Embedding of the ordinal segment $[0, \omega_1]$ into continuous images of Valdivia compacta

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Abstract. We prove in particular that a continuous image of a Valdivia compact space is Corson provided it contains no homeomorphic copy of the ordinal segment $[0, \omega_1]$. This generalizes a result of R. Deville and G. Godefroy who proved it for Valdivia compact spaces. We give also a refinement of their result which yields a pointwise version of retractions on a Valdivia compact space.

Keywords: Corson compact space, Valdivia compact space, continuous image, ordinal segment

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1. Introduction

Valdivia compact spaces play an important role in the study of non-separable Banach spaces. In particular, they are closely related with projectional resolutions of the identity and Markuševič bases — see e.g. [V1], [V2], [DG], [FGZ] and [K4]. Valdivia compacta are generalization of Corson compact spaces, and the distance between these two classes is in a way expressed by the result of [DG] that a Valdivia compact space which does not contain any homeomorphic copy of the ordinal segment $[0, \omega_1]$ is already Corson. We extend this result to the class of all continuous images of Valdivia compacta. This seems to be of some interest as the class of Valdivia compact spaces is not closed under continuous images [V3], in fact any compact space all continuous images of which are Valdivia is already Corson ([K2]). However, the class of continuous images of Valdivia compacta shares many properties of that of Valdivia compacta. Using results of [K2] it is easy to observe that this class is closed with respect to closed G_δ subspaces and arbitrary products (and, of course, to continuous images). Further $C(K)$ has an equivalent locally uniformly rotund norm whenever K is a continuous image of a Valdivia compact ([V1]). Our result shows another shared property.

Moreover, it gives a new method to prove that a given compact space is not continuous image of a Valdivia compact (cf. Remark 2 after Theorem 1). Further applications will be given in [K5], where it will be used to find a subspace of $C(K)$ with non-Valdivia dual unit ball for any non-Corson compact K which is a continuous image of a Valdivia compact.

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We also give a refined version of the mentioned result of [DG] which can be viewed as a pointwise version of the existence of a retractional resolution of the identity in a Valdivia compact space (cf. [DG]).

Let us start with basic definitions.

Definition 1. Let Γ be a set.

(1) For $x \in \mathbb{R}^\Gamma$ we denote $\operatorname{supp} x = \{\gamma \in \Gamma \mid x(\gamma) \neq 0\}$.
(2) We put $\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \operatorname{supp} x \text{ is countable}\}$.

Definition 2. Let K be a compact Hausdorff space.

(1) K is called a *Corson compact* space if K is homeomorphic to a subset of $\Sigma(\Gamma)$ for some Γ.
(2) K is called a *Valdivia compact* space if K is homeomorphic to a subset K' of $\mathbb{R}^\Gamma$ for some Γ such that $K' \cap \Sigma(\Gamma)$ is dense in K'.

It turned out to be useful to introduce the following auxiliary notion.

Definition 3. Let K be a compact Hausdorff space and $A \subset K$ be arbitrary. We say that A is a Σ*-subset* of K if there is a homeomorphic injection φ of K into $\mathbb{R}^\Gamma$ for some Γ such that $\varphi(A) = \varphi(K) \cap \Sigma(\Gamma)$.

In this setting a compact K is Valdivia if it has a dense Σ-subset.

2. Main results

Our main result is the following theorem generalizing [DG, Proposition III-1].

Theorem 1. *Let $f : K \to L$ be a continuous surjection between compact Hausdorff spaces and K be a Valdivia compact. Then the following assertions are equivalent.*

(1) *L is a Corson compact space.*
(2) *L contains no homeomorphic copy of the ordinal segment $[0, \omega_1]$.*
(3) *There is a dense Σ-subset A of K such that $f \restriction A$ is a closed mapping of A into K.*
(4) *For every dense Σ-subset A of K the restriction $f \restriction A$ is a closed mapping of A onto K.*

Remark 1. In fact, the main result is the implication (2) $\Rightarrow$ (1). The equivalence of the remaining three conditions follows from the results of [G].

Remark 2. Using Theorem 1 one can show that, for example, the "double-arrow space" (see e.g. [F, Section 2.3]) as well as any its non-metrizable modification from [K1] is not continuous image of a Valdivia compact, even though it is well-known that the space of continuous functions on each such space has an equivalent locally uniformly rotund norm.

Remark 3. The referee pointed out that Theorem 1 also implies that no separable non-metrizable Rosenthal compact is a continuous image of a Valdivia compact.

Rosenthal compact spaces form a class containing the above mentioned "double-arrow space". They are studied e.g. in [GT].

In the proof of Theorem 1 we will use the following result, which refines the proof of [DG, Proposition III-1]. This theorem can be viewed as a pointwise version of retractions. Some weaker versions were needed in [K2] and [K4].

Theorem 2. *Let $K \subset \mathbb{R}^\Gamma$ be a compact space such that $K \cap \Sigma(\Gamma)$ is dense in K. If $x \in K$ is such that $\operatorname{card}\operatorname{supp} x = \kappa$ is uncountable, then there is a continuous one-to-one map $\varphi : [0, \kappa] \to K$ such that the following holds.*

(i) $\operatorname{supp}\varphi(\alpha) \subset \operatorname{supp} x$ *for every* $\alpha < \kappa$*;*
(ii) $\varphi(\kappa) = x$*;*
(iii) $\operatorname{card}\operatorname{supp}\varphi(\alpha) \le \max(\operatorname{card}\alpha, \aleph_0)$ *for every* $\alpha < \kappa$*.*

3. Auxiliary results

In this section we give some results which will be used in the proofs of our main theorems. We will need the following notions.

Definition 4. Let κ be an uncountable cardinal.

(1) If Γ is a set, we put

$$\Sigma_\kappa(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \operatorname{card}(\operatorname{supp} x) < \kappa\}.$$

(2) We say that a subset A of a topological space X is *κ-closed* in X if $\overline{C} \subset A$ for every $C \subset A$ with $\operatorname{card} C < \kappa$.

(3) A topological space X is called *κ-compact* if $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F}$ is a centered family of closed subsets of X with $\operatorname{card}\mathcal{F} < \kappa$. Let us recall that a family is called *centered* if every finite subfamily has nonempty intersection.

Remark. It is clear that any compact space is κ-compact for any cardinal κ. $\aleph_1$-compact spaces are usually called *countably compact*. It is not difficult to prove, by transfinite induction on κ, that any κ-closed subset of a κ-compact (in particular compact) space is κ-compact. (Let it hold for any $\kappa < \lambda$ and A be a λ-closed subset of a λ-compact space K. Let $(F_\alpha \mid \alpha < \mu)$ be a centered family of relatively closed subsets of A, with μ being a cardinal strictly less than λ. By the induction hypothesis we can suppose without loss of generality that the transfinite sequence F_α is non-increasing. We finish by the standard argument, taking $f_\alpha \in F_\alpha$ for every $\alpha < \mu$ and by applying λ-compactness of K to the sets $\{f_\gamma : \gamma \in [\alpha, \mu)\}$, $\alpha < \mu$.)

The following lemma is a straightforward consequence of definitions.

Lemma 1. *The space $\Sigma_\kappa(\Gamma)$ is κ-closed in $\mathbb{R}^\Gamma$ for every set Γ and every uncountable regular cardinal κ.*

Next we will need a lemma on regular spaces.

Lemma 2. *Let K be a Hausdorff regular space. If λ is an ordinal and $(G_\alpha \mid \alpha < \lambda)$ a family of open subsets of K with $x \in \bigcap_{\alpha<\lambda} G_\alpha$, then there is a family $(H_\alpha \mid \alpha < \lambda)$ of subsets of K satisfying the following conditions.*

(i) *H_α is the intersection of at most* $\operatorname{card}\alpha$ *open subsets of K.*

(ii) *H_α is the intersection of an open and a closed subset of K.*

(iii) *$x \in H_\alpha \subset \overline{H_\alpha} \subset G_\alpha \cap \bigcap_{\beta<\alpha} H_\beta$.*

Proof: We will proceed by transfinite induction. Choose H_0 open such that $x \in H_0 \subset \overline{H_0} \subset G_0$. Suppose that $0 < \alpha < \lambda$ and we have constructed H_β for $\beta < \alpha$ such that the conditions (i)–(iii) are satisfied. We are going to construct H_α. There are two possibilities.

(a) $\alpha = \beta + 1$ for some β. Then by (ii) we have $H_\beta = F \cap G$ with F closed and G open. By (iii) and the assumptions on x we get $x \in F \cap G \cap G_\alpha$. As F is regular and closed in K, there is a relatively open subset $H_\alpha \subset F$ with $x \in H_\alpha \subset \overline{H_\alpha} \subset F \cap G \cap G_\alpha$. Then clearly the condition (iii) holds for α, too. The condition (ii) follows from the fact that H_α is relatively open in F, and the condition (i) from the fact that H_α is relatively open in H_β.

(b) α is limit. Put $\tilde{H}_\alpha = \bigcap_{\beta<\alpha} H_\beta$. Due to (i) the set $\tilde{H}_\alpha$ is the intersection of at most $\operatorname{card}\alpha$ open subsets of K. By (iii) we have $\bigcap_{\beta<\alpha} H_\beta = \bigcap_{\beta<\alpha} \overline{H_\beta}$, so $\tilde{H}_\alpha$ is closed. It follows from (iii) and the assumptions that $x \in G_\alpha \cap \tilde{H}_\alpha$. By the regularity of $\tilde{H}_\alpha$ there is a relatively open set $H_\alpha \subset \tilde{H}_\alpha$ with $x \in H_\alpha \subset \overline{H_\alpha} \subset \tilde{H}_\alpha \cap G_\alpha$. It is clear that (i)–(iii) hold for α too.

This completes the proof. ☐

The following lemma generalizes [K2, Lemma 2.3] which was widely used in [K2], [K3], [K4] and [FGZ].

Lemma 3. *Let K be a compact Hausdorff space, κ an infinite cardinal, and $A \subset K$ a dense κ^+-compact subset of K. Then $G \cap A$ is dense in G whenever $G \subset K$ is the intersection of at most κ open sets.*

Proof: It is clearly enough to show that $G \cap A \neq \emptyset$ whenever $G \subset K$ is a nonempty intersection of at most κ open sets. We will prove it by transfinite induction. First, by density of A, we have that $G \cap A \neq \emptyset$ whenever G is a nonempty intersection of finitely many open sets. Let us suppose that $\lambda \leq \kappa$ is an infinite cardinal such that $G \cap A \neq \emptyset$ whenever G is a nonempty intersection of less than λ open sets.

Let $G = \bigcap_{\alpha<\lambda} G_\alpha$, where each G_α is open in K and $x \in G$. Let H_α, $\alpha < \lambda$, be as in Lemma 2.

By the condition (i) of Lemma 2 and the induction hypothesis on λ we have that $H_\alpha \cap A \neq \emptyset$ for every $\alpha < \lambda$. It follows from the condition (iii) of Lemma 2

that
$$\bigcap_{\alpha<\lambda} H_\alpha \cap A = \bigcap_{\alpha<\lambda} \overline{H_\alpha} \cap A \neq \emptyset,$$
as A is κ^+-compact. Now it follows that $G \cap A \neq \emptyset$ which completes the proof. □

Finally we give the following lemma which generalizes the well-known fact that any continuous mapping between compact Hausdorff spaces is necessarily closed.

Lemma 4. *Let $f : K \to L$ be a continuous mapping between two compact Hausdorff spaces, and let κ be an uncountable cardinal. Then $f(A)$ is κ-closed in L whenever A is a κ-closed subset of K.*

Proof: Let $C \subset f(A)$ be such that $\operatorname{card} C < \kappa$. For each $c \in C$ choose some $b_c \in A$ with $f(b_c) = c$. Put $B = \{b_c \mid c \in C\}$. Then $B \subset A$ and $\operatorname{card} B = \operatorname{card} C < \kappa$, so $\overline{B} \subset A$. As $\overline{B}$ is compact, its image $f(\overline{B})$ is compact as well. Moreover, clearly $C \subset f(\overline{B}) \subset f(A)$. Therefore $\overline{C} \subset f(A)$ which completes the proof. □

4. Proofs of the main results

Proof of Theorem 2: Let $x \in K$ be such that $\kappa = \operatorname{card} \operatorname{supp} x$ is uncountable. Put $I = \operatorname{supp} x$ and fix an enumeration $I = \{i_\alpha \mid \alpha < \kappa\}$. We will construct by transfinite induction $x_\alpha \in K$ and $J_\alpha \subset \Gamma$ for $\alpha < \kappa$ such that the following conditions hold.

(i) $i_\alpha \in J_{\alpha+1}$, $\bigcup_{\beta<\alpha} \operatorname{supp} x_\beta \subset J_\alpha$, $\operatorname{card} J_\alpha \leq \max(\operatorname{card} \alpha, \aleph_0)$;

(ii) $J_\alpha \subset J_{\alpha+1}$, $\operatorname{supp} x_\alpha \cap I \subsetneqq J_{\alpha+1} \cap I$;

(iii) $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$ if α is limit;

(iv) $x_\alpha(i) = x(i)$ for every $i \in J_\alpha$ and $\operatorname{card} \operatorname{supp} x_\alpha \leq \max(\operatorname{card} \alpha, \aleph_0)$;

(v) $x_\alpha = \lim_{\beta<\alpha} x_\beta$ if α is limit.

Put $J_0 = \{i_0\}$ and choose $x_0 \in K \cap \Sigma(\Gamma)$ such that $x_0(i_0) = x(i_0)$. This is possible due to Lemma 3 as the set $\{y \in K \mid y(i_0) = x(i_0)\}$ is G_δ and $K \cap \Sigma(\Gamma)$ is $\aleph_1$-closed in K.

Suppose that we have J_α and x_α. As $\operatorname{card}(\operatorname{supp} x_\alpha) \leq \max(\operatorname{card} \alpha, \aleph_0) < \kappa$ and $\operatorname{card} I = \kappa$, there is some $j \in I \setminus \operatorname{supp} x_\alpha$. Put $J_{\alpha+1} = J_\alpha \cup \operatorname{supp} x_\alpha \cup \{i_\alpha, j\}$. Clearly $\operatorname{card} J_{\alpha+1} \leq \max(\operatorname{card} \alpha, \aleph_0)$. Since $\{y \in K \mid y(i) = x(i) \text{ for } i \in J_{\alpha+1}\}$ is the intersection of at most $\max(\operatorname{card} \alpha, \aleph_0)$ open sets, we can (due to Lemma 3 and Lemma 1) choose $x_{\alpha+1} \in K \cap \Sigma_{\max((\operatorname{card} \alpha)^+, \aleph_1)}(\Gamma)$ such that $x_{\alpha+1}(i) = x(i)$ for every $i \in J_{\alpha+1}$.

Now suppose that α is limit and we have constructed x_β and J_β for every $\beta < \alpha$. Put $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$. Let us remark, that the net x_β, $\beta < \alpha$, converges to some point, which will be denoted x_α. Indeed, if $i \in \Gamma \setminus J_\alpha$, then $x_\beta(i) = 0$ for every $\beta < \alpha$. If $i \in J_\alpha$, then $i \in J_\beta$ for some $\beta < \alpha$, and thus $x_\gamma(i) = x(i)$ for $\beta \le \gamma < \alpha$. So the mentioned net converges to the point x_α such that $x_\alpha(i) = x(i)$ for $i \in J_\alpha$ and $x_\alpha(i) = 0$ if $i \in \Gamma \setminus J_\alpha$. As K is compact, we have $x_\alpha \in K$.

This completes the construction.

Now put $J = \bigcup_{\alpha<\kappa} J_\alpha$ and $x_\kappa = \lim_{\alpha<\kappa} x_\alpha$. In the same way as in the limit induction step it can be shown that this limit exists and that $x_\kappa(i) = x(i)$ for $i \in J$ and $x_\kappa(i) = 0$ otherwise. Further, $J \supset I$, so $x = x_\kappa$.

It is clear from the construction that $\operatorname{supp} x_\alpha \subset \operatorname{supp} x$ for each limit $\alpha < \kappa$. If we denote by L the set of all limit ordinals from $[0,\kappa]$, there is clearly a bijective increasing mapping $\psi : [0,\kappa] \to L$. Let us define $\varphi : [0,\kappa] \to K$ by the formula $\varphi(\alpha) = x_{\psi(\alpha)}$. This mapping is one-to-one by the condition (ii) and the continuity follows easily from the condition (v) and the definition of x_κ. This completes the proof. $\square$

Proof of Theorem 1: (1) $\Rightarrow$ (2) follows from the well known fact that $[0,\omega_1]$ is not a Corson compact (see e.g. [DG]).

(1) $\Rightarrow$ (4) Let A be a dense Σ-subset of K and $F \subset A$ an arbitrary relatively closed subset. Then F is $\aleph_1$-closed in K (by Lemma 1), so $f(F)$ is $\aleph_1$-closed in L by Lemma 4. But as L is Corson, it is angelic (by [N, Theorem 2.1]), in particular any $\aleph_1$-closed subset of L is already closed. So $f \restriction A$ is a closed mapping. In particular $f(A)$ is closed in L and since it is dense (by continuity of f), necessarily $f(A) = L$.

(4) $\Rightarrow$ (3) This is clear since there is at least one dense Σ-subset of K as K is Valdivia.

(3) $\Rightarrow$ (1) It is clear that $f(A) = L$, so $f \restriction A$ is a continuous closed map of A onto L, in particular it is a quotient mapping. It remains to use the fact that countably compact subsets of $\Sigma(\Gamma)$ are stable with respect to quotient mappings. This is a result of Gul'ko [G]. The original proof is done by transfinite induction on cardinality of Γ, using the existence of certain families of retractions on closed subsets of $\Sigma(\Gamma)$. Another proof, less direct and dependent on Gul'ko result, using a Pol-like characterization of countably compact subsets of $\Sigma(\Gamma)$ is given in [K3, Theorem 2.20]. So, using the mentioned results, it follows that L is Corson.

(2) $\Rightarrow$ (3) Let A be a dense Σ-subset of K and $F \subset A$ a relatively closed subset. Then clearly F is a dense Σ-subset of $\overline{F}$. Let $h : \overline{F} \to \mathbb{R}^\Gamma$ be a homeomorphic embedding such that $h(F) = h(\overline{F}) \cap \Sigma(\Gamma)$. We will show that $f(F) = f(\overline{F})$. Suppose it is not the case. Let $\kappa = \min\{\operatorname{card} \operatorname{supp} h(x) \mid x \in \overline{F} \ \&\ f(x) \notin f(F)\}$ and let $x \in \overline{F}$ be such that $\operatorname{card} h(x) = \kappa$ and $f(x) \notin f(F)$. It is clear that κ is uncountable. Let $\varphi : [0,\kappa] \to \overline{F}$ be the mapping from Theorem 2.

If κ is singular, then there is an infinite cardinal $\lambda < \kappa$ and cardinals $(\tau_\gamma)_{\gamma<\lambda}$ with $\tau_\gamma < \kappa$ for $\gamma < \lambda$ and $\kappa = \sup_{\gamma<\lambda} \tau_\gamma$. By the definition of κ we have $f(\varphi(\tau_\gamma)) \in f(F)$. Moreover, $f(F) = f(\overline{F} \cap h^{-1}(\Sigma_{\lambda^+}(\Gamma)))$, so $f(F)$ is λ^+-closed in L (by

Lemma 1 and Lemma 4). In particular, $f(x) \in f(F)$, as it is in the closure of the set $\{f(\varphi(\tau_\gamma)) \mid \gamma < \lambda\}$. This is a contradiction.

Hence κ is a regular cardinal. Put $g = f \circ \varphi$. At first let us note that $\operatorname{card} g((\alpha, \kappa)) = \kappa$ for every $\alpha < \kappa$. Otherwise by regularity of κ, the set $g^{-1}(l)$ is unbounded in $[0, \kappa)$ for some $l \in f(F)$. But then $x = g(\kappa) \in f(F)$, as it is equal to l by continuity of g.

So we can choose by transfinite induction ordinals $\eta_\alpha < \kappa$ for $\alpha < \kappa$ such that

(a) $\eta_{\alpha+1} > \eta_\alpha$;

(b) $g(\eta_{\alpha+1}) \notin \{g(\eta_\beta) \mid \beta \leq \alpha\} \cup \{g(\kappa)\}$;

(c) $\eta_\alpha = \sup\limits_{\beta<\alpha} \eta_\beta$ for $\alpha < \kappa$ limit.

Now it is clear that L contains a homeomorphic copy of $[0, \kappa]$, and therefore also that of $[0, \omega_1]$, as κ is uncountable. $\square$

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