Products, the Baire category theorem, and the axiom of dependent choice

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Abstract. In **ZF** (i.e., Zermelo-Fraenkel set theory without the Axiom of Choice) the following statements are shown to be equivalent:

(1) The axiom of dependent choice.
(2) Products of compact Hausdorff spaces are Baire.
(3) Products of pseudocompact spaces are Baire.
(4) Products of countably compact, regular spaces are Baire.
(5) Products of regular-closed spaces are Baire.
(6) Products of Čech-complete spaces are Baire.
(7) Products of pseudo-complete spaces are Baire.

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Concerning the status of the Baire category theorem for compact Hausdorff respectively Čech-complete spaces in **ZF** the following results are known:

Theorem 1 ([1], [8]). *Čech-complete spaces are Baire if and only if the axiom of dependent choice holds.*

Theorem 2 ([7]). *Compact Hausdorff spaces are Baire if and only if the axiom of dependent multiple choice holds.*

Theorem 3 ([3], [11]). *Countable products of compact Hausdorff spaces are Baire if and only if the axiom of dependent choice holds.*

The natural question asking for the set-theoretical status of the statement "arbitrary products of compact Hausdorff (resp. Čech-complete) spaces are Baire" has been left open so far. The purpose of this note is to close this gap. Recall:

Definitions. (1) A topological space X is called *Baire* provided that in X the intersection of any sequence of dense open sets is dense.
(2) A filter[1] on a space is called *regular* provided that it has a closed base and an

[1]Filters on X are always supposed to be *proper* subsets of the power set of X.

open base.
(3) A topological space X is called *regular-closed* provided that X is regular and any regular filter on X has a non-empty intersection. See [10].
(4) A collection $\mathcal{B}$ of non-empty open sets of a topological space X is called a *regular pseudo-base* for X provided that $\mathcal{B}$ satisfies the following conditions:

(α) for each non-empty open set A in X there exists some $B \in \mathcal{B}$ with $clB \subset A$,
(β) if A is a non-empty open subset of some $B \in \mathcal{B}$, then $A \in \mathcal{B}$.

(5) A topological space X is called *pseudo-complete* provided that it has a sequence $(\mathcal{B}_n)_{n\in\mathbb{N}}$ of regular pseudo-bases such that every regular filter on X, that has a countable base and meets each $\mathcal{B}_n$, has a non-empty intersection. (See [Ox]). Such a sequence of regular pseudo-bases will be called *suitable* for X.

Remark. Each compact Hausdorff space is simultaneously

(a) countably compact and regular,
(b) pseudocompact,
(c) regular-closed,
(d) Čech-complete.

Moreover, each topological space that satisfies (a), (b), (c) or (d) is pseudo-complete.

Theorem 4. *The following conditions are equivalent:*

1. *The axiom of dependent choice.*
2. *Countable products of compact Hausdorff spaces are Baire.*
3. *Products of compact Hausdorff spaces are Baire.*
4. *Products of pseudocompact spaces are Baire.*
5. *Products of countably compact, regular spaces are Baire.*
6. *Products of regular-closed spaces are Baire.*
7. *Products of Čech-complete spaces are Baire.*
8. *Products of pseudo-complete spaces are Baire.*

Proof: In view of the above Remark, condition (8) implies the conditions (4), (5), (6), and (7), and moreover, each of the latter conditions implies condition (3). Since the implication (3) $\Rightarrow$ (2) holds trivially and the implication (2) $\Rightarrow$ (1) holds by Theorem 3, it remains to be shown that condition (1) implies condition (8).

Assume condition (1) to hold. Let $(X_i)_{i\in I}$ be a family of pseudo-complete spaces and let $X = \prod_{i\in I} X_i$ be the corresponding product with projections $\pi_i\colon X \longrightarrow X_i$.

Case 1: $X = \emptyset$.
Then X is Baire.

Case 2: $X \neq \emptyset$.
Let $x = (x_i)_{i\in I}$ be a fixed element of X. Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of dense open subsets of X and let B be a non-empty open subset of X. Consider the set Y of all quadruples

$$\Big(n, F, (B_i)_{i\in F},\ (\mathcal{B}_i)_{i\in F}\Big)$$

consisting of

a) a natural number n,
b) a finite subset F of I,
c) a family $(B_i)_{i\in F}$ of non-empty open subsets B_i of X_i,
d) a family $(\mathcal{B}_i)_{i\in F}$ of suitable sequences $(\mathcal{B}_i^n)_{n\in\mathbb{N}}$ of regular pseudo-bases for X_i,

subject to the following conditions:

e) $\bigcap_{i\in F}\pi_i^{-1}[B_i]\subset(B\cap D_n)$,
f) $B_i\in\mathcal{B}_i^m$ for each $i\in F$ and each $m\leq n$.

The fact that each $\mathcal{B}_i^m$ is a regular pseudo-base implies that Y is non-empty. Consider further the relation ϱ defined on Y by:

$$\begin{aligned}\text{If}\quad & y=\Big(n,F,(B_i)_{i\in F},\ (\mathcal{B}_i)_{i\in F}\Big)\\ \text{and}\quad & \tilde{y}=\Big(\tilde{n},\tilde{F},(\tilde{B}_i)_{i\in\tilde{F}},\ (\tilde{\mathcal{B}}_i)_{i\in\tilde{F}}\Big)\end{aligned}$$

then $y\varrho\tilde{y}$ iff the following conditions are satisfied:

α) $n+1=\tilde{n}$,
β) $F\subset\tilde{F}$,
γ) $cl_{X_i}\tilde{B}_i\subset B_i$ for each $i\in F$,
δ) $\mathcal{B}_i=\tilde{\mathcal{B}}_i$ for each $i\in F$.

The fact that each $\mathcal{B}_i^m$ is a regular pseudo-base implies that for each $y\in Y$ there exists some $\tilde{y}\in Y$ with $y\varrho\tilde{y}$. Thus condition (1) guarantees the existence of a sequence $(y_n)_{n\in\mathbb{N}}$ in Y with $y_n\varrho y_{n+1}$ for each n, and $y_n=\big(n,F_n,(B_i^n)_{i\in F_n},(\mathcal{B}_i)_{i\in F_n}\big)$. The set $F=\bigcup_{n\in\mathbb{N}}F_n$ is, by condition (1), as a countable union of finite sets at most countable. For each $i\in F$, consider $n_i=\min\{n\in\mathbb{N}\mid i\in F_n\}$. Then for each $i\in F$ the sequence $(B_i^n)_{n\geq n_i}$ is a base for a regular filter on X_i with $B_i^m\in\mathcal{B}_i^n$ for all $m\geq n_i$ and all $n\leq m$. Thus pseudo-completeness of the X_i's implies that, for each $i\in F$, the set $B_i=\bigcap_{n\geq n_i}B_i^n$ is non-empty. By countability of F and the fact that (1) implies the axiom of countable choice, there exists an element $(b_i)_{i\in F}$ in $\prod_{i\in F}B_i$. Thus the point $(y_i)_{i\in I}$, defined by $y_i=\begin{cases}b_i, & \text{if } i\in F\\ x_i, & \text{if } i\in(I\backslash F),\end{cases}$ belongs to $B\cap\bigcap_{n\in\mathbb{N}}D_n$. Consequently $\bigcap_{n\in\mathbb{N}}D_n$ is dense in X. □

Remarks. (1) That Case 1 in the above proof may occur even if all the X_i's are non-empty compact Hausdorff spaces is shown by the model $\mathcal{N}15$ in [12]. Thus in **ZF** the statement

(*) Products of non-empty compact Hausdorff spaces are non-empty and Baire

is properly stronger than the axiom of dependent choice.

(2) By Theorem 4 each of the statement (1)–(8) is a theorem in **ZFC** (i.e., Zermelo-Fraenkel set theory including the axiom of choice). In particular the following are known:

(a) Complete metric spaces are Baire. See Hausdorff [9].
(b) Products of completely metrizable spaces are Baire. See Bourbaki [2].
(c) Compact Hausdorff spaces are Baire. See R.L. Moore [13].
(d) (Countably) Čech-complete spaces are Baire. See Čech [4] and Goldblatt [8].
(e) Products of Čech-complete spaces are Baire. See Oxtoby [14].
(f) Countably compact, regular spaces are Baire. See Colmez [5].
(g) Pseudocompact spaces are Baire. See Colmez [5].
(h) Pseudo-complete spaces are Baire. See Oxtoby [14].

Observe that in **ZFC** none of the following properties is closed under the formation of products:

α) Baire (see, e.g., [6, 3.9.J.]),
β) pseudocompact (see, e.g., [6, Example 3.10.19.]),
γ) countably compact, regular (see, e.g., [6, Example 3.10.19.]),
δ) regular-closed (see [15]),
ϵ) Čech-complete (see, e.g., [6, 3.9.D.(a)]).

Observe further that in **ZFC** all the above results follow from Oxtoby's [14] results (h) above and

(i) Products of pseudo-complete spaces are pseudo-complete.

But, whereas (h) holds in **ZF** + **DC** (= the axiom of dependent choice), the result (i) seems to require far stronger selection principles. Thus each of the results (3)–(8), considered as a theorem in **ZF** + **DC**, is new.

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