokl. **21** (2) (1980), 734–737.

[K1] Kowalski O., *Generalized Symmetric Spaces*, Springer, Berlin, 1980.

[K2] Kowalski O., *Counter-example to the "Second Singer's Theorem"*, Ann. Global Anal. Geom. **8** (1990), 211–214.

[KT] Kowalski O., Tricerri F., *A canonical connection for locally homogeneous Riemannian manifolds*, Proc. Conf. Global Diff. Geom. and Global Analysis, Berlin, 1990, Lecture Notes in Math. **1481** (D. Ferus et al., eds.), Springer, Berlin, 1990, pp. 97–103.

[LT] Lastaria F., Tricerri F., *Curvature orbits and locally homogeneous Riemannian manifolds*, Ann. Mat. Pura Appl. **165** (1993), 121–131.

[No] Nomizu K., *Invariant affine connections on homogeneous spaces*, Amer. J. Math. **76** (1954), 33–65.

[NT] Nicolodi L., Tricerri F., *On two theorems of I.M. Singer about homogeneous spaces*, Ann. Global Anal. Geom. **8** (1990), 193–209.

[Se] Sekigawa K., *Notes on homogeneous almost Hermitian manifolds*, Hokkaido Math. J. **7** (1978), 206–213.

[Si] Singer I.M., *Infinitesimally homogeneous spaces*, Comm. Pure Appl. Math. **13** (1960), 685–697.

[Tr] Tricerri F., *Locally homogeneous Riemannian manifolds*, Rend. Sem. Mat. Univ. Politec. Torino **50/4** (1993), 411–426.

[TV] Tricerri F., Vanhecke L., *Homogeneous Structures on Riemannian Manifolds*, London Mathematical Society Lecture Notes Series 83, Cambridge University Press, Cambridge, 1983.

[TW] Tricerri F., Watanabe Y., *Infinitesimal models and locally homogeneous almost Hermitian manifolds*, Math. J. Toyama Univ. **18** (1995), 147–154.

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(*Received* September 30, 1998)